Name KEY Date__________ Period______

Worksheet 4.8—Inverse & Inverse Trig Functions

Show all work. No calculator unless otherwise stated.

1. Find the derivative with respect to the appropriate variable. Simplify your expression

(a) $y=\sec^{-1}\left(x^2\right)$

$$\frac{dy}{dx}=\frac{1}{|x^2|\sqrt{(x^2)^2-1}}\cdot(2x)$$

$$\frac{dy}{dx}=\frac{2x}{|x^2|\sqrt{x^4-1}}$$

or $$\boxed{\frac{dy}{dx}=\frac{2}{x\sqrt{x^4-1}}}$$

if $x>0$, $\frac{dy}{dx}>0$

if $x<0$ $\frac{dy}{dx}<0$

(b) $y=s\sqrt{1-s^2}+\arccos s$

$$y'=1\cdot\sqrt{1-s^2}+s\left(\tfrac{1}{2}\right)(1-s^2)^{-1/2}(-2s)+\frac{-1}{\sqrt{1-s^2}}$$

$$y'=\frac{\sqrt{1-s^2}}{1}-\frac{s^2}{\sqrt{1-s^2}}-\frac{1}{\sqrt{1-s^2}}$$

$$y'=\frac{1-s^2-s^2-1}{\sqrt{1-s^2}}$$

$$\boxed{y'=\frac{-2s^2}{\sqrt{1-s^2}}}$$

(c) $y=\dfrac{1}{\arcsin(2x)}$

$$y=(\arcsin(2x))^{-1}$$

$$y'=-1(\arcsin(2x))^{-2}\cdot\frac{1}{\sqrt{1-(2x)^2}}\cdot 2$$

$$\boxed{y'=\frac{-2}{(\arcsin(2x))^2\sqrt{1-4x^2}}}$$

(d) $y=\cot^{-1}\sqrt{t-1}$

$$y'=\frac{-1}{1+(\sqrt{t-1})^2}\cdot\left(\tfrac{1}{2}\right)(t-1)^{-1/2}\cdot(1)$$

$$y'=\frac{-1}{1+t-1}\left(\frac{1}{2\sqrt{t-1}}\right)$$

$$\boxed{y'=\frac{-1}{2t\sqrt{t-1}}}$$

(e) $y=\csc^{-1}\dfrac{x}{2}$

$$\frac{dy}{dx}=\frac{-1}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}}\cdot\left(\tfrac{1}{2}\right)$$

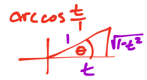
$$\frac{dy}{dx}=\frac{-1}{|x|\sqrt{\frac{x^2}{4}-1}}\cdot\frac{(2)}{\sqrt{4}}$$

$$\boxed{\frac{dy}{dx}=\frac{-2}{|x|\sqrt{x^2-4}}}$$

(f) $y=\sin(\arccos t)$

$$\frac{dy}{dx}=\cos(\arccos t)\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$$

$\arccos\frac{t}{1}$



$\cos(\arccos t)=t$

$$\boxed{\frac{dy}{dx}=\frac{-t}{\sqrt{1-t^2}}}$$

(g) $y=m\arctan m$

$$\frac{dy}{dm}=1\cdot\arctan m+m\left(\frac{1}{1+m^2}\right)$$

$$\boxed{\frac{dy}{dm}=\arctan m+\frac{m}{1+m^2}}$$

(h) $y=x\arcsin x+\sqrt{1+x^2}$

$$y'=1\cdot\arcsin x+x\cdot\left(\frac{1}{\sqrt{1-x^2}}\right)+\tfrac{1}{2}(1+x^2)^{-1/2}\cdot(2x)$$

$$\boxed{y'=\arcsin x+\frac{x}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1+x^2}}}$$

(i) $y=\dfrac{\arcsin 3x}{x}$

$$y'=\frac{(x)\left(\frac{1}{\sqrt{1-(3x)^2}}\cdot(3)\right)-\arcsin 3x\cdot(1)}{x^2}$$

$$y'=\frac{\frac{3x}{\sqrt{1-9x^2}}-\arcsin(3x)}{x^2}$$

$$y'=\frac{3x}{x^2\sqrt{1-9x^2}}-\frac{\arcsin 3x}{x^2}$$

$$\boxed{y'=\frac{3}{x\sqrt{1-9x^2}}-\frac{\arcsin 3x}{x^2}}$$

(j) $y = \frac{1}{2}\left(x\sqrt{4-x^2} + 4\arcsin\left(\frac{x}{2}\right)\right)$

$y' = \frac{1}{2}\left[(1)\sqrt{4-x^2} + x\left(\frac{1}{2}\right)(4-x^2)^{-1/2}(-2x) + \frac{4}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\left(\frac{1}{2}\right)\right]$

$y' = \frac{1}{2}\left[\sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} + \frac{2}{\sqrt{1-\frac{x^2}{4}}}\frac{(2)}{(\sqrt{4})}\right]$

$y' = \frac{1}{2}\left[\frac{\sqrt{4-x^2}}{1} - \frac{x^2}{\sqrt{4-x^2}} + \frac{4}{\sqrt{4-x^2}}\right]$

$y' = \frac{1}{2}\left[\frac{4-x^2-x^2+4}{\sqrt{4-x^2}}\right]$

$y' = \frac{1}{2}\left[\frac{8-2x^2}{\sqrt{4-x^2}}\right]$

$y' = \frac{(4-x^2)^1}{(4-x^2)^{1/2}}$

$\boxed{y' = \sqrt{4-x^2}}$ Nice!!

(k) $y = \sin^{-1} t + \cos^{-1} t$

$y' = \frac{1}{\sqrt{1-t^2}} + \frac{-1}{\sqrt{1-t^2}}$

$\boxed{y' = 0}$

2. If a particle's position is given by $x(t) = \tan^{-1}(t^2)$, find the particle's velocity at $t = 1$.

$x'(t) = v(t) = \frac{1}{1+(t^2)^2} = \frac{1}{1+t^4}(2t) = \frac{2t}{1+t^4}$

$x'(1) = v(1) = \frac{2}{1+1^4} = \boxed{1}$ Thanks, SKS.

3. (Calculator Permitted) Find the equation for the tangent line (in slope-intercept form) for the tangent to the graph of y at the indicated point.

(a) $y = \sec^{-1} x$ at $x = 2$

$y' = \frac{1}{|x|\sqrt{x^2-1}}$

$y(2) = \sec^{-1}(2) = \frac{\pi}{3}$

pt: $\left(2, \frac{\pi}{3}\right)$

$y'(2) = \frac{1}{2\sqrt{3}}$

tangent line:

$\boxed{y = \frac{\pi}{3} + \frac{1}{2\sqrt{3}}(x-2)}$

(b) $y = \sin^{-1}\left(\frac{x}{4}\right)$ at $x = 3$

$y(3) = \sin^{-1}\left(\frac{3}{4}\right)$

$y' = \frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}}\left(\frac{1}{4}\right)$

$y'(3) = \frac{1}{4\sqrt{1-\frac{9}{16}}} = \frac{1}{4\sqrt{\frac{7}{16}}}$

$y'(3) = \frac{1}{\cancel{4}\left(\frac{\sqrt{7}}{\cancel{4}}\right)} = \frac{1}{\sqrt{7}}$

$\boxed{\text{eq: } y = \sin^{-1}\left(\frac{3}{4}\right) + \frac{1}{\sqrt{7}}(x-3)}$

4. A particle moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \arctan t$.

(a) Prove that the particle is always moving to the right by analyzing the velocity function.

$v(t) = x'(t) = \frac{1}{1+t^2} > 0 \ \forall t \geq 0$

So particle moves right $\forall t \geq 0$.

(b) Prove that the particle's velocity is always decreasing by analyzing the acceleration function.

$x''(t) = v'(t) = a(t) = \frac{-1}{(1+t^2)^2}(2t)$

$a(t) = \frac{-2t}{(1+t^2)^2} < 0 \ \forall t \geq 0$

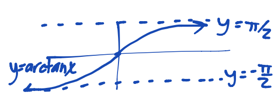
So velocity is decreasing $\forall t \geq 0$

(c) What is the limiting position of the particle as t approaches infinity.

$\lim\limits_{t \to \infty} x(t)$

$= \lim\limits_{t \to \infty} \arctan t$

$= \boxed{\frac{\pi}{2}}$



5. If $f(x) = x^5 + 2x^3 + x - 1$ and $\underbrace{f(g(x)) = x = g(f(x))}_{\text{so } f \,\&\, g \text{ are inverses}}$ find (

(a) $f(1)$

$f(1) = 1+2+1-1$

$= \boxed{3}$

(b) $g'(3)$.

$f'(x) = 5x^4 + 6x^2 + 1$

$f'(1) = 5+6+1 = 12$

$g'(3) = \frac{1}{f'(1)}$

$\boxed{g'(3) = \frac{1}{12}}$

6. If $h(x) = \cos x + 3x$, find a) $h(0)$ and b) $\left(h^{-1}\right)'(1)$.

(a) $h(0) = \cos 0 + 3(0)$
$= 1$

(b) $\left(h^{-1}\right)'(1) = \frac{1}{h'(0)}$

$\boxed{\left(h^{-1}\right)'(1) = \frac{1}{3}}$

$h'(x) = -\sin x + 3$
$h'(0) = -\sin 0 + 3 = 3$

7. Find the equation of the tangent line to the graph of $x^2 + x \arctan y = y - 1$ at $\left(-\frac{\pi}{4}, 1\right)$.

$\frac{d}{dx}\left[x^2 + x\arctan y\right] = \frac{d}{dx}\left[y - 1\right]$

$2x + 1 \cdot \arctan y + x\left(\frac{1}{1+y^2}\right)\left(\frac{dy}{dx}\right) = \frac{dy}{dx} - 0$

$2x + \arctan y = \frac{dy}{dx}\left[1 - \frac{x}{1+y^2}\right]$

$\frac{dy}{dx} = \frac{2x + \arctan y}{1 - \frac{x}{1+y^2}}$

$\left.\frac{dy}{dx}\right|_{\left(-\frac{\pi}{4}, 1\right)} = \frac{2\left(-\frac{\pi}{4}\right) + \arctan 1}{1 - \frac{-\pi/4}{1+1}}$

$= \frac{-\frac{\pi}{2} + \frac{\pi}{4}}{1 + \frac{\pi}{8}}\left(\frac{8}{8}\right)$

$= \frac{-4\pi + 2\pi}{8 + \pi} = \boxed{\frac{-2\pi}{8+\pi} = m}$

$\boxed{eq: y = 1 - \frac{2\pi}{8+\pi}\left(x + \frac{\pi}{4}\right)}$

8. If $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$, find $f^{-1}(x) = 8(x+3) = 8x + 24$, $g^{-1}(x) = \sqrt[3]{x}$

(a) $\left(f^{-1} \circ g^{-1}\right)(1)$

$f^{-1}\left(g^{-1}(1)\right)$
$= f^{-1}\left(\sqrt[3]{1}\right)$
$= f^{-1}(1)$
$= \boxed{32}$

(b) $\left(g^{-1} \circ g^{-1}\right)(-4)$

$g^{-1}\left(g^{-1}(-4)\right)$
$g^{-1}\left(\sqrt[3]{-4}\right)$
$\sqrt[3]{\sqrt[3]{-4}}$
$= (-4)^{1/9}$
$= \boxed{-\sqrt[9]{4}}$

(c) $\left(g^{-1} \circ f^{-1}\right)(-3)$

$g^{-1}\left(f^{-1}(-3)\right)$
$g^{-1}(0)$
$\sqrt[3]{0}$
$\boxed{0}$

C 9. Find the value of $f(1)$ when $f(x)=5\sin^{-1}x+6\tan^{-1}x$.

(A) 3π (B) 2π (C) 4π (D) $\frac{7\pi}{2}$ (E) $\frac{5\pi}{2}$

$f(1)=5\sin^{-1}1+6\tan^{-1}1$
$=5\left(\frac{\pi}{2}\right)+6\left(\frac{\pi}{4}\right)$
$=\frac{5\pi}{2}+\frac{3\pi}{2}$
$=\frac{8\pi}{2}=\boxed{4\pi}$

D 10. Simplify the expression $f(x)=\sin\left(\tan^{-1}x\right)$ by writing it in algebraic form.

(A) $f(x)=\frac{1}{\sqrt{1+x^2}}$ (B) $f(x)=\sqrt{1+x^2}$ (C) $f(x)=\frac{x}{\sqrt{1-x^2}}$ (D) $f(x)=\frac{x}{\sqrt{1+x^2}}$

(E) $f(x)=\frac{1}{\sqrt{1-x^2}}$ (F) $f(x)=\sqrt{1-x^2}$

$\tan^{-1}x$ $\sqrt{1+x^2}$, θ, x, 1 $\quad \sin(\tan^{-1}x)=\boxed{\frac{x}{\sqrt{1+x^2}}}$

E 11. Determine $f'(x)$ when $f(x)=\tan^{-1}\left(\frac{x}{\sqrt{6-x^2}}\right)$.

(A) $f'(x)=\frac{\sqrt{6}}{\sqrt{6+x^2}}$ (B) $f'(x)=\frac{x}{x^2+6}$ (C) $f'(x)=\frac{x}{\sqrt{x^2-6}}$ (D) $f'(x)=\frac{\sqrt{6}}{\sqrt{6-x^2}}$

(E) $f'(x)=\frac{1}{\sqrt{6-x^2}}$

$f'(x)=\frac{1}{1+\left(\frac{x}{\sqrt{6-x^2}}\right)^2}\cdot\left(\frac{\sqrt{6-x^2}(1)-x\left(\frac{1}{2}\right)(6-x^2)^{-1/2}(-2x)}{6-x^2}\right)$

$=\left(\frac{1}{1+\frac{x^2}{6-x^2}}\right)\left(\frac{\frac{\sqrt{6-x^2}}{1}+\frac{x^2}{\sqrt{6-x^2}}}{6-x^2}\right)$

$=\left(\frac{1}{1+\frac{x^2}{6-x^2}}\right)\left(\frac{6-x^2}{6-x^2}\right)\left(\frac{\frac{\sqrt{6-x^2}}{1}+\frac{x^2}{\sqrt{6-x^2}}}{6-x^2}\right)\left(\frac{\sqrt{6-x^2}}{\sqrt{6-x^2}}\right)$

$=\left(\frac{6-x^2}{6-x^2+x^2}\right)\left(\frac{6-x^2+x^2}{(6-x^2)(\sqrt{6-x^2})}\right)$

$=\left(\frac{\cancel{6-x^2}}{\cancel{6}}\right)\left(\frac{\cancel{6}}{\cancel{(6-x^2)}(\sqrt{6-x^2})}\right)=\boxed{\frac{1}{\sqrt{6-x^2}}}$

or $\tan^{-1}\left(\frac{x}{\sqrt{6-x^2}}\right)$ $\sqrt{6}$, θ, x, $\sqrt{6-x^2}$

$=\sin^{-1}\left(\frac{x}{\sqrt{6}}\right)$

$\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{\sqrt{6}}\right)\right]=\frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{6}}\right)^2}}\left(\frac{1}{\sqrt{6}}\right)$

$=\frac{1}{\sqrt{1-\frac{x^2}{6}}}\left(\frac{1}{\sqrt{6}}\right)$

$=\boxed{\frac{1}{\sqrt{6-x^2}}}$

A 12. Find the derivative of f when $f(x)=5\arcsin\frac{x}{5}+\sqrt{25-x^2}$

(A) $f'(x)=\sqrt{\frac{5-x}{5+x}}$ (B) $f'(x)=\frac{x}{\sqrt{25-x^2}}$ (C) $f'(x)=\frac{5}{\sqrt{25-x^2}}$ (D) $f'(x)=\frac{1}{\sqrt{5+x}}$

(E) $f'(x)=\frac{1}{\sqrt{5-x}}$ (F) $f'(x)=\sqrt{\frac{5+x}{5-x}}$

$$f'(x)=\frac{5}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\left(\frac{1}{5}\right)+\frac{1}{2}(25-x^2)^{-1/2}(-2x)$$

$$=\frac{1}{\sqrt{1-\frac{x^2}{25}}}\left(\frac{5}{\sqrt{25}}\right)-\frac{x}{\sqrt{25-x^2}}$$

$$=\frac{5}{\sqrt{25-x^2}}-\frac{x}{\sqrt{25-x^2}}$$

$$=\frac{5-x}{\sqrt{(5-x)(5+x)}}$$

$$=\frac{(5-x)}{\sqrt{5-x}\sqrt{5+x}}\left(\frac{\sqrt{5-x}}{\sqrt{5-x}}\right)$$

$$=\frac{(5-x)\sqrt{5-x}}{(5-x)\sqrt{5+x}}$$

$$=\sqrt{\frac{5-x}{5+x}}$$

A 13. Find the derivative of f when $f(x)=3\left(\sin^{-1}x\right)^2$.

(A) $f'(x)=\frac{6\sin^{-1}x}{\sqrt{1-x^2}}$ (B) $f'(x)=\frac{3\sin^{-1}x}{\sqrt{1-x^2}}$ (C) $f'(x)=\frac{6\sin^{-1}x}{1+x^2}$ (D) $f'(x)=\frac{6\cos^{-1}x}{\sqrt{1-x^2}}$

(E) $f'(x)=\frac{3\cos^{-1}x}{1+x^2}$ (F) $f'(x)=\frac{3\cos^{-1}x}{\sqrt{1-x^2}}$

$$f'(x)=6(\sin^{-1}x)'\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$f'(x)=\frac{6\sin^{-1}x}{\sqrt{1-x^2}}$$

A 14. Determine if $\lim_{x\to\infty} \sin^{-1}\left(\frac{1+x}{6+2x}\right)$ exists, and if it does, find its value.

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{2}$ (C) 0 (D) $\frac{\pi}{3}$ (E) $\frac{\pi}{4}$ (F) DNE

$\sin^{-1}\left(\lim_{x\to\infty}\left(\frac{x+1}{2x+6}\right)\right)$

$= \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

H 15. Let *f* be a twice-differentiable function and let *g* be its inverse. Consider the following equations:

I. $g(f(x)) = x$, $f(g(x)) = x$ → f & g are inverses, so $g'(x) = \frac{1}{f'(g(x))}$, so I & III are true, check $\boxed{\text{II}}$

II. $f''(g(x))(g'(x))^2 + f'(g(x))g''(x) = 0$

III. $g'(x) = \frac{1}{f'(g(x))}$

II: $\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$, $\frac{d}{dx}[f'(g(x))g'(x)] = f''(g(x))\cdot g'(x)\cdot g'(x) + f'(g(x))\cdot g''(x) = \frac{d}{dx}[1] = 0$

$= \frac{d}{dx}[x] = 1$

$= f''(g(x))\cdot(g'(x))^2 + f'(g(x))g''(x) = 0$ so II is true

Which one do *f*, *g* satisfy?

(A) none of them (B) I and II only (C) I only (D) III only (E) II only (F) II and III only (G) I and III only (H) all of them

A 16. Find the value of $g'(1)$ when *g* is the inverse of the function $f(x) = 2\sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

(A) $\frac{1}{\sqrt{3}}$ (B) -1 (C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{3}}$ (E) 1 (F) $-\frac{1}{\sqrt{2}}$

$f(x) = 2\sin x = 1$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$

so $f: \left(\frac{\pi}{6}, 1\right)$

$g: \left(1, \frac{\pi}{6}\right)$

so $g'(1) = \frac{1}{f'\left(\frac{\pi}{6}\right)}$

$g'(1) = \boxed{\frac{1}{\sqrt{3}}}$

$f'(x) = 2\cos x$

$f'\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} = \boxed{\sqrt{3}}$

D 17. Suppose g is the inverse function of a differentiable function f and $G(x) = \frac{1}{g(x)}$. If $f(3) = 7$ and $f'(3) = \frac{1}{9}$, find $G'(7)$.

(A) -5 (B) 4 (C) 6 (D) -1 (E) -4

$G(x) = (g(x))^{-1}$

$G'(x) = -1(g(x))^{-2} \cdot g'(x)$

$G'(x) = \frac{-g'(x)}{(g(x))^2}$

$G'(7) = \frac{-g'(7)}{(g(7))^2}$

$G'(7) = \frac{-9}{(3)^2} = \boxed{-1}$

$f: (3,7)$

$g: (7,3)$

$g'(7) = \frac{1}{f'(3)} = 9$

A 18. Find $\frac{dy}{dx}$ when $\tan(2x - y) = 2x$

(A) $\frac{dy}{dx} = \frac{8x^2}{1+4x^2}$ (B) $\frac{dy}{dx} = -\frac{8x^2}{1+4x^2}$ (C) $\frac{dy}{dx} = -\frac{4y^2}{2+x^2}$ (D) $\frac{dy}{dx} = \frac{4y^2}{2+x^2}$

(E) $\frac{dy}{dx} = -\frac{8x^2}{1+4y^2}$ (F) $\frac{dy}{dx} = \frac{8y^2}{2+x^2}$

*Notice no answer choices have trig in them, so we will solve explicitly for y 1st.

$\tan^{-1}(\tan(2x-y)) = \tan^{-1}(2x)$

$2x - y = \tan^{-1}(2x)$

$y = 2x - \tan^{-1}(2x)$

$\frac{dy}{dx} = 2 - \frac{1}{1+(2x)^2}(2)$

$\frac{dy}{dx} = 2 - \frac{2}{1+4x^2}$

$\frac{dy}{dx} = \frac{2(1+4x^2) - 2}{1+4x^2}$

$\frac{dy}{dx} = \frac{2+8x^2-2}{1+4x^2}$

$\boxed{\frac{dy}{dx} = \frac{8x^2}{1+4x^2}}$