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8 Variation and Polynomial Equations

8–1 Direct Variation and Proportion

Objective: To solve problems involving direct variation.

Vocabulary

Direct variation A linear function defined by an equation of the form $y = mx$ $(m \neq 0)$. The constant m in the equation is called the *constant of variation* or *constant of proportionality*. We say that *y varies directly as x* because if x increases, y also increases, and if x decreases, y also decreases.

Proportion An equality of ratios. A proportion can be written in the form $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, or $y_1 : x_1 = y_2 : x_2$, where the ordered pairs (x_1, y_1) and (x_2, y_2) are solutions of a direct variation, and $\frac{y_1}{x_1} = \frac{y_2}{x_2} = m$ (x_1 and $x_2 \neq 0$).

In a direct variation, y is often said to be *directly proportional* to x.

Means and extremes In the proportion $y_1 : x_1 = y_2 : x_2$, the means are the numbers x_1 and y_2, and the extremes are y_1 and x_2. In a proportion, *the product of the extremes equals the product of the means*, or $y_1 x_2 = x_1 y_2$.

Example 1 If y varies directly as x, and $y = 12$ when $x = 20$, find y when $x = 50$.

Solution First find m and write an equation of the direct variation.

$y = mx$ — Start with the general equation.

$12 = m(20)$ — Substitute $y = 12$ and $x = 20$.

$m = \frac{12}{20} = \frac{3}{5}$ — Solve for m.

$\therefore y = \frac{3}{5}x$ is an equation of the direct variation.

You can find y when $x = 50$ by substituting 50 for x in this equation.

$$y = \frac{3}{5}(50) = 30$$

Example 2 If a is directly proportional to $b + 3$, and $a = 6$ when $b = 15$, find b when $a = 7$.

Solution Since a is directly proportional to $b + 3$, you can write a proportion.

$\frac{a_1}{b_1 + 3} = \frac{a_2}{b_2 + 3}$ — Set up a proportion.

$\frac{6}{15 + 3} = \frac{7}{b_2 + 3}$ — Substitute values for variables.

$6(b_2 + 3) = 18(7)$ — Multiply the extremes, and the means.

$b_2 + 3 = 126 \div 6 = 21$ — Solve for b_2.

$b_2 = 18$

8-1 Direct Variation and Proportion *(continued)*

CAUTION It doesn't matter whether you solve direct variation problems by finding the equation first or by using the proportion method. However, if you decide to use the proportion method, be sure that the values in the numerators represent the same variable.

Solve.

1. If y varies directly as x, and $y = 6$ when $x = 4$, find y when $x = 12$.
2. If a is directly proportional to b, and $a = 25$ when $b = 35$, find b when $a = 40$.
3. If w varies directly as z, and $w = 4.5$ when $z = 3$, find z when $w = 1.5$.
4. If p is directly proportional to q^3, and $p = 3$ when $q = 2$, find p when $q = 4$.
5. If r varies directly as $s + 1$, and $r = 4$ when $s = 5$, find r when $s = 8$.
6. If a varies directly as $3b + 2$, and $a = 10$ when $b = 6$, find b when $a = 7$.

Example 3 If a car travels 70 km in 2 hours, how far can it travel in 4.5 hours, traveling at the same rate of speed?

Solution Let d be the required distance in kilometers.

Since the ratio $\frac{distance}{time} = rate$ is constant, a proportion can be written.

$$\frac{d_1}{t_1} = \frac{d_2}{t_2} \longrightarrow \frac{70}{2} = \frac{d}{4.5}$$

$$2d = 70(4.5)$$

$$d = 157.5$$

$\therefore$ the distance the car will travel is 157.5 km.

Solve.

7. If the sales tax on a \$38 purchase is \$2.85, what will the tax be on an \$84 purchase?
8. A survey showed that 52 out of 234 people questioned preferred hot cereal to cold cereal. In a school population of 1800, how many people are likely to prefer hot cereal?
9. A real estate agent received a commission of \$2232 on a piece of land that sold for \$124,000. At this rate, what commission will the agent receive for a piece of land that sold for \$160,000?

Mixed Review Exercises

Solve each equation over the real numbers.

1. $4x^2 - 7x + 2 = 0$
2. $\frac{a - 1}{a + 5} = 1 - \frac{3}{a}$
3. $\frac{3y^2}{8} + \frac{y}{4} = 1$
4. $|2a - 8| = 6$
5. $\sqrt{2m + 15} = m$
6. $4q^{-2} + 7q^{-1} = 2$
7. $6n^2 = 7n$
8. $(2x - 5)^2 = 18$

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8–2 Inverse and Joint Variation

Objective: To solve problems involving inverse and joint variation.

Vocabulary

Inverse variation A function defined by an equation of the form $xy = k$ or $y = \frac{k}{x}$ ($x \neq 0$, $k \neq 0$). The constant k is called the *constant of variation* or *constant of proportionality*. We say that *y varies inversely as x*, or *y is inversely proportional to x*, because if x increases, y decreases, and if x decreases, y increases.

Joint variation When a quantity varies directly as the product of two or more other quantities, the variation is called joint variation. Example: If m varies jointly as n and p, then $m = knp$ for some nonzero constant k. Another way to state this relationship is "m is jointly proportional to n and p."

Example 1 If y is inversely proportional to x, and $y = 8$ when $x = 6$, find x when $y = 15$.

Solution First find k and write an equation of the inverse variation.

$xy = k$	Start with the general equation.
$(6)(8) = k$	Substitute $x = 6$ and $y = 8$.
$k = 48$	Solve for k.

$\therefore$ an equation of the inverse variation is $xy = 48$.

To find x when $y = 15$, substitute this value in $xy = 48$.

$$x(15) = 48$$
$$x = 3.2$$

Solve.

1. If y varies inversely as x, and $y = 5$ when $x = 4$, find x when $y = 10$.
2. If p is inversely proportional to q, and $p = 10$ when $q = 5$, find q when $p = 2$.
3. If a is inversely proportional to b, and $b = 12$ when $a = 8$, find b when $a = 3$.
4. If x varies inversely as the square of y, and $x = 2$ when $y = 12$, find y when $x = 8$.

Example 2 If z varies jointly as x and the square of y, and $z = 12$ when $x = 6$ and $y = 2$, find z when $x = 10$ and $y = 3$.

Solution First find k and write an equation of the joint variation. Remember it's a *direct* variation.

$$z = kxy^2$$
$$12 = k(6)(2)^2$$
$$12 = 24k$$
$$k = \frac{1}{2}$$

$\therefore$ an equation of the joint variation is $z = \frac{1}{2}xy^2$.

Then, substitute $x = 10$ and $y = 3$ in this equation.

$$z = \frac{1}{2}xy^2$$
$$z = \frac{1}{2}(10)(3)^2$$
$$z = 45$$

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8–2 *Inverse and Joint Variation* *(continued)*

Solve.

5. If x varies jointly as y and z, and $x = 100$ when $y = 20$ and $z = 10$, find x when $y = 60$ and $z = 30$.

6. If a is jointly proportional to b and c, and $a = 48$ when $b = 6$ and $c = 4$, find c when $a = 540$ and $b = 18$.

7. If I varies jointly as p and r, and $I = 14$ when $p = 100$ and $r = 0.07$, find p when $I = 48$ and $r = 0.08$.

8. If x varies jointly as y and the square root of z, and $x = 20$ when $y = 5$ and $z = 9$, find x when $y = 6$ and $z = 25$.

Example 3 The intensity of light, measured in *lux*, is inversely proportional to the square of the distance between the light source and the object illuminated. A light meter 6.4 m from a light source registers 30 lux. What intensity would it register 16 m from the light source?

Solution Let I = the intensity of light in lux, and d = the distance in meters of the illuminated object from the light source. I varies inversely as the square of d. That is,

$$Id^2 = k.$$

Find k when $I = 30$ lux and $d = 6.4$ m.

$$30(6.4)^2 = k$$
$$k = 1228.8$$

$\therefore$ an equation of the variation is $Id^2 = 1228.8$.

Then, to find I when $d = 16$ m, substitute this value in $Id^2 = 1228.8$.

$$I(16)^2 = 1228.8$$
$$I = 4.8$$

$\therefore$ the intensity is 4.8 lux.

Solve.

9. A light meter 5.4 m from a light source registers 20 lux. What intensity would it measure 1.8 m from the light source?

10. The frequency of a radio signal varies inversely as the wavelength. A signal of frequency 1250 kilohertz (kHz) has a wavelength of 240 m. What frequency has a signal of wavelength 300 m?

11. The volume of a cylinder is jointly proportional to the height and the square of the radius of a base. A cylinder of height 12 cm and base radius 3 cm has volume 108π cm^3. Find the radius of the base of a cylinder if the height is 2 cm and the volume is 72π cm^3.

12. The stretch in a wire under a given tension varies directly as the length of the wire and inversely as the square of its diameter. If the length and the diameter of a wire are both doubled, what is the effect on the stretch of the wire?

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8–3 Dividing Polynomials

Objective: To divide one polynomial by another polynomial.

Vocabulary

Dividend A quantity to be divided. In long division, the number *under* the division symbol is the dividend. In a fraction, the numerator is the dividend.

Divisor The quantity by which another quantity is divided. In long division, the number *outside* the division symbol is the divisor. In a fraction, the denominator is the divisor.

Division algorithm (rule) $\dfrac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \dfrac{\text{Remainder}}{\text{Divisor}}$

or Dividend = Quotient × Divisor + Remainder

CAUTION Before using long division, always arrange the terms of both dividend and divisor in order of decreasing degree. Also, insert any "missing" terms by using zero as a coefficient. Example: $x - 3x^3 - 5$ becomes $-3x^3 + 0x^2 + x - 5$.

Example 1 Divide: $\dfrac{3x + 2x^3 - 5}{x + 2}$

Solution

1. Rewrite the problem in the form you would use to do long division.
2. Divide the first term of the dividend by the first term of the divisor. Write this monomial above the line as the first term of the quotient.
3. Multiply by the new quotient term.
4. Subtract.
5. Repeat Steps 2–4 until the remainder is 0 or the degree of the remainder is less than the degree of the divisor.

$$
\begin{array}{rrll}
 & 2x^2 - 4x + 11 & \longleftarrow & \textbf{quotient} \\
\textbf{divisor} \longrightarrow x + 2 & \big)\,2x^3 + 0x^2 + 3x - 5 & \longleftarrow & \textbf{dividend} \\
 & \underline{2x^3 + 4x^2} & \longleftarrow & \text{Multiply } x + 2 \text{ by } 2x^2. \\
 & -4x^2 + 3x & \longleftarrow & \text{Subtract.} \\
 & \underline{-4x^2 - 8x} & \longleftarrow & \text{Multiply } x + 2 \text{ by } -4x. \\
 & 11x - 5 & \longleftarrow & \text{Subtract.} \\
 & \underline{11x + 22} & \longleftarrow & \text{Multiply } x + 2 \text{ by } 11. \\
\textbf{remainder} \longrightarrow & -27 & \longleftarrow & \text{Subtract.}
\end{array}
$$

$$\therefore \underbrace{\frac{2x^3 + 3x - 5}{x + 2}}_{\frac{\text{Dividend}}{\text{Divisor}}} = \underbrace{2x^2 - 4x + 11}_{\text{Quotient}} + \underbrace{\frac{-27}{x + 2}}_{\frac{\text{Remainder}}{\text{Divisor}}}$$

Check: To check the result, use the following form of the division algorithm:

Quotient × Divisor + Remainder = Dividend

$$(2x^2 - 4x + 11)(x + 2) + (-27) \stackrel{?}{=} 2x^3 + 3x - 5$$

$$2x^3 - 4x^2 + 11x + 4x^2 - 8x + 22 - 27 \stackrel{?}{=} 2x^3 + 3x - 5$$

$$2x^3 + 3x - 5 = 2x^3 + 3x - 5 \quad \checkmark$$

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8-3 *Dividing Polynomials* *(continued)*

Divide. Additional answers are given at the back of this Answer Key.

1. $\frac{x^2 + 8x + 12}{x + 4}$

2. $\frac{-68 + 17x - x^2}{6 - x}$

3. $\frac{x^2 - 16x}{x - 4}$

4. $\frac{9x^2 - 6x + 1}{3x + 1}$

5. $\frac{6x^2 - 13x + 6}{2x - 3}$

6. $\frac{10x^2 + x - 3}{5x + 3}$

7. $\frac{2x^3 - 15x + 8}{x - 2}$

8. $\frac{x^3 - 4x - 8}{x - 2}$

9. $\frac{6x^3 - 19x^2 + 15}{3x - 5}$

Example 2 Divide: $\frac{x^4 + x^2 + 4}{x^2 + x + 1}$

Solution Insert the "missing" terms, $0x^3$ and $0x$, in the dividend.

$$\begin{array}{r|l}
 & x^2 - x + 1 \\ \hline
x^2 + x + 1 & x^4 + 0x^3 + x^2 + 0x + 4 \\
 & \underline{x^4 + 1x^3 + x^2} \\
 & -x^3 + 0x^2 + 0x \\
 & \underline{-x^3 - x^2 - x} \\
 & x^2 + x + 4 \\
 & \underline{x^2 + x + 1} \\
 & 3
\end{array}$$

$\therefore$ the quotient is $x^2 - x + 1$, and the remainder is 3.

$$\therefore \frac{x^4 + x^2 + 4}{x^2 + x + 1} = x^2 - x + 1 + \frac{3}{x^2 + x + 1}$$

The check is left for you.

Divide.

10. $\frac{x^3 - 6x^2 + 12x - 8}{x^2 - 4x + 4}$

11. $\frac{x^4 - x^3 + 7x + 5}{x^2 + 2x + 1}$

12. $\frac{3x^4 + x^3 - 2x + 7}{x^2 - x + 1}$

13. $\frac{4x^4 + 5x^2 + 12x + 1}{2x^2 - x - 3}$

14. $\frac{6x^4 + 5x^3 - 9x^2 + 5}{3x^2 + x - 2}$

15. $\frac{10x^4 - 2x^3 + x^2 + x - 3}{2x^2 - 1}$

Mixed Review Exercises

In Exercises 1–4, assume that y varies directly as x.

1. If $y = 10$ when $x = 4$, find y when $x = 5$.

2. If $y = 2$ when $x = 0.5$, find y when $x = 6$.

3. If $y = 7$ when $x = \sqrt{3}$, find y when $x = \sqrt{15}$.

4. If $y = 4.2$ when $x = 0.4$, find y when $x = 2.5$.

5–8. Solve 1–4, assuming that y varies inversely as x.

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8–4 Synthetic Division

Objective: To use synthetic division to divide a polynomial by a first-degree binomial.

Vocabulary

Synthetic division An efficient way to divide a polynomial in x by a binomial of the form $x - c$. It uses only the coefficients of the polynomials involved.

Example 1 Use synthetic division to divide $x^4 - 3x^3 + 2x^2 - 4$ by $x + 2$.

Solution

1. Arrange the terms of the polynomial in order from the largest to the smallest exponent: $x^4 - 3x^3 + 2x^2 + 0x - 4$. Then write the coefficients of the terms in a row. The coefficient of x^4 is 1. Use a 0 as the coefficient of any power of x that is "missing."
2. Write the constant term c of the binomial divisor $x - c$ to the left of the coefficients. *Note:* Since $x + 2 = x - (-2)$, you use -2 for c.
3. Bring down the first coefficient.
4. Multiply what you've brought down by c and write the result below the next coefficient.
5. Add.
6. Repeat Steps 4 and 5 for every coefficient.

Step 2 ↓						
−2 |	1	−3	2	0	−4	← *Step 1*
	↓	−2	10	−24	48	
Step 3 →	1	−5	12	−24	| 44	
		↑ *Steps 4, 5*			↑ *Remainder*	

The resulting numbers 1, −5, 12, and −24 are the coefficients of the quotient. The degree of the quotient will be 1 less than the degree of the dividend, or 3. The quotient is $1x^3 - 5x^2 + 12x - 24$, and the remainder is 44.

$$\therefore \frac{x^4 - 3x^3 + 2x^2 - 4}{x + 2} = x^3 - 5x^2 + 12x - 24 + \frac{44}{x + 2}$$

Example 2 Express as a polynomial (a) the divisor, (b) the dividend, (c) the quotient, and (d) the remainder for the synthetic division shown below.

3 |	1	2	−10	0	−41
		3	15	15	45
	1	5	5	15	| 4

(Solution is on the next page.)

8–4 Synthetic Division *(continued)*

Solution You need to put the variables back in place.

a. The divisor has the form $x - c$, and $c = 3$. Thus, the divisor is $x - 3$.

b. There are 5 coefficients in the top row. The coefficient -41 is a constant. The dividend is the *fourth*-degree polynomial $x^4 + 2x^3 - 10x^2 + 0x - 41$, or $x^4 + 2x^3 - 10x^2 - 41$.

c. The first four coefficients in the bottom row form the quotient. The coefficient 15 is a constant. Thus, the quotient is the *third*-degree polynomial $x^3 + 5x^2 + 5x + 15$.

d. The remainder is the last number in the bottom row, or 4.

For each synthetic division shown below, express as a polynomial (a) the divisor, (b) the dividend, (c) the quotient, and (d) the remainder. Use x as the variable.

1.

-3	1	5	7	3
		-3	-6	-3
	1	2	1	0

2.

4	1	-6	0	18	-10
		4	-8	-32	-56
	1	-2	-8	-14	-66

Example 3 Divide using synthetic division: $\dfrac{x^4 - 16}{x - 2}$

Solution Use 2 for c. Insert zeros for the missing terms x^3, x^2, and x in the dividend.

2	1	0	0	0	-16
		2	4	8	16
	1	2	4	8	0

The quotient is $1x^3 + 2x^2 + 4x + 8$, and the remainder is 0.

$$\therefore \frac{x^4 - 16}{x - 2} = x^3 + 2x^2 + 4x + 8$$

Divide using synthetic division.

3. $\dfrac{x^3 + 2x^2 - 2x - 4}{x + 1}$

4. $\dfrac{x^3 + 2x^2 - 2x - 1}{x - 1}$

5. $\dfrac{x^3 + x^2 - 8x - 12}{x + 3}$

6. $\dfrac{2x^3 - 5x^2 + 4x - 3}{x - 2}$

7. $\dfrac{x^4 + 2x^3 + 3x^2 - x - 3}{x + 1}$

8. $\dfrac{3x^4 + 10x^3 - 7x^2 + 5x + 6}{x + 4}$

9. $\dfrac{x^4 + 3x^2 - x - 5}{x + 2}$

10. $\dfrac{2x^4 - 7x^3 - x + 10}{x - 3}$

11. $\dfrac{x^5 - 32}{x - 2}$

12. $\dfrac{x^7 - 1}{x - 1}$

13. $\dfrac{x^6 + 2x^3 - 7x^2 + 8x + 4}{x + 2}$

14. $\dfrac{x^6 - 3x^5 + 14x + 4}{x - 2}$