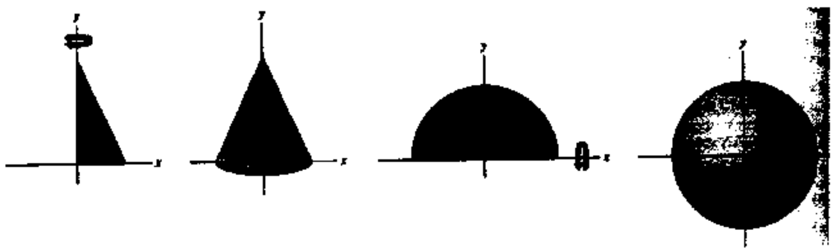
Volumes of Revolution
AP Calculus

Answers

We've been finding areas between curves. Now let's find volumes of three-dimensional solids! We'll begin by looking at "solids of revolution," which are basically solids that are created by rotating (or revolving) a region in the plane about an axis.

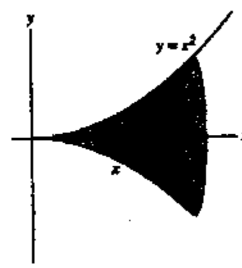
A right circular cone is an example of a solid of revolution as is a sphere. See below.



See over for volumes →

You can create all sorts of fun solids by rotating an area between curves over an axis or another line.

Here's an example of a solid created by rotating the region under $y = x^2$ about the x-axis for $0 \le x \le 2$

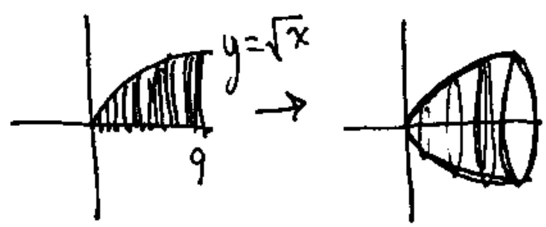


Your mission:

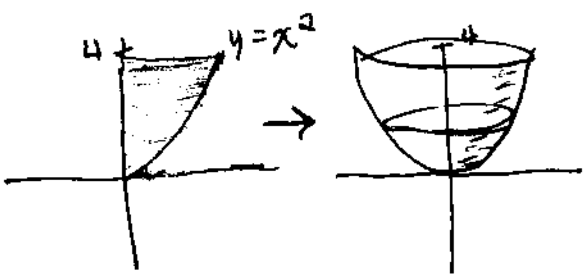
Draw a sketch of each curve described below. Then make a sketch of what the 3-D solid would look like after the region is rotated over the given axis or line. As you do this, think about how you might be able to find the volumes of these solids.

1) $y = x^2$ rotated over the y-axis for $0 \le y \le 4$
2) $y = x + 2$ rotated over the x-axis for $0 \le x \le 4$
3) $y = \sqrt{x}$ rotated over the x-axis for $0 \le x \le 9$
4) $y = \sqrt{x}$ rotated over the line $y = -2$ for $0 \le x \le 9$

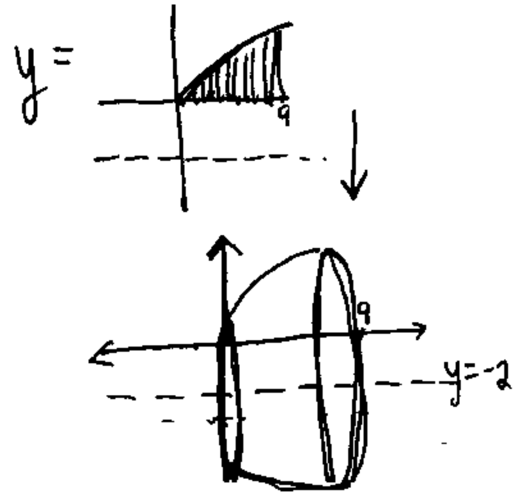
3)



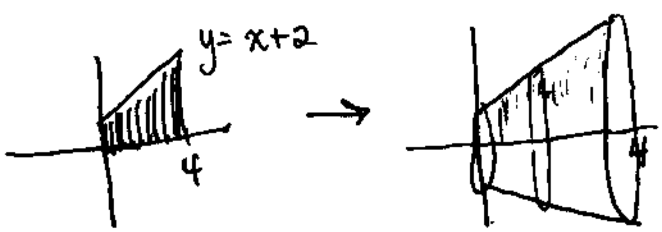
1)

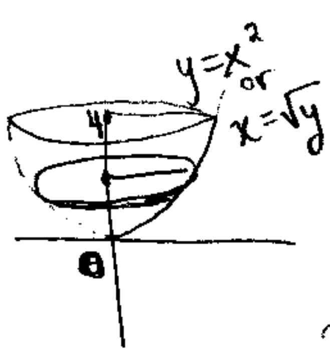


4)



2)



1) 

#1 will be in terms of y. Why? Each cross section is a circle (or disc) so find the area of each circle (πr^2), multiply each by a "thickness" of dy, & then add up the infinite number of discs!

$$\text{Volume} = \pi\int_a^b r^2\,dy = \pi\int_0^4 \sqrt{y}^{\,2}\,dy = \pi\int_0^4 y\,dy$$

$$= \pi\left[\tfrac{1}{2}y^2\right]_0^4 = \pi\left(\tfrac{1}{2}(16)-\tfrac{1}{2}(0)\right) = \boxed{8\pi}$$

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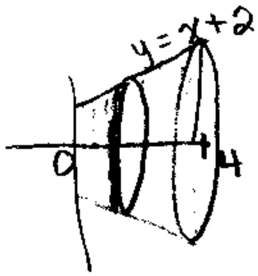
2) 

Each cross section ($\perp$ to x-axis) is a circular disc w/thickness "dx". The integral adds up all the discs to get the total volume.

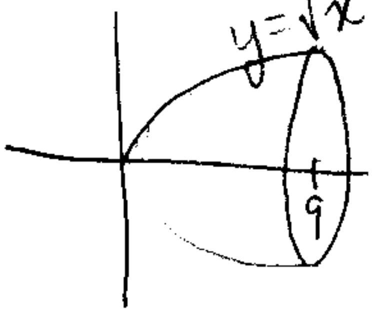
$$\text{Volume} = \underbrace{\pi\int_0^4 (x+2)^2}_{\pi r^2}\,dx = \pi\left[\tfrac{1}{3}(x+2)^3\right]_0^4$$

$$= \pi\left(\tfrac{1}{3}(6)^3-\tfrac{1}{3}(2)^3\right) = \boxed{\pi\left(72-\tfrac{8}{3}\right)}$$

or $\frac{208}{3}\pi$

$\longrightarrow$ over

#3

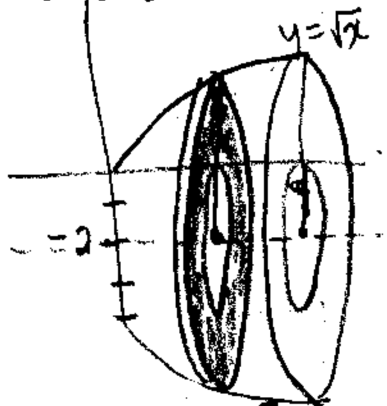


$$V = \pi \int_0^9 (\sqrt{x})^2\,dx = \pi \int_0^9 x\,dx$$

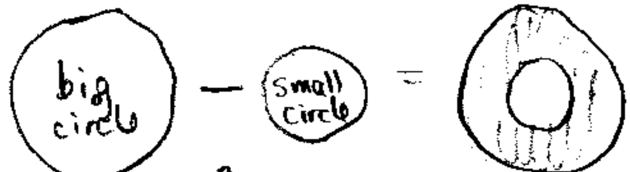
$$= \pi \left[\tfrac{1}{2}x^2\right]_0^9 = \pi\left(\tfrac{1}{2}(81) - \tfrac{1}{2}(0)\right)$$

$$= \boxed{\frac{81\pi}{2}}$$

#4 This is trickier. And (as always) we could do this in various ways. I'll use what's often called the "washer method" Each cross section is a <u>washer</u>, ie a disc with a hole in it.



Let's find the area of each washer



$$\pi\cdot\left(\text{big radius}\right)^2 - \pi\left(\text{small radius}\right)^2 = \text{Area of washer (shaded)}$$

$$V = \pi\int_0^9 \left((r_1)^2 - (r_2)^2\right)dx = \pi\int_0^9 \left((\sqrt{x}-(-2))^2 - (0-(-2))^2\right)dx$$

$$= \pi\int_0^9 \left((\sqrt{x}+2)^2 + 4\right)dx = \pi\int_0^9 x + 4x^{\frac{1}{2}} + 8\,dx = \pi\left[\tfrac{1}{2}x^2 + \tfrac{8}{3}x^{\frac{3}{2}} + 8x\right]_0^9$$

$$= \boxed{184.5\pi}$$