(b) Rewrite the differential equation as

$$\frac{\pi}{\sqrt{2g}B}\left(2Ry^{1/2} - y^{3/2}\right)dy = -dt,$$

and then integrate both sides to obtain

$$\frac{2\pi}{\sqrt{2g}B}\left(\frac{2}{3}Ry^{3/2} - \frac{1}{5}y^{5/2}\right) = C - t,$$

where C is an arbitrary constant. Simplifying gives

$$\frac{2\pi}{15B\sqrt{2g}}(10Ry^{3/2} - 3y^{5/2}) = C - t \qquad (*)$$

(c) From Equation (*) we see that $y = 0$ when $t = C$. It follows that $C = t_e$, the time at which the tank is empty. Moreover, the initial condition $y(0) = R$ allows us to determine the value of C:

$$\frac{2\pi}{15B\sqrt{2g}}(10R^{5/2} - 3R^{5/2}) = \frac{14\pi}{15B\sqrt{2g}}R^{5/2} = C$$

(d) From part (c),

$$t_e = \frac{14\pi}{15\sqrt{2g}} \cdot \frac{R^{5/2}}{B},$$

from which it is clear that t_e is proportional to $R^{5/2}$ and inversely proportional to B.

9.2 Models Involving $y' = k(y - b)$

Preliminary Questions

1. Write down a solution to $y' = 4(y - 5)$ that tends to $-\infty$ as $t \to \infty$.

SOLUTION The general solution is $y(t) = 5 + Ce^{4t}$ for any constant C; thus the solution tends to $-\infty$ as $t \to \infty$ whenever $C < 0$. One specific example is $y(t) = 5 - e^{4t}$.

2. Does $y' = -4(y - 5)$ have a solution that tends to ∞ as $t \to \infty$?

SOLUTION The general solution is $y(t) = 5 + Ce^{-4t}$ for any constant C. As $t \to \infty$, $y(t) \to 5$. Thus, there is no solution of $y' = -4(y - 5)$ that tends to ∞ as $t \to \infty$.

3. True or false? If $k > 0$, then all solutions of $y' = -k(y - b)$ approach the same limit as $t \to \infty$.

SOLUTION True. The general solution of $y' = -k(y - b)$ is $y(t) = b + Ce^{-kt}$ for any constant C. If $k > 0$, then $y(t) \to b$ as $t \to \infty$.

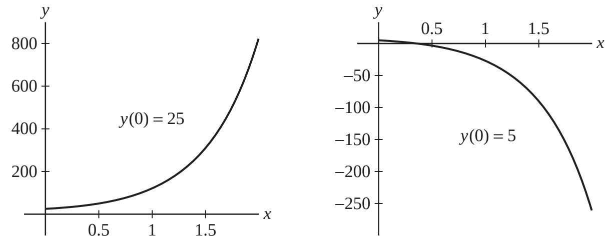
4. As an object cools, its rate of cooling slows. Explain how this follows from Newton's Law of Cooling.

SOLUTION Newton's Law of Cooling states that $y' = -k(y - T_0)$ where $y(t)$ is the temperature and T_0 is the ambient temperature. Thus as $y(t)$ gets closer to T_0, $y'(t)$, the rate of cooling, gets smaller and the rate of cooling slows.

Exercises

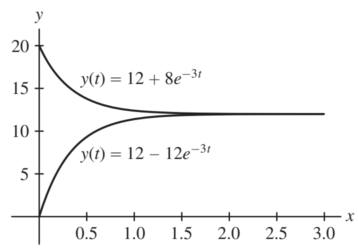
1. Find the general solution of $y' = 2(y - 10)$. Then find the two solutions satisfying $y(0) = 25$ and $y(0) = 5$, and sketch their graphs.

SOLUTION The general solution of $y' = 2(y - 10)$ is $y(t) = 10 + Ce^{2t}$ for any constant C. If $y(0) = 25$, then $10 + C = 25$, or $C = 15$; therefore, $y(t) = 10 + 15e^{2t}$. On the other hand, if $y(0) = 5$, then $10 + C = 5$, or $C = -5$; therefore, $y(t) = 10 - 5e^{2t}$. Graphs of these two functions are given below.



2. Verify directly that $y = 12 + Ce^{-3t}$ satisfies $y' = -3(y - 12)$ for all C. Then find the two solutions satisfying $y(0) = 20$ and $y(0) = 0$, and sketch their graphs.

SOLUTION The general solution of $y' = -3(y - 12)$ is $y(t) = 12 + Ce^{-3t}$ for any constant C. If $y(0) = 20$, then $12 + C = 20$, or $C = 8$; therefore, $y(t) = 12 + 8e^{-3t}$. On the other hand, if $y(0) = 0$, then $12 + C = 0$, or $C = -12$; therefore, $y(t) = 12 - 12e^{-3t}$. Graphs of these two functions are given below.



3. Solve $y' = 4y + 24$ subject to $y(0) = 5$.

SOLUTION Rewrite

$$y' = 4y + 24 \quad \text{as} \quad \frac{1}{4y + 24}\,dy = 1\,dt$$

Integrating gives

$$\begin{aligned}\frac{1}{4}\ln|4y + 24| &= t + C\\ \ln|4y + 24| &= 4t + C\\ 4y + 24 &= \pm e^{4t+C}\\ y &= Ae^{4t} - 6\end{aligned}$$

where $A = \pm e^{C}/4$ is any constant. Since $y(0) = 5$ we have $5 = A - 6$ so that $A = 11$, and the solution is $y = 11e^{4t} - 6$.

4. Solve $y' + 6y = 12$ subject to $y(2) = 10$.

SOLUTION Rewrite

$$y' + 6y = 12 \quad \text{as} \quad \frac{dy}{dt} = 12 - 6y \quad \text{and then as} \quad \frac{1}{12 - 6y}\,dy = 1\,dt$$

Integrate both sides:

$$\begin{aligned}-\frac{1}{6}\ln|12 - 6y| &= t + C\\ \ln|12 - 6y| &= -6t + C\\ 12 - 6y &= \pm e^{-6t+C}\\ y &= Ae^{-6t} + 2\end{aligned}$$

where $A = \pm e^{C}/6$ is any constant. Since $y(2) = 10$ we have $10 = Ae^{-12} + 2$ so that $A = 8e^{12}$, and the solution is $y = 8e^{12-6t} + 2$.

In Exercises 5–12, use Newton's Law of Cooling.

5. A hot anvil with cooling constant $k = 0.02\ \text{s}^{-1}$ is submerged in a large pool of water whose temperature is 10°C. Let $y(t)$ be the anvil's temperature t seconds later.

(a) What is the differential equation satisfied by $y(t)$?

(b) Find a formula for $y(t)$, assuming the object's initial temperature is 100°C.

(c) How long does it take the object to cool down to 20°?

SOLUTION

(a) By Newton's Law of Cooling, the differential equation is

$$y' = -0.02(y - 10)$$

(b) Separating variables gives

$$\frac{1}{y-10}\,dy = -0.02\,dt$$

Integrate to get

$$\ln|y-10| = -0.02t + C$$
$$y - 10 = \pm e^{-0.02t+C}$$
$$y = 10 + Ae^{-0.02t}$$

where $A = \pm e^C$ is a constant. Since the initial temperature is 100°C, we have $y(0) = 100 = 10 + A$ so that $A = 90$, and $y = 10 + 90e^{-0.02t}$.

(c) We must find the value of t such that $y(t) = 20$, so we need to solve $20 = 10 + 90e^{-0.02t}$. Thus

$$10 = 90e^{-0.02t} \quad\Rightarrow\quad \frac{1}{9} = e^{-0.02t} \quad\Rightarrow\quad -\ln 9 = -0.02t \quad\Rightarrow\quad t = 50\ln 9 \approx 109.86 \text{ s}$$

6. Frank's automobile engine runs at 100°C. On a day when the outside temperature is 21°C, he turns off the ignition and notes that five minutes later, the engine has cooled to 70°C.

(a) Determine the engine's cooling constant k.

(b) What is the formula for $y(t)$?

(c) When will the engine cool to 40°C?

SOLUTION

(a) The differential equation is

$$y' = -k(y-21)$$

Rewriting gives $\frac{1}{y-21}\,dy = -k\,dt$. Integrate to get

$$\ln|y-21| = -kt + C$$
$$y - 21 = \pm e^{C-kt}$$
$$y = 21 + Ae^{-kt}$$

where $A = \pm e^C$ is a constant. The initial temperature is 100°C, so $y(0) = 100$. Thus $100 = 21 + A$ and $A = 79$, so that $y = 21 + 79e^{-kt}$. The second piece of information tells us that $y(5) = 70 = 21 + 79e^{-5k}$. Solving for k gives

$$k = -\frac{1}{5}\ln\frac{49}{79} \approx 0.0955$$

(b) From part (b), the equation is $y = 21 + 79e^{-0.0955t}$.

(c) The engine has cooled to 40°C when $y(t) = 40$; solving gives

$$40 = 21 + 79e^{-0.0955t} \quad\Rightarrow\quad e^{-0.0955t} = \frac{19}{79} \quad\Rightarrow\quad t = -\frac{1}{0.0955}\ln\frac{19}{79} \approx 14.92 \text{ m}$$

7. At 10:30 AM, detectives discover a dead body in a room and measure its temperature at 26°C. One hour later, the body's temperature had dropped to 24.8°C. Determine the time of death (when the body temperature was a normal 37°C), assuming that the temperature in the room was held constant at 20°C.

SOLUTION Let $t = 0$ be the time when the person died, and let t_0 denote 10:30AM. The differential equation satisfied by the body temperature, $y(t)$, is

$$y' = -k(y-20)$$

by Newton's Law of Cooling. Separating variables gives $\frac{1}{y-20}\,dy = -k\,dt$. Integrate to get

$$\ln|y-20| = -kt + C$$
$$y - 20 = \pm e^{-kt+C}$$
$$y = 20 + Ae^{-kt}$$

where $A = \pm e^C$ is a constant. Since normal body temperature is 37°C, we have $y(0) = 37 = 20 + A$ so that $A = 17$. To determine k, note that

$$26 = 20 + 17e^{-kt_0} \qquad \text{and} \qquad 24.8 = 20 + 17e^{-k(t_0+1)}$$

$$kt_0 = -\ln\frac{6}{17} \qquad\qquad kt_0 + k = -\ln\frac{4.8}{17}$$

Subtracting these equations gives

$$k = \ln\frac{6}{17} - \ln\frac{4.8}{17} = \ln\frac{6}{4.8} \approx 0.223$$

We thus have

$$y = 20 + 17e^{-0.223t}$$

as the equation for the body temperature at time t. Since $y(t_0) = 26$, we have

$$26 = 20 + 17e^{-0.223t} \quad \Rightarrow \quad e^{-0.223t} = \frac{6}{17} \quad \Rightarrow \quad t = -\frac{1}{0.223}\ln\frac{6}{17} \approx 4.667 \text{ h}$$

so that the time of death was approximately 4 hours and 40 minutes ago.

8. A cup of coffee with cooling constant $k = 0.09 \text{ min}^{-1}$ is placed in a room at temperature 20°C.

(a) How fast is the coffee cooling (in degrees per minute) when its temperature is $T = 80$°C?

(b) Use the Linear Approximation to estimate the change in temperature over the next 6 s when $T = 80$°C.

(c) If the coffee is served at 90°C, how long will it take to reach an optimal drinking temperature of 65°C?

SOLUTION

(a) According to Newton's Law of Cooling, the coffee will cool at the rate $k(T - T_0)$, where k is the cooling constant of the coffee, T is the current temperature of the coffee and T_0 is the temperature of the surroundings. With $k = 0.09 \text{ min}^{-1}$, $T = 80$°C and $T_0 = 20$°C, the coffee is cooling at the rate

$$0.09(80 - 20) = 5.4^\circ\text{C/min}.$$

(b) Using the result from part (a) and the Linear Approximation, we estimate that the coffee will cool

$$(5.4^\circ\text{C/min})(0.1 \text{ min}) = 0.54^\circ\text{C}$$

over the next 6 seconds.

(c) With $T_0 = 20$°C and an initial temperature of 90°C, the temperature of the coffee at any time t is $T(t) = 20 + 70e^{-0.09t}$. Solving $20 + 70e^{-0.09t} = 65$ for t yields

$$t = -\frac{1}{0.09}\ln\left(\frac{45}{70}\right) \approx 4.91 \text{ minutes.}$$

9. A cold metal bar at −30°C is submerged in a pool maintained at a temperature of 40°C. Half a minute later, the temperature of the bar is 20°C. How long will it take for the bar to attain a temperature of 30°C?

SOLUTION With $T_0 = 40$°C, the temperature of the bar is given by $F(t) = 40 + Ce^{-kt}$ for some constants C and k. From the initial condition, $F(0) = 40 + C = -30$, so $C = -70$. After 30 seconds, $F(30) = 40 - 70e^{-30k} = 20$, so

$$k = -\frac{1}{30}\ln\left(\frac{20}{70}\right) \approx 0.0418 \text{ seconds}^{-1}.$$

To attain a temperature of 30°C we must solve $40 - 70e^{-0.0418t} = 30$ for t. This yields

$$t = \frac{\ln\left(\frac{10}{70}\right)}{-0.0418} \approx 46.55 \text{ seconds.}$$

10. When a hot object is placed in a water bath whose temperature is 25°C, it cools from 100°C to 50°C in 150 s. In another bath, the same cooling occurs in 120 s. Find the temperature of the second bath.

SOLUTION With $T_0 = 25$°C, the temperature of the object is given by $F(t) = 25 + Ce^{-kt}$ for some constants C and k. From the initial condition, $F(0) = 25 + C = 100$, so $C = 75$. After 150 seconds, $F(150) = 25 + 75e^{-150k} = 50$, so

$$k = -\frac{1}{150}\ln\left(\frac{25}{75}\right) \approx 0.0073 \text{ seconds}^{-1}.$$

If we place the same object with a temperature of 100°C into a second bath whose temperature is T_0, then the temperature of the object is given by

$$F(t) = T_0 + (100 - T_0)e^{-0.0073t}.$$

To cool from 100°C to 50°C in 120 seconds, T_0 must satisfy

$$T_0 + (100 - T_0)e^{-0.0073(120)} = 50.$$

Thus, $T_0 = 14.32$°C.

11. GU Objects A and B are placed in a warm bath at temperature $T_0 = 40$°C. Object A has initial temperature -20°C and cooling constant $k = 0.004\ \text{s}^{-1}$. Object B has initial temperature 0°C and cooling constant $k = 0.002\ \text{s}^{-1}$. Plot the temperatures of A and B for $0 \le t \le 1000$. After how many seconds will the objects have the same temperature?

SOLUTION With $T_0 = 40$°C, the temperature of A and B are given by

$$A(t) = 40 + C_A e^{-0.004t} \qquad B(t) = 40 + C_B e^{-0.002t}$$

Since $A(0) = -20$ and $B(0) = 0$, we have

$$A(t) = 40 - 60e^{-0.004t} \qquad B(t) = 40 - 40e^{-0.002t}$$

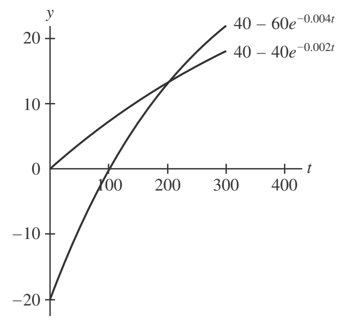
The two objects will have the same temperature whenever $A(t) = B(t)$, so we must solve

$$40 - 60e^{-0.004t} = 40 - 40e^{-0.002t} \quad \Rightarrow \quad 3e^{-0.004t} = 2e^{-0.002t}$$

Take logs to get

$$-0.004t + \ln 3 = -0.002t + \ln 2 \quad \Rightarrow \quad t = \frac{\ln 3 - \ln 2}{0.002} \approx 202.7 \text{ s}$$

or about 3 minutes 22 seconds.



12. In Newton's Law of Cooling, the constant $\tau = 1/k$ is called the "characteristic time." Show that τ is the time required for the temperature difference $(y - T_0)$ to decrease by the factor $e^{-1} \approx 0.37$. For example, if $y(0) = 100$°C and $T_0 = 0$°C, then the object cools to $100/e \approx 37$°C in time τ, to $100/e^2 \approx 13.5$°C in time 2τ, and so on.

SOLUTION If $y' = -k(y - T_0)$, then $y(t) = T_0 + Ce^{-kt}$. But then

$$\frac{y(t+\tau) - T_0}{y(t) - T_0} = \frac{Ce^{-k(t+\tau)}}{Ce^{-kt}} = e^{-k\tau} = e^{-k \cdot 1/k} = e^{-1}$$

Thus after time τ starting from any time t, the temperature difference will have decreased by a factor of e^{-1}.

In Exercises 13–16, use Eq. (3) as a model for free-fall with air resistance.

13. A 60-kg skydiver jumps out of an airplane. What is her terminal velocity, in meters per second, assuming that $k = 10$ kg/s for free-fall (no parachute)?

SOLUTION The free-fall terminal velocity is

$$\frac{-gm}{k} = \frac{-9.8(60)}{10} = -58.8 \text{ m/s}.$$

14. Find the terminal velocity of a skydiver of weight $w = 192$ lb if $k = 1.2$ lb-s/ft. How long does it take him to reach half of his terminal velocity if his initial velocity is zero? Mass and weight are related by $w = mg$, and Eq. (3) becomes $v' = -(kg/w)(v + w/k)$ with $g = 32$ ft/s^2.

SOLUTION The skydiver's velocity $v(t)$ satisfies the differential equation

$$v' = -\frac{kg}{w}\left(v + \frac{w}{k}\right),$$

where

$$\frac{kg}{w} = \frac{(1.2)(32)}{192} = 0.2 \text{ sec}^{-1} \quad \text{and} \quad \frac{w}{k} = \frac{192}{1.2} = 160 \text{ ft/sec}.$$

The general solution to this equation is $v(t) = -160 + Ce^{-0.2t}$, for some constant C. From the initial condition $v(0) = 0$, we find $0 = -160 + C$, or $C = 160$. Therefore,

$$v(t) = -160 + 160e^{-0.2t} = -160(1 - e^{-0.2t}).$$

Now, the terminal velocity of the skydiver is

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} -160(1 - e^{-0.2t}) = -160 \text{ ft/sec}.$$

To determine how long it takes for the skydiver to reach half this terminal velocity, we must solve the equation $v(t) = -80$ for t:

$$\begin{aligned} -160(1 - e^{-0.2t}) &= -80 \\ 1 - e^{-0.2t} &= \frac{1}{2} \\ e^{-0.2t} &= \frac{1}{2} \\ t &= -\frac{1}{0.2}\ln\frac{1}{2} \approx 3.47 \text{ sec}. \end{aligned}$$

15. A 80-kg skydiver jumps out of an airplane (with zero initial velocity). Assume that $k = 12$ kg/s with a closed parachute and $k = 70$ kg/s with an open parachute. What is the skydiver's velocity at $t = 25$ s if the parachute opens after 20 s of free fall?

SOLUTION We first compute the skydiver's velocity after 20 s of free fall, then use that as the initial velocity to calculate her velocity after an additional 5 s of restrained fall. We have $m = 80$ and $g = 9.8$; for free fall, $k = 12$, so

$$\frac{k}{m} = \frac{12}{80} = 0.15, \qquad \frac{-mg}{k} = \frac{-80 \cdot 9.8}{12} \approx -65.33$$

The general solution is thus $v(t) = -65.33 + Ce^{-0.15t}$. Since $v(0) = 0$, we have $C = 65.33$, so that

$$v(t) = -65.33(1 - e^{-0.15t})$$

After 20 s of free fall, the diver's velocity is thus

$$v(20) = -65.33(1 - e^{-0.15\cdot 20}) \approx -62.08 \text{ m/s}$$

Once the parachute opens, $k = 70$, so

$$\frac{k}{m} = \frac{70}{80} = 0.875, \qquad \frac{mg}{k} = \frac{80 \cdot 9.8}{70} = 11.2$$

so that the general solution for the restrained fall model is $v_r(t) = -11.2 + Ce^{-0.875t}$. Here $v_r(0) = -62.08$, so that $C = 11.2 - 62.08 = -50.88$ and $v_r(t) = -11.20 - 50.88e^{-0.875t}$. After 5 additional seconds, the diver's velocity is therefore

$$v_r(5) = -11.20 - 50.88e^{-0.875\cdot 5} \approx -11.84 \text{ m/s}$$

16. Does a heavier or a lighter skydiver reach terminal velocity faster?

SOLUTION The velocity of a skydiver is

$$v(t) = -\frac{gm}{k} + Ce^{-kt/m}.$$

As m decreases, the fraction $-k/m$ becomes more negative and $e^{-(k/m)t}$ approaches zero more rapidly. Thus, a lighter skydiver approaches terminal velocity faster.

17. A continuous annuity with withdrawal rate $N = \$5000$/year and interest rate $r = 5\%$ is funded by an initial deposit of $P_0 = \$50{,}000$.

(a) What is the balance in the annuity after 10 years?

(b) When will the annuity run out of funds?

SOLUTION

(a) From Equation 7, the value of the annuity is given by

$$P(t) = \frac{5000}{0.05} + Ce^{0.05t} = 100{,}000 + Ce^{0.05t}$$

for some constant C. Since $P(0) = 50{,}000$, we have $C = -50{,}000$ and $P(t) = 100{,}000 - 50{,}000e^{0.05t}$. After ten years, then, the balance in the annuity is

$$P(10) = 100{,}000 - 50{,}000e^{0.05\cdot 10} = 100{,}000 - 50{,}000e^{0.5} \approx \$17{,}563.94$$

(b) The annuity will run out of funds when $P(t) = 0$:

$$0 = 100{,}000 - 50{,}000e^{0.05t} \quad \Rightarrow \quad e^{0.05t} = 2 \quad \Rightarrow \quad t = \frac{\ln 2}{0.05} \approx 13.86$$

The annuity will run out of funds after approximately 13 years 10 months.

18. Show that a continuous annuity with withdrawal rate $N = \$5000$/year and interest rate $r = 8\%$, funded by an initial deposit of $P_0 = \$75{,}000$, never runs out of money.

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{5000}{0.08} + Ce^{0.08t} = 62500 + Ce^{0.08t}$$

for some constant C. If $P_0 = 75{,}000$, then $75{,}000 = 62{,}500 + C$ and $C = 12{,}500$. Thus, $P(t) = 62{,}500 + 12{,}500e^{0.08t}$. As $t \to \infty$, $P(t) \to \infty$, so the annuity lives forever. Note the annuity will live forever for any $P_0 \geq \$62{,}500$.

19. Find the minimum initial deposit P_0 that will allow an annuity to pay out \$6000/year indefinitely if it earns interest at a rate of 5%.

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{6000}{0.05} + Ce^{0.05t} = 120{,}000 + Ce^{0.05t}$$

for some constant C. To fund the annuity indefinitely, we must have $C \geq 0$. If the initial deposit is P_0, then $P_0 = 120{,}000 + C$ and $C = P_0 - 120{,}000$. Thus, to fund the annuity indefinitely, we must have $P_0 \geq \$120{,}000$.

20. Find the minimum initial deposit P_0 necessary to fund an annuity for 20 years if withdrawals are made at a rate of \$10,000/year and interest is earned at a rate of 7%.

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{10{,}000}{0.07} + Ce^{0.07t} = 142{,}857.14 + Ce^{0.07t}$$

for some constant C. If the initial deposit is P_0, then $P_0 = 142{,}857.14 + C$ and $C = 142{,}857.14 - P_0$. To fund the annuity for 20 years, we need

$$P(20) = 142{,}857.14 + (P_0 - 142{,}857.14)e^{0.07(20)} \geq 0.$$

Hence,

$$P_0 \geq 142{,}857.14(1 - e^{-1.4}) = \$107{,}629.00.$$

21. An initial deposit of 100,000 euros are placed in an annuity with a French bank. What is the minimum interest rate the annuity must earn to allow withdrawals at a rate of 8000 euros/year to continue indefinitely?

SOLUTION Let $P(t)$ denote the balance of the annuity at time t measured in years. Then

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{8000}{r} + Ce^{rt}$$

for some constant C. To fund the annuity indefinitely, we need $C \geq 0$. If the initial deposit is 100,000 euros, then $100{,}000 = \frac{8000}{r} + C$ and $C = 100{,}000 - \frac{8000}{r}$. Thus, to fund the annuity indefinitely, we need $100{,}000 - \frac{8000}{r} \geq 0$, or $r \geq 0.08$. The bank must pay at least 8%.

22. Show that a continuous annuity never runs out of money if the initial balance is greater than or equal to N/r, where N is the withdrawal rate and r the interest rate.

SOLUTION With a withdrawal rate of N and an interest rate of r, the balance in the annuity is $P(t) = \frac{N}{r} + Ce^{rt}$ for some constant C. Let P_0 denote the initial balance. Then $P_0 = P(0) = \frac{N}{r} + C$ and $C = P_0 - \frac{N}{r}$. If $P_0 \geq \frac{N}{r}$, then $C \geq 0$ and the annuity lives forever.

23. Sam borrows \$10,000 from a bank at an interest rate of 9% and pays back the loan continuously at a rate of N dollars per year. Let $P(t)$ denote the amount still owed at time t.

(a) Explain why $P(t)$ satisfies the differential equation

$$y' = 0.09y - N$$

(b) How long will it take Sam to pay back the loan if $N = \$1200$?

(c) Will the loan ever be paid back if $N = \$800$?

SOLUTION

(a)

$$\text{Rate of Change of Loan} = (\text{Amount still owed})(\text{Interest rate}) - (\text{Payback rate})$$
$$= P(t) \cdot r - N = r\left(P - \frac{N}{r}\right).$$

Therefore, if $y = P(t)$,

$$y' = r\left(y - \frac{N}{r}\right) = ry - N$$

(b) From the differential equation derived in part (a), we know that $P(t) = \frac{N}{r} + Ce^{rt} = 13{,}333.33 + Ce^{0.09t}$. Since \$10,000 was initially borrowed, $P(0) = 13{,}333.33 + C = 10{,}000$, and $C = -3333.33$. The loan is paid off when $P(t) = 13{,}333.33 - 3333.33e^{0.09t} = 0$. This yields

$$t = \frac{1}{0.09}\ln\left(\frac{13{,}333.33}{3333.33}\right) \approx 15.4 \text{ years.}$$

(c) If the annual rate of payment is \$800, then $P(t) = 800/0.09 + Ce^{0.09t} = 8888.89 + Ce^{0.09t}$. With $P(0) = 8888.89 + C = 10{,}000$, it follows that $C = 1111.11$. Since $C > 0$ and $e^{0.09t} \to \infty$ as $t \to \infty$, $P(t) \to \infty$, and the loan will never be paid back.

24. April borrows \$18,000 at an interest rate of 5% to purchase a new automobile. At what rate (in dollars per year) must she pay back the loan, if the loan must be paid off in 5 years? *Hint:* Set up the differential equation as in Exercise 23).

SOLUTION As in Exercise 23, the differential equation is

$$P(t)' = rP(t) - N = r\left(P(t) - \frac{N}{r}\right)$$

where r is the interest rate and N is the payment amount, so that here

$$P(t)' = 0.05\left(P(t) - \frac{N}{0.05}\right) \quad \Rightarrow \quad P(t) = \frac{N}{0.05} + Ce^{0.05t}$$

Since $P(0) = 18{,}000$, we have $C = 18{,}000 - \frac{N}{0.05}$, so that

$$P(t) = \frac{N}{0.05} + \left(18{,}000 - \frac{N}{0.05}\right)e^{0.05t}$$

If the loan is to be paid back in 5 years, we must have

$$P(5) = 0 = \frac{N}{0.05} + \left(18{,}000 - \frac{N}{0.05}\right)e^{0.05\cdot 5}$$

Solving for N gives

$$N = \frac{900}{1 - e^{-0.25}} \approx 4068.73$$

so the payments must be at least \$4068.73 per year.

25. Let $N(t)$ be the fraction of the population who have heard a given piece of news t hours after its initial release. According to one model, the rate $N'(t)$ at which the news spreads is equal to k times the fraction of the population that has not yet heard the news, for some constant $k > 0$.

(a) Determine the differential equation satisfied by $N(t)$.

(b) Find the solution of this differential equation with the initial condition $N(0) = 0$ in terms of k.

(c) Suppose that half of the population is aware of an earthquake 8 hours after it occurs. Use the model to calculate k and estimate the percentage that will know about the earthquake 12 hours after it occurs.

SOLUTION

(a) $N'(t) = k(1 - N(t)) = -k(N(t) - 1)$.

(b) The general solution of the differential equation from part (a) is $N(t) = 1 + Ce^{-kt}$. The initial condition determines the value of C: $N(0) = 1 + C = 0$ so $C = -1$. Thus, $N(t) = 1 - e^{-kt}$.

(c) Knowing that $N(8) = 1 - e^{-8k} = \frac{1}{2}$, we find that

$$k = -\frac{1}{8}\ln\left(\frac{1}{2}\right) \approx 0.0866 \text{ hours}^{-1}.$$

With the value of k determined, we estimate that

$$N(12) = 1 - e^{-0.0866(12)} \approx 0.6463 = 64.63\%$$

of the population will know about the earthquake after 12 hours.

26. Current in a Circuit When the circuit in Figure 1 (which consists of a battery of V volts, a resistor of R ohms, and an inductor of L henries) is connected, the current $I(t)$ flowing in the circuit satisfies

$$L\frac{dI}{dt} + RI = V$$

with the initial condition $I(0) = 0$.

(a) Find a formula for $I(t)$ in terms of L, V, and R.

(b) Show that $\lim_{t\to\infty} I(t) = V/R$.

(c) Show that $I(t)$ reaches approximately 63% of its maximum value at the "characteristic time" $\tau = L/R$.

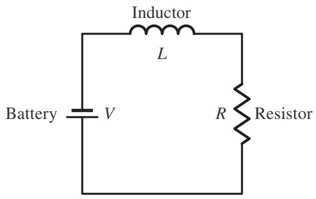


FIGURE 1 Current flow approaches the level $I_{\max} = V/R$.

SOLUTION

(a) Solve the differential equation for $\frac{dI}{dt}$:

$$\frac{dI}{dt} = -\frac{1}{L}(RI - V) = -\frac{R}{L}\left(I - \frac{V}{R}\right)$$

so that the general solution is

$$I(t) = \frac{V}{R} + Ce^{-(R/L)t}$$

The initial condition $I(0) = 0$ gives $C = -\frac{V}{R}$, so that

$$I(t) = \frac{V}{R}(1 - e^{-(R/L)t})$$

(b) As $t \to \infty$, $e^{-(R/L)t} \to 0$, so that $I(t) \to \frac{V}{R}$.

(c) When $t = \tau = L/R$,

$$I(\tau) = \frac{V}{R}(1 - e^{-(R/L)\tau}) = \frac{V}{R}(1 - e^{-(R/L)(L/R)}) = \frac{V}{R}(1 - e^{-1}) \approx 0.63\frac{V}{R}$$

which is 63% of the maximum value of V/R.

Further Insights and Challenges

27. Show that the cooling constant of an object can be determined from two temperature readings $y(t_1)$ and $y(t_2)$ at times $t_1 \neq t_2$ by the formula

$$k = \frac{1}{t_1 - t_2}\ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right)$$

SOLUTION We know that $y(t_1) = T_0 + Ce^{-kt_1}$ and $y(t_2) = T_0 + Ce^{-kt_2}$. Thus, $y(t_1) - T_0 = Ce^{-kt_1}$ and $y(t_2) - T_0 = Ce^{-kt_2}$. Dividing the latter equation by the former yields

$$e^{-kt_2 + kt_1} = \frac{y(t_2) - T_0}{y(t_1) - T_0},$$

so that

$$k(t_1 - t_2) = \ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right) \quad \text{and} \quad k = \frac{1}{t_1 - t_2}\ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right).$$

28. Show that by Newton's Law of Cooling, the time required to cool an object from temperature A to temperature B is

$$t = \frac{1}{k}\ln\left(\frac{A - T_0}{B - T_0}\right)$$

where T_0 is the ambient temperature.

SOLUTION At any time t, the temperature of the object is $y(t) = T_0 + Ce^{-kt}$ for some constant C. Suppose the object is initially at temperature A and reaches temperature B at time t. Then $A = T_0 + C$, so $C = A - T_0$. Moreover,

$$B = T_0 + Ce^{-kt} = T_0 + (A - T_0)e^{-kt}.$$

Solving this last equation for t yields

$$t = \frac{1}{k}\ln\left(\frac{A - T_0}{B - T_0}\right).$$

29. Air Resistance A projectile of mass $m = 1$ travels straight up from ground level with initial velocity v_0. Suppose that the velocity v satisfies $v' = -g - kv$.

(a) Find a formula for $v(t)$.

(b) Show that the projectile's height $h(t)$ is given by

$$h(t) = C(1 - e^{-kt}) - \frac{g}{k}t$$

where $C = k^{-2}(g + kv_0)$.

(c) Show that the projectile reaches its maximum height at time $t_{max} = k^{-1}\ln(1 + kv_0/g)$.

(d) In the absence of air resistance, the maximum height is reached at time $t = v_0/g$. In view of this, explain why we should expect that

$$\lim_{k\to 0}\frac{\ln(1 + \frac{kv_0}{g})}{k} = \frac{v_0}{g} \qquad \boxed{8}$$

(e) Verify Eq. (8). *Hint:* Use Theorem 2 in Section 5.8 to show that $\lim_{k\to 0}\left(1 + \frac{kv_0}{g}\right)^{1/k} = e^{v_0/g}$ or use L'Hôpital's Rule.

SOLUTION

(a) Since $v' = -g - kv = -k\left(v - \frac{-g}{k}\right)$ it follows that $v(t) = \frac{-g}{k} + Be^{-kt}$ for some constant B. The initial condition $v(0) = v_0$ determines B: $v_0 = -\frac{g}{k} + B$, so $B = v_0 + \frac{g}{k}$. Thus,

$$v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}.$$

(b) $v(t) = h'(t)$ so

$$h(t) = \int\left(-\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}\right)dt = -\frac{g}{k}t - \frac{1}{k}\left(v_0 + \frac{g}{k}\right)e^{-kt} + D.$$

The initial condition $h(0) = 0$ determines

$$D = \frac{1}{k}\left(v_0 + \frac{g}{k}\right) = \frac{1}{k^2}(v_0k + g).$$

Let $C = \frac{1}{k^2}(v_0k + g)$. Then

$$h(t) = C(1 - e^{-kt}) - \frac{g}{k}t.$$

(c) The projectile reaches its maximum height when $v(t) = 0$. This occurs when

$$-\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt} = 0,$$

or

$$t = \frac{1}{-k}\ln\left(\frac{g}{kv_0 + g}\right) = \frac{1}{k}\ln\left(1 + \frac{kv_0}{g}\right).$$

(d) Recall that k is the proportionality constant for the force due to air resistance. Thus, as $k \to 0$, the effect of air resistance disappears. We should therefore expect that, as $k \to 0$, the time at which the maximum height is achieved from part (c) should approach v_0/g. In other words, we should expect

$$\lim_{k\to 0}\frac{1}{k}\ln\left(1 + \frac{kv_0}{g}\right) = \frac{v_0}{g}.$$

(e) Recall that

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

If we substitute $x = v_0/g$ and $k = 1/n$, we find

$$e^{v_0/g} = \lim_{k\to 0}\left(1+\frac{v_0 k}{g}\right)^{1/k}.$$

Then

$$\lim_{k\to 0}\frac{1}{k}\ln\left(1+\frac{kv_0}{g}\right) = \lim_{k\to 0}\ln\left(1+\frac{v_0 k}{g}\right)^{1/k} = \ln\left(\lim_{k\to 0}\left(1+\frac{v_0 k}{g}\right)^{1/k}\right) = \ln(e^{v_0/g}) = \frac{v_0}{g}.$$

9.3 Graphical and Numerical Methods

Preliminary Questions

1. What is the slope of the segment in the slope field for $\dot{y} = ty + 1$ at the point $(2, 3)$?

SOLUTION The slope of the segment in the slope field for $\dot{y} = ty + 1$ at the point $(2, 3)$ is $(2)(3) + 1 = 7$.

2. What is the equation of the isocline of slope $c = 1$ for $\dot{y} = y^2 - t$?

SOLUTION The isocline of slope $c = 1$ has equation $y^2 - t = 1$, or $y = \pm\sqrt{1+t}$.

3. For which of the following differential equations are the slopes at points on a vertical line $t = C$ all equal?

(a) $\dot{y} = \ln y$ **(b)** $\dot{y} = \ln t$

SOLUTION Only for the equation in part (b). The slope at a point is simply the value of $\dot{y}$ at that point, so for part (a), the slope depends on y, while for part (b), the slope depends only on t.

4. Let $y(t)$ be the solution to $\dot{y} = F(t, y)$ with $y(1) = 3$. How many iterations of Euler's Method are required to approximate $y(3)$ if the time step is $h = 0.1$?

SOLUTION The initial condition is specified at $t = 1$ and we want to obtain an approximation to the value of the solution at $t = 3$. With a time step of $h = 0.1$,

$$\frac{3-1}{0.1} = 20$$

iterations of Euler's method are required.

Exercises

1. Figure 1 shows the slope field for $\dot{y} = \sin y \sin t$. Sketch the graphs of the solutions with initial conditions $y(0) = 1$ and $y(0) = -1$. Show that $y(t) = 0$ is a solution and add its graph to the plot.

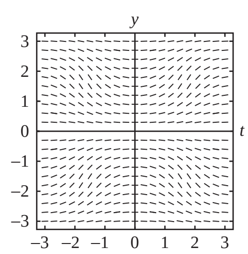
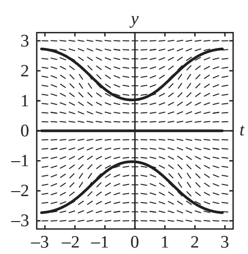


FIGURE 1 Slope field for $\dot{y} = \sin y \sin t$.

SOLUTION The sketches of the solutions appear below.



If $y(t) = 0$, then $y' = 0$; moreover, $\sin 0 \sin t = 0$. Thus, $y(t) = 0$ is a solution of $\dot{y} = \sin y \sin t$.