5.6 Substitution Method

Preliminary Questions

1. Which of the following integrals is a candidate for the Substitution Method?

(a) $\int 5x^4 \sin(x^5)\, dx$ **(b)** $\int \sin^5 x \cos x\, dx$ **(c)** $\int x^5 \sin x\, dx$

SOLUTION The function in **(c)**: $x^5 \sin x$ is not of the form $g(u(x))u'(x)$. The function in **(a)** meets the prescribed pattern with $g(u) = \sin u$ and $u(x) = x^5$. Similarly, the function in **(b)** meets the prescribed pattern with $g(u) = u^5$ and $u(x) = \sin x$.

2. Find an appropriate choice of u for evaluating the following integrals by substitution:

(a) $\int x(x^2+9)^4\, dx$ **(b)** $\int x^2 \sin(x^3)\, dx$ **(c)** $\int \sin x \cos^2 x\, dx$

SOLUTION

(a) $x(x^2+9)^4 = \frac{1}{2}(2x)(x^2+9)^4$; hence, $c = \frac{1}{2}$, $f(u) = u^4$, and $u(x) = x^2+9$.
(b) $x^2 \sin(x^3) = \frac{1}{3}(3x^2)\sin(x^3)$; hence, $c = \frac{1}{3}$, $f(u) = \sin u$, and $u(x) = x^3$.
(c) $\sin x \cos^2 x = -(-\sin x)\cos^2 x$; hence, $c = -1$, $f(u) = u^2$, and $u(x) = \cos x$.

3. Which of the following is equal to $\int_0^2 x^2(x^3+1)\, dx$ for a suitable substitution?

(a) $\frac{1}{3}\int_0^2 u\, du$ **(b)** $\int_0^9 u\, du$ **(c)** $\frac{1}{3}\int_1^9 u\, du$

SOLUTION With the substitution $u = x^3+1$, the definite integral $\int_0^2 x^2(x^3+1)\, dx$ becomes $\frac{1}{3}\int_1^9 u\, du$. The correct answer is **(c)**.

Exercises

In Exercises 1–6, calculate du.

1. $u = x^3 - x^2$

SOLUTION Let $u = x^3 - x^2$. Then $du = (3x^2 - 2x)\, dx$.

2. $u = 2x^4 + 8x^{-1}$

SOLUTION Let $u = 2x^4 + 8x^{-1}$. Then $du = (8x^3 - 8x^{-2})\, dx$.

3. $u = \cos(x^2)$

SOLUTION Let $u = \cos(x^2)$. Then $du = -\sin(x^2)\cdot 2x\, dx = -2x\sin(x^2)\, dx$.

4. $u = \tan x$

SOLUTION Let $u = \tan x$. Then $du = \sec^2 x\, dx$.

5. $u = e^{4x+1}$

SOLUTION Let $u = e^{4x+1}$. Then $du = 4e^{4x+1}\, dx$.

6. $u = \ln(x^4+1)$

SOLUTION Let $u = \ln(x^4+1)$. Then $du = \dfrac{4x^3}{x^4+1}\, dx$.

In Exercises 7–22, write the integral in terms of u and du. Then evaluate.

7. $\int (x-7)^3\, dx, \quad u = x-7$

SOLUTION Let $u = x-7$. Then $du = dx$. Hence

$$\int (x-7)^3\, dx = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{1}{4}(x-7)^4 + C.$$

8. $\int (x+25)^{-2}\, dx, \quad u = x+25$

SOLUTION Let $u = x+25$. Then $du = dx$ and

$$\int (x+25)^{-2}\, dx = \int u^{-2}\, du = -u^{-1} + C = -\frac{1}{x+25} + C.$$

9. $\displaystyle\int t\sqrt{t^2+1}\,dt, \quad u = t^2+1$

SOLUTION Let $u = t^2+1$. Then $du = 2t\,dt$. Hence,

$$\int t\sqrt{t^2+1}\,dt = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(t^2+1)^{3/2} + C.$$

10. $\displaystyle\int (x^3+1)\cos(x^4+4x)\,dx, \quad u = x^4+4x$

SOLUTION Let $u = x^4+4x$. Then $du = (4x^3+4)\,dx = 4(x^3+1)\,dx$ and

$$\int (x^3+1)\cos(x^4+4x)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+4x) + C.$$

11. $\displaystyle\int \frac{t^3}{(4-2t^4)^{11}}\,dt, \quad u = 4-2t^4$

SOLUTION Let $u = 4-2t^4$. Then $du = -8t^3\,dt$. Hence,

$$\int \frac{t^3}{(4-2t^4)^{11}}\,dt = -\frac{1}{8}\int u^{-11}\,du = \frac{1}{80}u^{-10} + C = \frac{1}{80}(4-2t^4)^{-10} + C.$$

12. $\displaystyle\int \sqrt{4x-1}\,dx, \quad u = 4x-1$

SOLUTION Let $u = 4x-1$. Then $du = 4\,dx$ or $\frac{1}{4}du = dx$. Hence

$$\int \sqrt{4x-1}\,dx = \frac{1}{4}\int u^{1/2}\,du = \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C = \frac{1}{6}(4x-1)^{3/2} + C.$$

13. $\displaystyle\int x(x+1)^9\,dx, \quad u = x+1$

SOLUTION Let $u = x+1$. Then $x = u-1$ and $du = dx$. Hence

$$\int x(x+1)^9\,dx = \int (u-1)u^9\,du = \int (u^{10}-u^9)\,du$$
$$= \frac{1}{11}u^{11} - \frac{1}{10}u^{10} + C = \frac{1}{11}(x+1)^{11} - \frac{1}{10}(x+1)^{10} + C.$$

14. $\displaystyle\int x\sqrt{4x-1}\,dx, \quad u = 4x-1$

SOLUTION Let $u = 4x-1$. Then $x = \frac{1}{4}(u+1)$ and $du = 4\,dx$ or $\frac{1}{4}\,du = dx$. Hence,

$$\int x\sqrt{4x-1}\,dx = \frac{1}{16}\int (u+1)u^{1/2}\,du = \frac{1}{16}\int (u^{3/2}+u^{1/2})\,du$$
$$= \frac{1}{16}\left(\frac{2}{5}u^{5/2}\right) + \frac{1}{16}\left(\frac{2}{3}u^{3/2}\right) + C$$
$$= \frac{1}{40}(4x-1)^{5/2} + \frac{1}{24}(4x-1)^{3/2} + C.$$

15. $\displaystyle\int x^2\sqrt{x+1}\,dx, \quad u = x+1$

SOLUTION Let $u = x+1$. Then $x = u-1$ and $du = dx$. Hence

$$\int x^2\sqrt{x+1}\,dx = \int (u-1)^2u^{1/2}\,du = \int (u^{5/2}-2u^{3/2}+u^{1/2})\,du$$
$$= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C.$$

16. $\displaystyle\int \sin(4\theta-7)\,d\theta, \quad u = 4\theta-7$

SOLUTION Let $u = 4\theta-7$. Then $du = 4\,d\theta$ and

$$\int \sin(4\theta-7)\,d\theta = \frac{1}{4}\int \sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos(4\theta-7) + C.$$

17. $\int \sin^2\theta \cos\theta\, d\theta, \quad u = \sin\theta$

SOLUTION Let $u = \sin\theta$. Then $du = \cos\theta\, d\theta$. Hence,

$$\int \sin^2\theta\cos\theta\, d\theta = \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C.$$

18. $\int \sec^2 x \tan x\, dx, \quad u = \tan x$

SOLUTION Let $u = \tan x$. Then $du = \sec^2 x\, dx$. Hence

$$\int \sec^2 x\tan x\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C.$$

19. $\int xe^{-x^2}\, dx, \quad u = -x^2$

SOLUTION Let $u = -x^2$. Then $du = -2x\, dx$ or $-\frac{1}{2}\, du = x\, dx$. Hence,

$$\int xe^{-x^2}\, dx = -\frac{1}{2}\int e^u\, du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

20. $\int (\sec^2 t)e^{\tan t}\, dt, \quad u = \tan t$

SOLUTION Let $u = \tan t$. Then $du = \sec^2 t\, dt$ and

$$\int (\sec^2 t)e^{\tan t}\, dt = \int e^u\, du = e^u + C = e^{\tan t} + C.$$

21. $\int \frac{(\ln x)^2\, dx}{x}, \quad u = \ln x$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$, and

$$\int \frac{(\ln x)^2}{x}\, dx = \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$$

22. $\int \frac{(\tan^{-1} x)^2\, dx}{x^2+1}, \quad u = \tan^{-1} x$

SOLUTION Let $u = \tan^{-1} x$. Then $du = \frac{1}{1+x^2}\, dx$, and

$$\int \frac{(\tan^{-1} x)^2}{x^2+1}\, dx = \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}(\tan^{-1} x)^3 + C.$$

In Exercises 23–26, evaluate the integral in the form $a\sin(u(x)) + C$ for an appropriate choice of $u(x)$ and constant a.

23. $\int x^3\cos(x^4)\, dx$

SOLUTION Let $u = x^4$. Then $du = 4x^3\, dx$ or $\frac{1}{4}\, du = x^3 dx$. Hence

$$\int x^3\cos(x^4)\, dx = \frac{1}{4}\int \cos u\, du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4) + C.$$

24. $\int x^2\cos(x^3+1)\, dx$

SOLUTION Let $u = x^3 + 1$. Then $du = 3x^2\, dx$ or $\frac{1}{3}\, du = x^2\, dx$. Hence

$$\int x^2\cos(x^3+1)\, dx = \frac{1}{3}\int \cos u\, du = \frac{1}{3}\sin u + C.$$

25. $\int x^{1/2}\cos(x^{3/2})\, dx$

SOLUTION Let $u = x^{3/2}$. Then $du = \frac{3}{2}x^{1/2}\, dx$ or $\frac{2}{3}\, du = x^{1/2}\, dx$. Hence

$$\int x^{1/2}\cos(x^{3/2})\, dx = \frac{2}{3}\int \cos u\, du = \frac{2}{3}\sin u + C = \frac{2}{3}\sin(x^{3/2}) + C.$$

26. $\int \cos x \cos(\sin x)\,dx$

SOLUTION Let $u = \sin x$. Then $du = \cos x\,dx$. Hence

$$\int \cos x \cos(\sin x)\,dx = \int \cos u\,du = \sin u + C.$$

In Exercises 27–72, evaluate the indefinite integral.

27. $\int (4x+5)^9\,dx$

SOLUTION Let $u = 4x + 5$. Then $du = 4\,dx$ and

$$\int (4x+5)^9\,dx = \frac{1}{4}\int u^9\,du = \frac{1}{40}u^{10} + C = \frac{1}{40}(4x+5)^{10} + C.$$

28. $\int \frac{dx}{(x-9)^5}$

SOLUTION Let $u = x - 9$. Then $du = dx$ and

$$\int \frac{dx}{(x-9)^5} = \int u^{-5}\,du = -\frac{1}{4}u^{-4} + C = -\frac{1}{4(x-9)^4} + C.$$

29. $\int \frac{dt}{\sqrt{t+12}}$

SOLUTION Let $u = t + 12$. Then $du = dt$ and

$$\int \frac{dt}{\sqrt{t+12}} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{t+12} + C.$$

30. $\int (9t+2)^{2/3}\,dt$

SOLUTION Let $u = 9t + 2$. Then $du = 9\,dt$ and

$$\int (9t+2)^{2/3}\,dt = \frac{1}{9}\int u^{2/3}\,du = \frac{1}{9}\cdot\frac{3}{5}u^{5/3} + C = \frac{1}{15}(9t+2)^{5/3} + C.$$

31. $\int \frac{x+1}{(x^2+2x)^3}\,dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx$ or $\frac{1}{2}du = (x+1)\,dx$. Hence

$$\int \frac{x+1}{(x^2+2x)^3}\,dx = \frac{1}{2}\int \frac{1}{u^3}\,du = \frac{1}{2}\left(-\frac{1}{2}u^{-2}\right) + C = -\frac{1}{4}(x^2+2x)^{-2} + C = \frac{-1}{4(x^2+2x)^2} + C.$$

32. $\int (x+1)(x^2+2x)^{3/4}\,dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx = 2(x+1)\,dx$ and

$$\int (x+1)(x^2+2x)^{3/4}\,dx = \frac{1}{2}\int u^{3/4}\,du = \frac{1}{2}\cdot\frac{4}{7}u^{7/4} + C$$
$$= \frac{2}{7}(x^2+2x)^{7/4} + C.$$

33. $\int \frac{x}{\sqrt{x^2+9}}\,dx$

SOLUTION Let $u = x^2 + 9$. Then $du = 2x\,dx$ or $\frac{1}{2}du = x\,dx$. Hence

$$\int \frac{x}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\frac{\sqrt{u}}{\frac{1}{2}} + C = \sqrt{x^2+9} + C.$$

34. $\int \frac{2x^2+x}{(4x^3+3x^2)^2}\,dx$

SOLUTION Let $u = 4x^3 + 3x^2$. Then $du = (12x^2+6x)\,dx$ or $\frac{1}{6}du = (2x^2+x)\,dx$. Hence

$$\int (4x^3+3x^2)^{-2}(2x^2+x)\,dx = \frac{1}{6}\int u^{-2}\,du = -\frac{1}{6}u^{-1} + C = -\frac{1}{6}(4x^3+3x^2)^{-1} + C.$$

35. $\int (3x^2+1)(x^3+x)^2\,dx$

SOLUTION Let $u = x^3 + x$. Then $du = (3x^2+1)\,dx$. Hence

$$\int (3x^2+1)(x^3+x)^2\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(x^3+x)^3 + C.$$

36. $\int \frac{5x^4+2x}{(x^5+x^2)^3}\,dx$

SOLUTION Let $u = x^5 + x^2$. Then $du = (5x^4+2x)\,dx$. Hence

$$\int \frac{5x^4+2x}{(x^5+x^2)^3}\,dx = \int \frac{1}{u^3}\,du = -\frac{1}{2}\frac{1}{u^2} + C = -\frac{1}{2}\frac{1}{(x^5+x^2)^2} + C.$$

37. $\int (3x+8)^{11}\,dx$

SOLUTION Let $u = 3x + 8$. Then $du = 3\,dx$ and

$$\int (3x+8)^{11}\,dx = \frac{1}{3}\int u^{11}\,du = \frac{1}{36}u^{12} + C = \frac{1}{36}(3x+8)^{12} + C.$$

38. $\int x(3x+8)^{11}\,dx$

SOLUTION Let $u = 3x + 8$. Then $du = 3\,dx$, $x = \frac{u-8}{3}$, and

$$\begin{aligned}\int x(3x+8)^{11}\,dx &= \frac{1}{9}\int (u-8)u^{11}\,du = \frac{1}{9}\int (u^{12} - 8u^{11})\,du\\ &= \frac{1}{9}\left(\frac{1}{13}u^{13} - \frac{2}{3}u^{12}\right) + C\\ &= \frac{1}{117}(3x+8)^{13} - \frac{2}{27}(3x+8)^{12} + C.\end{aligned}$$

39. $\int x^2\sqrt{x^3+1}\,dx$

SOLUTION Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ and

$$\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int u^{1/2}\,du = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C.$$

40. $\int x^5\sqrt{x^3+1}\,dx$

SOLUTION Let $u = x^3 + 1$. Then $du = 3x^2\,dx$, $x^3 = u - 1$ and

$$\begin{aligned}\int x^5\sqrt{x^3+1}\,dx &= \frac{1}{3}\int (u-1)\sqrt{u}\,du = \frac{1}{3}\int (u^{3/2} - u^{1/2})\,du\\ &= \frac{1}{3}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C\\ &= \frac{2}{15}(x^3+1)^{5/2} - \frac{2}{9}(x^3+1)^{3/2} + C.\end{aligned}$$

41. $\int \frac{dx}{(x+5)^3}$

SOLUTION Let $u = x + 5$. Then $du = dx$ and

$$\int \frac{dx}{(x+5)^3} = \int u^{-3}\,du = -\frac{1}{2}u^{-2} + C = -\frac{1}{2}(x+5)^{-2} + C.$$

42. $\int \frac{x^2\,dx}{(x+5)^3}$

SOLUTION Let $u = x + 5$. Then $du = dx$, $x = u - 5$ and

$$\int \frac{x^2\,dx}{(x+5)^3} = \int \frac{(u-5)^2}{u^3}\,du = \int (u^{-1} - 10u^{-2} + 25u^{-3})\,du$$

$$= \ln|u| + 10u^{-1} - \frac{25}{2}u^{-2} + C$$

$$= \ln|x+5| + \frac{10}{x+5} - \frac{25}{2(x+5)^2} + C.$$

43. $\int z^2(z^3+1)^{12}\,dz$

SOLUTION Let $u = z^3 + 1$. Then $du = 3z^2\,dz$ and

$$\int z^2(z^3+1)^{12}\,dz = \frac{1}{3}\int u^{12}\,du = \frac{1}{39}u^{13} + C = \frac{1}{39}(z^3+1)^{13} + C.$$

44. $\int (z^5 + 4z^2)(z^3+1)^{12}\,dz$

SOLUTION Let $u = z^3 + 1$. Then $du = 3z^2\,dz$, $z^3 = u - 1$ and

$$\int (z^5 + 4z^2)(z^3+1)^{12}\,dz = \frac{1}{3}\int (u+3)u^{12}\,du = \frac{1}{3}\int (u^{13} + 3u^{12})\,du$$

$$= \frac{1}{3}\left(\frac{1}{14}u^{14} + \frac{3}{13}u^{13}\right) + C$$

$$= \frac{1}{42}(z^3+1)^{14} + \frac{1}{13}(z^3+1)^{13} + C.$$

45. $\int (x+2)(x+1)^{1/4}\,dx$

SOLUTION Let $u = x + 1$. Then $x = u - 1$, $du = dx$ and

$$\int (x+2)(x+1)^{1/4}\,dx = \int (u+1)u^{1/4}\,du = \int (u^{5/4} + u^{1/4})\,du$$

$$= \frac{4}{9}u^{9/4} + \frac{4}{5}u^{5/4} + C$$

$$= \frac{4}{9}(x+1)^{9/4} + \frac{4}{5}(x+1)^{5/4} + C.$$

46. $\int x^3(x^2-1)^{3/2}\,dx$

SOLUTION Let $u = x^2 - 1$. Then $u + 1 = x^2$ and $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence

$$\int x^3(x^2-1)^{3/2}\,dx = \int x^2 \cdot x(x^2-1)^{3/2}\,dx$$

$$= \frac{1}{2}\int (u+1)u^{3/2}\,du = \frac{1}{2}\int (u^{5/2} + u^{3/2})\,du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2}\right) + \frac{1}{2}\left(\frac{2}{5}u^{5/2}\right) + C = \frac{1}{7}(x^2-1)^{7/2} + \frac{1}{5}(x^2-1)^{5/2} + C.$$

47. $\int \sin(8 - 3\theta)\,d\theta$

SOLUTION Let $u = 8 - 3\theta$. Then $du = -3\,d\theta$ and

$$\int \sin(8-3\theta)\,d\theta = -\frac{1}{3}\int \sin u\,du = \frac{1}{3}\cos u + C = \frac{1}{3}\cos(8-3\theta) + C.$$

48. $\int \theta\sin(\theta^2)\,d\theta$

SOLUTION Let $u = \theta^2$. Then $du = 2\theta\,d\theta$ and

$$\int \theta\sin(\theta^2)\,d\theta = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(\theta^2) + C.$$

49. $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$

SOLUTION Let $u = \sqrt{t} = t^{1/2}$. Then $du = \frac{1}{2}t^{-1/2}\,dt$ and

$$\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{t} + C.$$

50. $\int x^2 \sin(x^3+1)\,dx$

SOLUTION Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ or $\frac{1}{3}du = x^2\,dx$. Hence

$$\int x^2 \sin(x^3+1)\,dx = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(x^3+1) + C.$$

51. $\int \tan(4\theta + 9)\,d\theta$

SOLUTION Let $u = 4\theta + 9$. Then $du = 4\,d\theta$ and

$$\int \tan(4\theta+9)\,d\theta = \frac{1}{4}\int \tan u\,du = \frac{1}{4}\ln|\sec u| + C = \frac{1}{4}\ln|\sec(4\theta+9)| + C.$$

52. $\int \sin^8\theta\cos\theta\,d\theta$

SOLUTION Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$ and

$$\int \sin^8\theta\cos\theta\,d\theta = \int u^8\,du = \frac{1}{9}u^9 + C = \frac{1}{9}\sin^9\theta + C.$$

53. $\int \cot x\,dx$

SOLUTION Let $u = \sin x$. Then $du = \cos x\,dx$, and

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$$

54. $\int (x^{-1/5})\tan x^{4/5}\,dx$

SOLUTION Let $u = x^{4/5}$. Then $du = \frac{4}{5}x^{-1/5}\,dx$ and

$$\int (x^{-1/5})\tan x^{4/5}\,dx = \frac{5}{4}\int \tan u\,du = \frac{5}{4}\ln|\sec u| + C = \frac{5}{4}\ln|\sec x^{4/5}| + C.$$

55. $\int \sec^2(4x+9)\,dx$

SOLUTION Let $u = 4x + 9$. Then $du = 4\,dx$ or $\frac{1}{4}\,du = dx$. Hence

$$\int \sec^2(4x+9)\,dx = \frac{1}{4}\int \sec^2 u\,du = \frac{1}{4}\tan u + C = \frac{1}{4}\tan(4x+9) + C.$$

56. $\int \sec^2 x\tan^4 x\,dx$

SOLUTION Let $u = \tan x$. Then $du = \sec^2 x\,dx$. Hence

$$\int \sec^2 x\tan^4 x\,dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}\tan^5 x + C.$$

57. $\int \frac{\sec^2(\sqrt{x})\,dx}{\sqrt{x}}$

SOLUTION Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$ or $2\,du = \frac{1}{\sqrt{x}}\,dx$. Hence,

$$\int \frac{\sec^2(\sqrt{x})\,dx}{\sqrt{x}} = 2\int \sec^2 u\,dx = 2\tan u + C = 2\tan(\sqrt{x}) + C.$$

58. $\int \frac{\cos 2x}{(1+\sin 2x)^2}\,dx$

SOLUTION Let $u = 1 + \sin 2x$. Then $du = 2\cos 2x$ or $\frac{1}{2}du = \cos 2x\,dx$. Hence

$$\int (1+\sin 2x)^{-2}\cos 2x\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}u^{-1} + C = -\frac{1}{2}(1+\sin 2x)^{-1} + C.$$

59. $\int \sin 4x\sqrt{\cos 4x + 1}\,dx$

SOLUTION Let $u = \cos 4x + 1$. Then $du = -4\sin 4x$ or $-\frac{1}{4}du = \sin 4x$. Hence

$$\int \sin 4x\sqrt{\cos 4x + 1}\,dx = -\frac{1}{4}\int u^{1/2}\,du = -\frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{6}(\cos 4x + 1)^{3/2} + C.$$

60. $\int \cos x(3\sin x - 1)\,dx$

SOLUTION Let $u = 3\sin x - 1$. Then $du = 3\cos x\,dx$ or $\frac{1}{3}du = \cos x\,dx$. Hence

$$\int (3\sin x - 1)\cos x\,dx = \frac{1}{3}\int u\,du = \frac{1}{3}\left(\frac{1}{2}u^2\right) + C = \frac{1}{6}(3\sin x - 1)^2 + C.$$

61. $\int \sec\theta\tan\theta(\sec\theta - 1)\,d\theta$

SOLUTION Let $u = \sec\theta - 1$. Then $du = \sec\theta\tan\theta\,d\theta$ and

$$\int \sec\theta\tan\theta(\sec\theta - 1)\,d\theta = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sec\theta - 1)^2 + C.$$

62. $\int \cos t\cos(\sin t)\,dt$

SOLUTION Let $u = \sin t$. Then $du = \cos t\,dt$ and

$$\int \cos t\cos(\sin t)\,dt = \int \cos u\,du = \sin u + C = \sin(\sin t) + C.$$

63. $\int e^{14x-7}\,dx$

SOLUTION Let $u = 14x - 7$. Then $du = 14\,dx$ or $\frac{1}{14}\,du = dx$. Hence,

$$\int e^{14x-7}\,dx = \frac{1}{14}\int e^u\,du = \frac{1}{14}e^u + C = \frac{1}{14}e^{14x-7} + C.$$

64. $\int (x+1)e^{x^2+2x}\,dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx$ or $\frac{1}{2}\,du = (x+1)\,dx$. Hence,

$$\int (x+1)e^{x^2+2x}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2+2x} + C.$$

65. $\int \frac{e^x\,dx}{(e^x+1)^4}$

SOLUTION Let $u = e^x + 1$. Then $du = e^x\,dx$, and

$$\int \frac{e^x}{(e^x+1)^4}\,dx = \int u^{-4}\,du = -\frac{1}{3u^3} + C = -\frac{1}{3(e^x+1)^3} + C.$$

66. $\int (\sec^2\theta)\,e^{\tan\theta}\,d\theta$

SOLUTION Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$, and

$$\int (\sec^2\theta)\,e^{\tan\theta}\,d\theta = \int e^u\,du = e^u + C = e^{\tan\theta} + C.$$

67. $\int \frac{e^t\,dt}{e^{2t} + 2e^t + 1}$

SOLUTION Let $u = e^t$. Then $du = e^t\,dt$, and

$$\int \frac{e^t\,dt}{e^{2t} + 2e^t + 1} = \int \frac{du}{u^2 + 2u + 1} = \int \frac{du}{(u+1)^2} = -\frac{1}{u+1} + C = -\frac{1}{e^t+1} + C.$$

68. $\int \frac{dx}{x(\ln x)^2}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, and

$$\int \frac{dx}{x(\ln x)^2} = \int u^{-2}\,du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C.$$

69. $\displaystyle\int \frac{(\ln x)^4\,dx}{x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, and

$$\int \frac{(\ln x)^4}{x}\,dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}(\ln x)^5 + C.$$

70. $\displaystyle\int \frac{dx}{x\ln x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, and

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C.$$

71. $\displaystyle\int \frac{\tan(\ln x)}{x}\,dx$

SOLUTION Let $u = \cos(\ln x)$. Then $du = -\frac{1}{x}\sin(\ln x)\,dx$ or $-du = \frac{1}{x}\sin(\ln x)\,dx$. Hence,

$$\int \frac{\tan(\ln x)}{x}\,dx = \int \frac{\sin(\ln x)}{x\cos(\ln x)}\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(\ln x)| + C.$$

72. $\displaystyle\int (\cot x)\ln(\sin x)\,dx$

SOLUTION Let $u = \ln(\sin x)$. Then

$$du = \frac{1}{\sin x}\cos x = \cot x,$$

and

$$\int (\cot x)\ln(\sin x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln(\sin x))^2 + C.$$

73. Evaluate $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$ using $u = 1 + \sqrt{x}$. *Hint:* Show that $dx = 2(u-1)du$.

SOLUTION Let $u = 1 + \sqrt{x}$. Then

$$du = \frac{1}{2\sqrt{x}}\,dx \quad \text{or} \quad dx = 2\sqrt{x}\,du = 2(u-1)\,du.$$

Hence,

$$\int \frac{dx}{(1+\sqrt{x})^3} = 2\int \frac{u-1}{u^3}\,du = 2\int (u^{-2} - u^{-3})\,du$$
$$= -2u^{-1} + u^{-2} + C = -\frac{2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C.$$

74. Can They Both Be Right? Hannah uses the substitution $u = \tan x$ and Akiva uses $u = \sec x$ to evaluate $\int \tan x \sec^2 x\,dx$. Show that they obtain different answers, and explain the apparent contradiction.

SOLUTION With the substitution $u = \tan x$, Hannah finds $du = \sec^2 x\,dx$ and

$$\int \tan x \sec^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}\tan^2 x + C_1.$$

On the other hand, with the substitution $u = \sec x$, Akiva finds $du = \sec x \tan x\,dx$ and

$$\int \tan x \sec^2 x\,dx = \int \sec x(\tan x \sec x)\,dx = \frac{1}{2}\sec^2 x + C_2$$

Hannah and Akiva have each found a correct antiderivative. To resolve what appears to be a contradiction, recall that any two antiderivatives of a specified function differ by a constant. To show that this is true in their case, note that

$$\left(\frac{1}{2}\sec^2 x + C_2\right) - \left(\frac{1}{2}\tan^2 x + C_1\right) = \frac{1}{2}(\sec^2 x - \tan^2 x) + C_2 - C_1$$
$$= \frac{1}{2}(1) + C_2 - C_1 = \frac{1}{2} + C_2 - C_1, \text{ a constant}$$

Here we used the trigonometric identity $\tan^2 x + 1 = \sec^2 x$.

75. Evaluate $\int \sin x \cos x\, dx$ using substitution in two different ways: first using $u = \sin x$ and then using $u = \cos x$. Reconcile the two different answers.

SOLUTION First, let $u = \sin x$. Then $du = \cos x\, dx$ and

$$\int \sin x \cos x\, dx = \int u\, du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}\sin^2 x + C_1.$$

Next, let $u = \cos x$. Then $du = -\sin x\, dx$ or $-du = \sin x\, dx$. Hence,

$$\int \sin x \cos x\, dx = -\int u\, du = -\frac{1}{2}u^2 + C_2 = -\frac{1}{2}\cos^2 x + C_2.$$

To reconcile these two seemingly different answers, recall that any two antiderivatives of a specified function differ by a constant. To show that this is true here, note that $(\frac{1}{2}\sin^2 x + C_1) - (-\frac{1}{2}\cos^2 x + C_2) = \frac{1}{2} + C_1 - C_2$, a constant. Here we used the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

76. Some Choices Are Better Than Others Evaluate

$$\int \sin x\, \cos^2 x\, dx$$

twice. First use $u = \sin x$ to show that

$$\int \sin x\, \cos^2 x\, dx = \int u\sqrt{1-u^2}\, du$$

and evaluate the integral on the right by a further substitution. Then show that $u = \cos x$ is a better choice.

SOLUTION Consider the integral $\int \sin x \cos^2 x\, dx$. If we let $u = \sin x$, then $\cos x = \sqrt{1-u^2}$ and $du = \cos x\, dx$. Hence

$$\int \sin x \cos^2 x\, dx = \int u\sqrt{1-u^2}\, du.$$

Now let $w = 1 - u^2$. Then $dw = -2u\, du$ or $-\frac{1}{2}dw = u\, du$. Therefore,

$$\begin{aligned}\int u\sqrt{1-u^2}\, du &= -\frac{1}{2}\int w^{1/2}\, dw = -\frac{1}{2}\left(\frac{2}{3}w^{3/2}\right) + C\\ &= -\frac{1}{3}w^{3/2} + C = -\frac{1}{3}(1-u^2)^{3/2} + C\\ &= -\frac{1}{3}(1-\sin^2 x)^{3/2} + C = -\frac{1}{3}\cos^3 x + C.\end{aligned}$$

A better substitution choice is $u = \cos x$. Then $du = -\sin x\, dx$ or $-du = \sin x\, dx$. Hence

$$\int \sin x \cos^2 x\, dx = -\int u^2\, du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cos^3 x + C.$$

77. What are the new limits of integration if we apply the substitution $u = 3x + \pi$ to the integral $\int_0^\pi \sin(3x+\pi)\, dx$?

SOLUTION The new limits of integration are $u(0) = 3 \cdot 0 + \pi = \pi$ and $u(\pi) = 3\pi + \pi = 4\pi$.

78. Which of the following is the result of applying the substitution $u = 4x - 9$ to the integral $\int_2^8 (4x-9)^{20}\, dx$?

(a) $\displaystyle\int_2^8 u^{20}\, du$

(b) $\displaystyle\frac{1}{4}\int_2^8 u^{20}\, du$

(c) $\displaystyle 4\int_{-1}^{23} u^{20}\, du$

(d) $\displaystyle\frac{1}{4}\int_{-1}^{23} u^{20}\, du$

SOLUTION Let $u = 4x - 9$. Then $du = 4\, dx$ or $\frac{1}{4}\, du = dx$. Furthermore, when $x = 2$, $u = -1$, and when $x = 8$, $u = 23$. Hence

$$\int_2^8 (4x-9)^{20}\, dx = \frac{1}{4}\int_{-1}^{23} u^{20}\, du.$$

The answer is therefore **(d)**.

In Exercises 79–90, use the Change-of-Variables Formula to evaluate the definite integral.

79. $\displaystyle\int_1^3 (x+2)^3\, dx$

SOLUTION Let $u = x + 2$. Then $du = dx$. Hence

$$\int_1^3 (x+2)^3\,dx = \int_3^5 u^3\,du = \frac{1}{4}u^4\Big|_3^5 = \frac{5^4}{4} - \frac{3^4}{4} = 136.$$

80. $\int_1^6 \sqrt{x+3}\,dx$

SOLUTION Let $u = x + 3$. Then $du = dx$. Hence

$$\int_1^6 \sqrt{x+3}\,dx = \int_4^9 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_4^9 = \frac{2}{3}(27-8) = \frac{38}{3}.$$

81. $\int_0^1 \frac{x}{(x^2+1)^3}\,dx$

SOLUTION Let $u = x^2 + 1$. Then $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence

$$\int_0^1 \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u^3}\,du = \frac{1}{2}\left(-\frac{1}{2}u^{-2}\right)\Big|_1^2 = -\frac{1}{16} + \frac{1}{4} = \frac{3}{16} = 0.1875.$$

82. $\int_{-1}^2 \sqrt{5x+6}\,dx$

SOLUTION Let $u = 5x + 6$. Then $du = 5\,dx$ or $\frac{1}{5}\,du = dx$. Hence

$$\int_{-1}^2 \sqrt{5x+6}\,dx = \frac{1}{5}\int_1^{16} \sqrt{u}\,du = \frac{1}{5}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^{16} = \frac{2}{15}(64-1) = \frac{42}{5}.$$

83. $\int_0^4 x\sqrt{x^2+9}\,dx$

SOLUTION Let $u = x^2 + 9$. Then $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence

$$\int_0^4 x\sqrt{x^2+9}\,dx = \frac{1}{2}\int_9^{25} \sqrt{u}\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_9^{25} = \frac{1}{3}(125-27) = \frac{98}{3}.$$

84. $\int_1^2 \frac{4x+12}{(x^2+6x+1)^2}\,dx$

SOLUTION Let $u = x^2 + 6x + 1$. Then $du = (2x+6)\,dx$ and

$$\int_1^2 \frac{4x+12}{(x^2+6x+1)^2}\,dx = 2\int_8^{17} u^{-2}\,du = -\frac{2}{u}\Big|_8^{17}$$
$$= -\frac{2}{17} + \frac{1}{4} = \frac{9}{68}.$$

85. $\int_0^1 (x+1)(x^2+2x)^5\,dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx = 2(x+1)\,dx$, and

$$\int_0^1 (x+1)(x^2+2x)^5\,dx = \frac{1}{2}\int_0^3 u^5\,du = \frac{1}{12}u^6\Big|_0^3 = \frac{729}{12} = \frac{243}{4}.$$

86. $\int_{10}^{17} (x-9)^{-2/3}\,dx$

SOLUTION Let $u = x - 9$. Then $du = dx$. Hence

$$\int_{10}^{17} (x-9)^{-2/3}\,dx = \int_1^8 u^{-2/3}\,dx = 3u^{1/3}\Big|_1^8 = 3\,(2-1) = 3.$$

87. $\int_0^1 \theta\tan(\theta^2)\,d\theta$

SOLUTION Let $u = \cos\theta^2$. Then $du = -2\theta\sin\theta^2\,d\theta$ or $-\frac{1}{2}du = \theta\sin\theta^2\,d\theta$. Hence,

$$\int_0^1 \theta\tan(\theta^2)\,d\theta = \int_0^1 \frac{\theta\sin(\theta^2)}{\cos(\theta^2)}\,d\theta = -\frac{1}{2}\int_1^{\cos 1} \frac{du}{u} = -\frac{1}{2}\ln|u|\Big|_1^{\cos 1} = -\frac{1}{2}\left[\ln(\cos 1) + \ln 1\right] = \frac{1}{2}\ln(\sec 1).$$

88. $\int_0^{\pi/6} \sec^2\left(2x - \frac{\pi}{6}\right) dx$

SOLUTION Let $u = 2x - \frac{\pi}{6}$. Then $du = 2\,dx$ and

$$\int_0^{\pi/6} \sec^2\left(2x - \frac{\pi}{6}\right) dx = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \sec^2 u\,du = \frac{1}{2}\tan u\Big|_{-\pi/6}^{\pi/6}$$
$$= \frac{1}{2}\left(\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}.$$

89. $\int_0^{\pi/2} \cos^3 x \sin x\,dx$

SOLUTION Let $u = \cos x$. Then $du = -\sin x\,dx$. Hence

$$\int_0^{\pi/2} \cos^3 x \sin x\,dx = -\int_1^0 u^3\,du = \int_0^1 u^3\,du = \frac{1}{4}u^4\Big|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}.$$

90. $\int_{\pi/3}^{\pi/2} \cot^2 \frac{x}{2} \csc^2 \frac{x}{2}\,dx$

SOLUTION Let $u = \cot \frac{x}{2}$. Then $du = -\frac{1}{2}\csc^2 \frac{x}{2}$ and

$$\int_{\pi/3}^{\pi/2} \cot^2 \frac{x}{2} \csc^2 \frac{x}{2}\,dx = -2\int_{\sqrt{3}}^1 u^2\,du$$
$$= -\frac{2}{3}u^3\Big|_{\sqrt{3}}^1 = \frac{2}{3}(3\sqrt{3} - 1).$$

91. Evaluate $\int_0^2 r\sqrt{5 - \sqrt{4 - r^2}}\,dr$.

SOLUTION Let $u = 5 - \sqrt{4 - r^2}$. Then

$$du = \frac{r\,dr}{\sqrt{4 - r^2}} = \frac{r\,dr}{5 - u}$$

so that

$$r\,dr = (5 - u)\,du.$$

Hence, the integral becomes:

$$\int_0^2 r\sqrt{5 - \sqrt{4 - r^2}}\,dr = \int_3^5 \sqrt{u}(5 - u)\,du = \int_3^5 \left(5u^{1/2} - u^{3/2}\right)\,du = \left(\frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_3^5$$
$$= \left(\frac{50}{3}\sqrt{5} - 10\sqrt{5}\right) - \left(10\sqrt{3} - \frac{18}{5}\sqrt{3}\right) = \frac{20}{3}\sqrt{5} - \frac{32}{5}\sqrt{3}.$$

92. Find numbers a and b such that

$$\int_a^b (u^2 + 1)\,du = \int_{-\pi/4}^{\pi/4} \sec^4 \theta\,d\theta$$

and evaluate. *Hint:* Use the identity $\sec^2 \theta = \tan^2 \theta + 1$.

SOLUTION Let $u = \tan \theta$. Then $u^2 + 1 = \tan^2 \theta + 1 = \sec^2 \theta$ and $du = \sec^2 \theta\,d\theta$. Moreover, because

$$\tan\left(-\frac{\pi}{4}\right) = -1 \quad \text{and} \quad \tan \frac{\pi}{4} = 1,$$

it follows that $a = -1$ and $b = 1$. Thus,

$$\int_{-\pi/4}^{\pi/4} \sec^4 \theta\,d\theta = \int_{-1}^1 (u^2 + 1)\,du = \left(\frac{1}{3}u^3 + u\right)\Big|_{-1}^1 = \frac{8}{3}.$$

93. Wind engineers have found that wind speed v (in meters/second) at a given location follows a **Rayleigh distribution** of the type

$$W(v) = \frac{1}{32} v e^{-v^2/64}$$

This means that at a given moment in time, the probability that v lies between a and b is equal to the shaded area in Figure 1.

(a) Show that the probability that $v \in [0, b]$ is $1 - e^{-b^2/64}$.

(b) Calculate the probability that $v \in [2, 5]$.

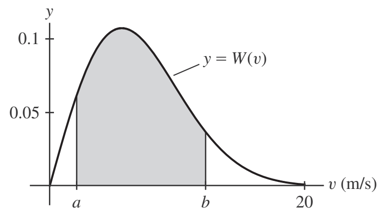

FIGURE 1 The shaded area is the probability that v lies between$a$ and b.

SOLUTION

(a) The probability that $v \in [0, b]$ is

$$\int_0^b \frac{1}{32} v e^{-v^2/64}\, dv.$$

Let $u = -v^2/64$. Then $du = -v/32\, dv$ and

$$\int_0^b \frac{1}{32} v e^{-v^2/64}\, dv = -\int_0^{-b^2/64} e^u\, du = -e^u \Big|_0^{-b^2/64} = -e^{-b^2/64} + 1.$$

(b) The probability that $v \in [2, 5]$ is the probability that $v \in [0, 5]$ minus the probability that $v \in [0, 2]$. By part (a), the probability that $v \in [2, 5]$ is

$$\left(1 - e^{-25/64}\right) - \left(1 - e^{-1/16}\right) = e^{-1/16} - e^{-25/64}.$$

94. Evaluate $\int_0^{\pi/2} \sin^n x \cos x\, dx$ for $n \geq 0$.

SOLUTION Let $u = \sin x$. Then $du = \cos x\, dx$. Hence

$$\int_0^{\pi/2} \sin^n x \cos x\, dx = \int_0^1 u^n\, du = \frac{u^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}.$$

In Exercises 95–96, use substitution to evaluate the integral in terms of $f(x)$.

95. $\int f(x)^3 f'(x)\, dx$

SOLUTION Let $u = f(x)$. Then $du = f'(x)\, dx$. Hence

$$\int f(x)^3 f'(x)\, dx = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{1}{4} f(x)^4 + C.$$

96. $\int \frac{f'(x)}{f(x)^2}\, dx$

SOLUTION Let $u = f(x)$. Then $du = f'(x)\, dx$. Hence

$$\int \frac{f'(x)}{f(x)^2}\, dx = \int u^{-2}\, du = -u^{-1} + C = \frac{-1}{f(x)} + C.$$

97. Show that $\int_0^{\pi/6} f(\sin\theta)\, d\theta = \int_0^{1/2} f(u) \frac{1}{\sqrt{1 - u^2}}\, du.$

SOLUTION Let $u = \sin\theta$. Then $u(\pi/6) = 1/2$ and $u(0) = 0$, as required. Furthermore, $du = \cos\theta\, d\theta$, so that

$$d\theta = \frac{du}{\cos\theta}.$$

If $\sin\theta = u$, then $u^2 + \cos^2\theta = 1$, so that $\cos\theta = \sqrt{1-u^2}$. Therefore $d\theta = du/\sqrt{1-u^2}$. This gives

$$\int_0^{\pi/6} f(\sin\theta)\, d\theta = \int_0^{1/2} f(u)\frac{1}{\sqrt{1-u^2}}\, du.$$

Further Insights and Challenges

98. Use the substitution $u = 1 + x^{1/n}$ to show that

$$\int \sqrt{1+x^{1/n}}\, dx = n\int u^{1/2}(u-1)^{n-1}\, du$$

Evaluate for $n = 2, 3$.

SOLUTION Let $u = 1 + x^{1/n}$. Then $x = (u-1)^n$ and $dx = n(u-1)^{n-1}\, du$. Accordingly, $\int \sqrt{1+x^{1/n}}\, dx = n\int u^{1/2}(u-1)^{n-1}\, du$.

For $n = 2$, we have

$$\int \sqrt{1+x^{1/2}}\, dx = 2\int u^{1/2}(u-1)^1\, du = 2\int (u^{3/2} - u^{1/2})\, du$$
$$= 2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C = \frac{4}{5}(1+x^{1/2})^{5/2} - \frac{4}{3}(1+x^{1/2})^{3/2} + C.$$

For $n = 3$, we have

$$\int \sqrt{1+x^{1/3}}\, dx = 3\int u^{1/2}(u-1)^2\, du = 3\int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$$
$$= 3\left(\frac{2}{7}u^{7/2} - (2)\left(\frac{2}{5}\right)u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$
$$= \frac{6}{7}(1+x^{1/3})^{7/2} - \frac{12}{5}(1+x^{1/3})^{5/2} + 2(1+x^{1/3})^{3/2} + C.$$

99. Evaluate $I = \int_0^{\pi/2} \frac{d\theta}{1+\tan^{6,000}\theta}$. *Hint:* Use substitution to show that I is equal to $J = \int_0^{\pi/2} \frac{d\theta}{1+\cot^{6,000}\theta}$ and then check that $I + J = \int_0^{\pi/2} d\theta$.

SOLUTION To evaluate

$$I = \int_0^{\pi/2} \frac{dx}{1+\tan^{6000} x},$$

we substitute $t = \pi/2 - x$. Then $dt = -dx$, $x = \pi/2 - t$, $t(0) = \pi/2$, and $t(\pi/2) = 0$. Hence,

$$I = \int_0^{\pi/2} \frac{dx}{1+\tan^{6000} x} = -\int_{\pi/2}^{0} \frac{dt}{1+\tan^{6000}(\pi/2 - t)} = \int_0^{\pi/2} \frac{dt}{1+\cot^{6000} t}.$$

Let $J = \int_0^{\pi/2} \frac{dt}{1+\cot^{6000}(t)}$. We know $I = J$, so $I + J = 2I$. On the other hand, by the definition of I and J and the linearity of the integral,

$$I + J = \int_0^{\pi/2} \frac{dx}{1+\tan^{6000} x} + \frac{dx}{1+\cot^{6000} x} = \int_0^{\pi/2} \left(\frac{1}{1+\tan^{6000} x} + \frac{1}{1+\cot^{6000} x}\right) dx$$
$$= \int_0^{\pi/2} \left(\frac{1}{1+\tan^{6000} x} + \frac{1}{1+(1/\tan^{6000} x)}\right) dx$$
$$= \int_0^{\pi/2} \left(\frac{1}{1+\tan^{6000} x} + \frac{1}{(\tan^{6000} x + 1)/\tan^{6000} x}\right) dx$$