# 9 INTRODUCTION TO DIFFERENTIAL EQUATIONS

## 9.1 Solving Differential Equations

### Preliminary Questions

**1.** Determine the order of the following differential equations:

**(a)** $x^5 y' = 1$

**(b)** $(y')^3 + x = 1$

**(c)** $y''' + x^4 y' = 2$

**(d)** $\sin(y'') + x = y$

**SOLUTION**

**(a)** The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

**(b)** The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

**(c)** The highest order derivative that appears in this equation is a third derivative, so this is a third order equation.

**(d)** The highest order derivative that appears in this equation is a second derivative, so this is a second order equation.

**2.** Is $y'' = \sin x$ a linear differential equation?

**SOLUTION** Yes.

**3.** Give an example of a nonlinear differential equation of the form $y' = f(y)$.

**SOLUTION** One possibility is $y' = y^2$.

**4.** Can a nonlinear differential equation be separable? If so, give an example.

**SOLUTION** Yes. An example is $y' = y^2$.

**5.** Give an example of a linear, nonseparable differential equation.

**SOLUTION** One example is $y' + y = x$.

### Exercises

**1.** Which of the following differential equations are first-order?

**(a)** $y' = x^2$

**(b)** $y'' = y^2$

**(c)** $(y')^3 + yy' = \sin x$

**(d)** $x^2 y' - e^x y = \sin y$

**(e)** $y'' + 3y' = \dfrac{y}{x}$

**(f)** $yy' + x + y = 0$

**SOLUTION**

**(a)** The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

**(b)** The highest order derivative that appears in this equation is a second derivative, so this is not a first order equation.

**(c)** The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

**(d)** The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

**(e)** The highest order derivative that appears in this equation is a second derivative, so this is not a first order equation.

**(f)** The highest order derivative that appears in this equation is a first derivative, so this is a first order equation.

**2.** Which of the equations in Exercise 1 are linear?

**SOLUTION**

**(a)** Linear; $(1)y' - x^2 = 0$.

**(b)** Not linear; $y^2$ is not a linear function of $y$.

**(c)** Not linear; $(y')^3$ is not a linear function of $y'$.

**(d)** Not linear; $\sin y$ is not a linear function of $y$.

**(e)** Linear; $(1)y'' + (3)y' - \frac{1}{x}y = 0$.

**(f)** Not linear. $yy'$ cannot be expressed as $a(x)y^{(n)}$.

*In Exercises 3–8, verify that the given function is a solution of the differential equation.*

**3.** $y' - 8x = 0, \quad y = 4x^2$

**SOLUTION** Let $y = 4x^2$. Then $y' = 8x$ and

$$y' - 8x = 8x - 8x = 0.$$

**4.** $yy' + 4x = 0, \quad y = \sqrt{12 - 4x^2}$

**SOLUTION** Let $y = \sqrt{12 - 4x^2}$. Then

$$y' = \frac{-4x}{\sqrt{12 - 4x^2}},$$

and

$$yy' + 4x = \sqrt{12 - 4x^2}\frac{-4x}{\sqrt{12 - 4x^2}} + 4x = -4x + 4x = 0.$$

**5.** $y' + 4xy = 0, \quad y = 25e^{-2x^2}$

**SOLUTION** Let $y = 25e^{-2x^2}$. Then $y' = -100xe^{-2x^2}$, and

$$y' + 4xy = -100xe^{-2x^2} + 4x(25e^{-2x^2}) = 0.$$

**6.** $(x^2 - 1)y' + xy = 0, \quad y = 4(x^2 - 1)^{-1/2}$

**SOLUTION** Let $y = 4(x^2 - 1)^{-1/2}$. Then $y' = -4x(x^2 - 1)^{-3/2}$, and

$$\begin{aligned}(x^2 - 1)y' + xy &= (x^2 - 1)(-4x)(x^2 - 1)^{-3/2} + 4x(x^2 - 1)^{-1/2} \\ &= -4x(x^2 - 1)^{-1/2} + 4x(x^2 - 1)^{-1/2} = 0.\end{aligned}$$

**7.** $y'' - 2xy' + 8y = 0, \quad y = 4x^4 - 12x^2 + 3$

**SOLUTION** Let $y = 4x^4 - 12x^2 + 3$. Then $y' = 16x^3 - 24x$, $y'' = 48x^2 - 24$, and

$$\begin{aligned}y'' - 2xy' + 8y &= (48x^2 - 24) - 2x(16x^3 - 24x) + 8(4x^4 - 12x^2 + 3) \\ &= 48x^2 - 24 - 32x^4 + 48x^2 + 32x^4 - 96x^2 + 24 = 0.\end{aligned}$$

**8.** $y'' - 2y' + 5y = 0, \quad y = e^x \sin 2x$

**SOLUTION** Let $y = e^x \sin 2x$. Then

$$\begin{aligned}y' &= 2e^x \cos 2x + e^x \sin 2x, \\ y'' &= -4e^x \sin 2x + 2e^x \cos 2x + 2e^x \cos 2x + e^x \sin 2x = -3e^x \sin 2x + 4e^x \cos 2x,\end{aligned}$$

and

$$\begin{aligned}y'' - 2y' + 5y &= -3e^x \sin 2x + 4e^x \cos 2x - 4e^x \cos 2x - 2e^x \sin 2x + 5e^x \sin 2x \\ &= (-3e^x - 2e^x + 5e^x) \sin 2x + (4e^x - 4e^x) \cos 2x = 0.\end{aligned}$$

**9.** Which of the following equations are separable? Write those that are separable in the form $y' = f(x)g(y)$ (but do not solve).

**(a)** $xy' - 9y^2 = 0$

**(b)** $\sqrt{4 - x^2}y' = e^{3y} \sin x$

**(c)** $y' = x^2 + y^2$

**(d)** $y' = 9 - y^2$

**SOLUTION**

**(a)** $xy' - 9y^2 = 0$ is separable:

$$\begin{aligned}xy' - 9y^2 &= 0 \\ xy' &= 9y^2 \\ y' &= \frac{9}{x}y^2\end{aligned}$$

**(b)** $\sqrt{4 - x^2}y' = e^{3y} \sin x$ is separable:

$$\begin{aligned}\sqrt{4 - x^2}y' &= e^{3y} \sin x \\ y' &= e^{3y} \frac{\sin x}{\sqrt{4 - x^2}}.\end{aligned}$$

**(c)** $y' = x^2 + y^2$ is not separable; $y'$ is already isolated, but is not equal to a product $f(x)g(y)$.

**(d)** $y' = 9 - y^2$ is separable: $y' = (1)(9 - y^2)$.

**10.** The following differential equations appear similar but have very different solutions.

$$\frac{dy}{dx} = x, \qquad \frac{dy}{dx} = y$$

Solve both subject to the initial condition $y(1) = 2$.

**SOLUTION** For the first differential equation, we have $y' = x$ so that, integrating,

$$y = \frac{x^2}{2} + C$$

Since $y(1) = 2$, $C = \frac{3}{2}$, so that

$$y = \frac{x^2 + 3}{2}$$

The second equation is separable: $y^{-1}\,dy = 1\,dx$, so that $\ln|y| = x + C$ and $y = Ce^x$. Since $y(1) = 2$, we have $2 = Ce$ or $C = 2e^{-1}$. Thus $y = 2e^{x-1}$.

**11.** Consider the differential equation $y^3 y' - 9x^2 = 0$.

**(a)** Write it as $y^3\,dy = 9x^2\,dx$.

**(b)** Integrate both sides to obtain $\frac{1}{4}y^4 = 3x^3 + C$.

**(c)** Verify that $y = (12x^3 + C)^{1/4}$ is the general solution.

**(d)** Find the particular solution satisfying $y(1) = 2$.

**SOLUTION** Solving $y^3 y' - 9x^2 = 0$ for $y'$ gives $y' = 9x^2 y^{-3}$.

**(a)** Separating variables in the equation above yields

$$y^3\,dy = 9x^2\,dx$$

**(b)** Integrating both sides gives

$$\frac{y^4}{4} = 3x^3 + C$$

**(c)** Simplify the equation above to get $y^4 = 12x^3 + C$, or $y = (12x^3 + C)^{1/4}$.

**(d)** Solve $2 = (12 \cdot 1^3 + C)^{1/4}$ to get $16 = 12 + C$, or $C = 4$. Thus the particular solution is $y = (12x^3 + 4)^{1/4}$.

**12.** Verify that $x^2 y' + e^{-y} = 0$ is separable.

**(a)** Write it as $e^y\,dy = -x^{-2}\,dx$.

**(b)** Integrate both sides to obtain $e^y = x^{-1} + C$.

**(c)** Verify that $y = \ln(x^{-1} + C)$ is the general solution.

**(d)** Find the particular solution satisfying $y(2) = 4$.

**SOLUTION** Solving $x^2 y' + e^{-y} = 0$ for $y'$ yields

$$y' = -x^{-2}e^{-y}.$$

**(a)** Separating variables in the last equation yields

$$e^y dy = -x^{-2}dx.$$

**(b)** Integrating both sides of the result of part (a), we find

$$\int e^y dy = -\int x^{-2}dx$$
$$e^y + C_1 = x^{-1} + C_2$$
$$e^y = x^{-1} + C$$

**(c)** Solving the last expression from part (b) for $y$, we find

$$y = \ln\left|x^{-1} + C\right|$$

**(d)** $y(2) = 4$ yields $4 = \ln\left|\frac{1}{2} + C\right|$, or $e^4 = C + \frac{1}{2}$. Thus the particular solution is

$$y = \ln\left|\frac{1}{x} - \frac{1}{2} + e^4\right|$$

*In Exercises 13–28, use separation of variables to find the general solution.*

**13.** $y' + 4xy^2 = 0$

**SOLUTION** Rewrite

$$y' + 4xy^2 = 0 \quad \text{as} \quad \frac{dy}{dx} = -4xy^2 \quad \text{and then as} \quad y^{-2}\,dy = -4x\,dx$$

Integrating both sides of this equation gives

$$\int y^{-2}\,dy = -4\int x\,dx$$
$$-y^{-1} = -2x^2 + C$$
$$y^{-1} = 2x^2 + C$$

Solving for $y$ gives

$$y = \frac{1}{2x^2 + C}$$

where $C$ is an arbitrary constant.

**14.** $y' + x^2y = 0$

**SOLUTION** Rewrite

$$y' + x^2y = 0 \quad \text{as} \quad \frac{dy}{dx} = -x^2y \quad \text{and then as} \quad y^{-1}\,dy = -x^2\,dx$$

Integrating both sides of this equation gives

$$\int y^{-1}\,dy = -\int x^2\,dx$$
$$\ln|y| = -\frac{x^3}{3} + C_1$$

Solve for $y$ to get

$$y = \pm e^{-x^3/3 + C_1} = Ce^{-x^3/3}$$

where $C = \pm e^{C_1}$ is an arbitrary constant.

**15.** $\dfrac{dy}{dt} - 20t^4e^{-y} = 0$

**SOLUTION** Rewrite

$$\frac{dy}{dt} - 20t^4e^{-y} = 0 \quad \text{as} \quad \frac{dy}{dt} = 20t^4e^{-y} \quad \text{and then as} \quad e^y\,dy = 20t^4\,dt$$

Integrating both sides of this equation gives

$$\int e^y\,dy = \int 20t^4\,dt$$
$$e^y = 4t^5 + C$$

Solve for $y$ to get $y = \ln(4t^5 + C)$, where $C$ is an arbitrary constant.

**16.** $t^3y' + 4y^2 = 0$

**SOLUTION** Rewrite

$$t^3y' + 4y^2 = 0 \quad \text{as} \quad \frac{dy}{dt} = -4y^2t^{-3} \quad \text{and then as} \quad y^{-2}\,dy = -4t^{-3}\,dt$$

Integrating both sides of this equation gives

$$\int y^{-2}\,dy = -4\int t^{-3}\,dt$$
$$-y^{-1} = 2t^{-2} + C$$

Solve for $y$ to get

$$y = \frac{-1}{2t^{-2} + C} = \frac{-t^2}{2 + Ct^2}$$

where $C$ is an arbitrary constant.

**17.** $2y' + 5y = 4$

**SOLUTION** Rewrite

$$2y' + 5y = 4 \quad \text{as} \quad y' = 2 - \frac{5}{2}y \quad \text{and then as} \quad (4-5y)^{-1}\,dy = \frac{1}{2}\,dx$$

Integrating both sides and solving for $y$ gives

$$\begin{aligned}
\int \frac{dy}{4-5y} &= \frac{1}{2}\int 1\,dx \\
-\frac{1}{5}\ln|4-5y| &= \frac{1}{2}x + C_1 \\
\ln|4-5y| &= C_2 - \frac{5}{2}x \\
4-5y &= C_3 e^{-5x/2} \\
5y &= 4 - C_3 e^{-5x/2} \\
y &= Ce^{-5x/2} + \frac{4}{5}
\end{aligned}$$

where $C$ is an arbitrary constant.

**18.** $\dfrac{dy}{dt} = 8\sqrt{y}$

**SOLUTION** Rewrite

$$\frac{dy}{dt} = 8\sqrt{y} \qquad \text{as} \qquad \frac{dy}{\sqrt{y}} = 8\,dt.$$

Integrating both sides of this equation yields

$$\begin{aligned}
\int \frac{dy}{\sqrt{y}} &= 8\int dt \\
2\sqrt{y} &= 8t + C.
\end{aligned}$$

Solving for $y$, we find

$$\begin{aligned}
\sqrt{y} &= 4t + C \\
y &= (4t+C)^2,
\end{aligned}$$

where $C$ is an arbitrary constant.

**19.** $\sqrt{1-x^2}\,y' = xy$

**SOLUTION** Rewrite

$$\sqrt{1-x^2}\,\frac{dy}{dx} = xy \qquad \text{as} \qquad \frac{dy}{y} = \frac{x}{\sqrt{1-x^2}}\,dx.$$

Integrating both sides of this equation yields

$$\begin{aligned}
\int \frac{dy}{y} &= \int \frac{x}{\sqrt{1-x^2}}\,dx \\
\ln|y| &= -\sqrt{1-x^2} + C.
\end{aligned}$$

Solving for $y$, we find

$$\begin{aligned}
|y| &= e^{-\sqrt{1-x^2}+C} = e^C e^{-\sqrt{1-x^2}} \\
y &= \pm e^C e^{-\sqrt{1-x^2}} = Ae^{-\sqrt{1-x^2}},
\end{aligned}$$

where $A$ is an arbitrary constant.

**20.** $y' = y^2(1-x^2)$

**SOLUTION** Rewrite

$$\frac{dy}{dx} = y^2(1-x^2) \qquad \text{as} \qquad \frac{dy}{y^2} = (1-x^2)\,dx.$$

Integrating both sides of this equation yields

$$\int \frac{dy}{y^2} = \int (1-x^2)\,dx$$
$$-y^{-1} = x - \frac{1}{3}x^3 + C.$$

Solving for $y$, we find

$$y^{-1} = \frac{1}{3}x^3 - x + C$$
$$y = \frac{1}{\frac{1}{3}x^3 - x + C},$$

where $C$ is an arbitrary constant.

**21.** $yy' = x$

**SOLUTION** Rewrite

$$y\frac{dy}{dx} = x \qquad \text{as} \qquad y\,dy = x\,dx.$$

Integrating both sides of this equation yields

$$\int y\,dy = \int x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C.$$

Solving for $y$, we find

$$y^2 = x^2 + 2C$$
$$y = \pm\sqrt{x^2 + A},$$

where $A = 2C$ is an arbitrary constant.

**22.** $(\ln y)y' - ty = 0$

**SOLUTION** Rewrite

$$(\ln y)y' - ty = 0 \quad \text{as} \quad (\ln y)\frac{dy}{dt} = ty \quad \text{and then as} \quad \frac{\ln y}{y}\,dy = t\,dt$$

Integrating both sides of this equation gives

$$\int \frac{\ln y}{y}\,dy = \int t\,dt$$
$$\frac{1}{2}\ln^2 y = \frac{1}{2}t^2 + C_1$$
$$\ln^2 y = t^2 + C$$
$$\ln y = \pm\sqrt{t^2 + C}$$
$$y = e^{\pm\sqrt{t^2+C}}$$

**23.** $\frac{dx}{dt} = (t+1)(x^2+1)$

**SOLUTION** Rewrite

$$\frac{dx}{dt} = (t+1)(x^2+1) \qquad \text{as} \qquad \frac{1}{x^2+1}\,dx = (t+1)\,dt.$$

Integrating both sides of this equation yields

$$\int \frac{1}{x^2+1}\,dx = \int (t+1)\,dt$$
$$\tan^{-1} x = \frac{1}{2}t^2 + t + C.$$

Solving for $x$, we find

$$x = \tan\left(\frac{1}{2}t^2 + t + C\right).$$

where $A = \tan C$ is an arbitrary constant.

**24.** $(1+x^2)y' = x^3 y$

**SOLUTION** Rewrite

$$(1+x^2)\frac{dy}{dx} = x^3 y \qquad \text{as} \qquad \frac{1}{y}\,dy = \frac{x^3}{1+x^2}\,dx.$$

Integrating both sides of this equation yields

$$\int \frac{1}{y}\,dy = \int \frac{x^3}{1+x^2}\,dx.$$

To integrate $\frac{x^3}{1+x^2}$, note

$$\frac{x^3}{1+x^2} = \frac{(x^3+x)-x}{1+x^2} = x - \frac{x}{1+x^2}.$$

Thus,

$$\ln|y| = \frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + C$$
$$|y| = e^C \frac{e^{x^2/2}}{\sqrt{x^2+1}}$$
$$y = \pm e^C \frac{e^{x^2/2}}{\sqrt{x^2+1}} = A\frac{e^{x^2/2}}{\sqrt{x^2+1}},$$

where $A = \pm e^C$ is an arbitrary constant.

**25.** $y' = x \sec y$

**SOLUTION** Rewrite

$$\frac{dy}{dx} = x \sec y \qquad \text{as} \qquad \cos y\,dy = x\,dx.$$

Integrating both sides of this equation yields

$$\int \cos y\,dy = \int x\,dx$$
$$\sin y = \frac{1}{2}x^2 + C.$$

Solving for $y$, we find

$$y = \sin^{-1}\left(\frac{1}{2}x^2 + C\right),$$

where $C$ is an arbitrary constant.

**26.** $\dfrac{dy}{d\theta} = \tan y$

**SOLUTION** Rewrite

$$\frac{dy}{d\theta} = \tan y \qquad \text{as} \qquad \cot y\,dy = d\theta.$$

Integrating both sides of this equation yields

$$\int \frac{\cos y}{\sin y}\,dy = \int d\theta$$
$$\ln|\sin y| = \theta + C.$$

Solving for $y$, we have

$$|\sin y| = e^{\theta+C} = e^C e^\theta$$
$$\sin y = \pm e^C e^\theta$$
$$y = \sin^{-1}\left(Ae^\theta\right),$$

where $A = \pm e^C$ is an arbitrary constant.

**27.** $\dfrac{dy}{dt} = y\tan t$

**SOLUTION** Rewrite

$$\frac{dy}{dt} = y\tan t \qquad \text{as} \qquad \frac{1}{y}\,dy = \tan t\,dt.$$

Integrating both sides of this equation yields

$$\int \frac{1}{y}\,dy = \int \tan t\,dt$$
$$\ln|y| = \ln|\sec t| + C.$$

Solving for $y$, we find

$$|y| = e^{\ln|\sec t|+C} = e^C\,|\sec t|$$
$$y = \pm e^C \sec t = A\sec t,$$

where $A = \pm e^C$ is an arbitrary constant.

**28.** $\dfrac{dx}{dt} = t\tan x$

**SOLUTION** Rewrite

$$\frac{dx}{dt} = t\tan x \qquad \text{as} \qquad \cot x\,dx = t\,dt.$$

Integrating both sides of this equation yields

$$\int \cot x\,dx = \int t\,dt$$
$$\ln|\sin x| = \frac{1}{2}t^2 + C.$$

Solving for $y$, we find

$$|\sin x| = e^{\frac{1}{2}t^2+C} = e^C e^{\frac{1}{2}t^2}$$
$$\sin x = \pm e^C e^{\frac{1}{2}t^2}$$
$$x = \sin^{-1}\left(Ae^{\frac{1}{2}t^2}\right),$$

where $A = \pm e^C$ is an arbitrary constant.

*In Exercises 29–42, solve the initial value problem.*

**29.** $y' + 2y = 0, \quad y(\ln 5) = 3$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} + 2y = 0 \qquad \text{as} \qquad \frac{1}{y}\,dy = -2\,dx,$$

and then integrate to obtain

$$\ln |y| = -2x + C.$$

Thus,

$$y = Ae^{-2x},$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(\ln 5) = 3$ allows us to determine the value of $A$.

$$3 = Ae^{-2(\ln 5)}; \quad 3 = A\frac{1}{25}; \quad \text{so} \quad 75 = A.$$

Finally,

$$y = 75e^{-2x}.$$

**30.** $y' - 3y + 12 = 0, \quad y(2) = 1$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} - 3y + 12 = 0 \qquad \text{as} \qquad \frac{1}{3y - 12}\,dy = 1\,dx,$$

and then integrate to obtain

$$\frac{1}{3}\ln|3y - 12| = x + C.$$

Thus,

$$y = Ae^{3x} + 4,$$

where $A = \pm\frac{1}{3}e^{3C}$ is an arbitrary constant. The initial condition $y(2) = 1$ allows us to determine the value of $A$.

$$1 = Ae^6 + 4; \quad -3 = Ae^6; \quad \text{so} \quad -3e^{-6} = A.$$

Finally,

$$y = -3e^{-6}e^{3x} + 4 = -3e^{3x-6} + 4$$

**31.** $yy' = xe^{-y^2}, \quad y(0) = -2$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$y\frac{dy}{dx} = xe^{-y^2} \qquad \text{as} \qquad ye^{y^2}\,dy = x\,dx,$$

and then integrate to obtain

$$\frac{1}{2}e^{y^2} = \frac{1}{2}x^2 + C.$$

Thus,

$$y = \pm\sqrt{\ln(x^2 + A)},$$

where $A = 2C$ is an arbitrary constant. The initial condition $y(0) = -2$ allows us to determine the value of $A$. Since $y(0) < 0$, we have $y = -\sqrt{\ln(x^2 + A)}$, and

$$-2 = -\sqrt{\ln(A)}; \quad 4 = \ln(A); \quad \text{so} \quad e^4 = A.$$

Finally,

$$y = -\sqrt{\ln(x^2 + e^4)}.$$

**32.** $y^2\frac{dy}{dx} = x^{-3}, \quad y(1) = 0$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$y^2\frac{dy}{dx} = x^{-3} \qquad \text{as} \qquad y^2\,dy = x^{-3}\,dx,$$

and then integrate to obtain

$$\frac{1}{3}y^3 = -\frac{1}{2}x^{-2} + C.$$

Thus,

$$y = \left(A - \frac{3}{2}x^{-2}\right)^{1/3},$$

where $A = 3C$ is an arbitrary constant. The initial condition $y(1) = 0$ allows us to determine the value of $A$.

$$0 = \left(A - \frac{3}{2}1^{-2}\right)^{1/3}; \quad 0 = \left(A - \frac{3}{2}\right)^{1/3}; \quad \text{so} \quad A = \frac{3}{2}.$$

Finally,

$$y = \left(\frac{3}{2} - \frac{3}{2}x^{-2}\right)^{1/3}.$$

**33.** $y' = (x-1)(y-2), \quad y(2) = 4$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = (x-1)(y-2) \qquad \text{as} \qquad \frac{1}{y-2}\,dy = (x-1)\,dx,$$

and then integrate to obtain

$$\ln|y-2| = \frac{1}{2}x^2 - x + C.$$

Thus,

$$y = Ae^{(1/2)x^2 - x} + 2,$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(2) = 4$ allows us to determine the value of $A$.

$$4 = Ae^0 + 2 \quad \text{so} \quad A = 2.$$

Finally,

$$y = 2e^{(1/2)x^2 - x} + 2.$$

**34.** $y' = (x-1)(y-2), \quad y(2) = 2$

**SOLUTION** First (as in the previous problem), we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = (x-1)(y-2) \qquad \text{as} \qquad \frac{1}{y-2}\,dy = (x-1)\,dx,$$

and then integrate to obtain

$$\ln|y-2| = \frac{1}{2}x^2 - x + C.$$

Thus,

$$y = Ae^{(1/2)x^2 - x} + 2,$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(2) = 2$ allows us to determine the value of $A$.

$$2 = Ae^0 + 2 \quad \text{so} \quad A = 0.$$

Finally,

$$y = 2.$$

**35.** $y' = x(y^2 + 1), \quad y(0) = 0$

**SOLUTION** First, find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = x(y^2 + 1) \qquad \text{as} \qquad \frac{1}{y^2 + 1}\,dy = x\,dx$$

and integrate to obtain

$$\tan^{-1} y = \frac{1}{2}x^2 + C$$

so that

$$y = \tan\left(\frac{1}{2}x^2 + C\right)$$

where $C$ is an arbitrary constant. The initial condition $y(0) = 0$ allows us to determine the value of $C$: $0 = \tan(C)$, so $C = 0$. Finally,

$$y = \tan\left(\frac{1}{2}x^2\right)$$

**36.** $(1 - t)\dfrac{dy}{dt} - y = 0, \quad y(2) = -4$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$(1 - t)\frac{dy}{dt} = y \qquad \text{as} \qquad \frac{1}{y}\,dy = \frac{-1}{t - 1}\,dt,$$

and then integrate to obtain

$$\ln|y| = -\ln|t - 1| + C.$$

Thus,

$$y = \frac{A}{t - 1},$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(2) = -4$ allows us to determine the value of $A$.

$$-4 = \frac{A}{2 - 1} = A.$$

Finally,

$$y = \frac{-4}{t - 1}.$$

**37.** $\dfrac{dy}{dt} = ye^{-t}, \quad y(0) = 1$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dt} = ye^{-t} \qquad \text{as} \qquad \frac{1}{y}\,dy = e^{-t}\,dt,$$

and then integrate to obtain

$$\ln|y| = -e^{-t} + C.$$

Thus,

$$y = Ae^{-e^{-t}},$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(0) = 1$ allows us to determine the value of $A$.

$$1 = Ae^{-1} \quad \text{so} \quad A = e.$$

Finally,

$$y = (e)e^{-e^{-t}} = e^{1-e^{-t}}.$$

**38.** $\dfrac{dy}{dt} = te^{-y}, \quad y(1) = 0$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dt} = te^{-y} \qquad \text{as} \qquad e^y\, dy = t\, dt,$$

and then integrate to obtain

$$e^y = \frac{1}{2}t^2 + C.$$

Thus,

$$y = \ln\left(\frac{1}{2}t^2 + C\right),$$

where $C$ is an arbitrary constant. The initial condition $y(1) = 0$ allows us to determine the value of $C$.

$$0 = \ln\left(\frac{1}{2} + C\right); \quad 1 = \frac{1}{2} + C; \quad \text{so} \quad C = \frac{1}{2}.$$

Finally,

$$y = \ln\left(\frac{1}{2}t^2 + \frac{1}{2}\right).$$

**39.** $t^2\dfrac{dy}{dt} - t = 1 + y + ty, \quad y(1) = 0$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$t^2\frac{dy}{dt} = 1 + t + y + ty = (1 + t)(1 + y)$$

as

$$\frac{1}{1+y}\, dy = \frac{1+t}{t^2}\, dt,$$

and then integrate to obtain

$$\ln|1 + y| = -t^{-1} + \ln|t| + C.$$

Thus,

$$y = A\frac{t}{e^{1/t}} - 1,$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(1) = 0$ allows us to determine the value of $A$.

$$0 = A\left(\frac{1}{e}\right) - 1 \quad \text{so} \quad A = e.$$

Finally,

$$y = \frac{et}{e^{1/t}} - 1.$$

**40.** $\sqrt{1 - x^2}\, y' = y^2 + 1, \quad y(0) = 0$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$\sqrt{1-x^2}\,\frac{dy}{dx} = y^2 + 1 \qquad \text{as} \qquad \frac{1}{y^2+1}\, dy = \frac{1}{\sqrt{1-x^2}}\, dx,$$

and then integrate to obtain

$$\tan^{-1} y = \sin^{-1} x + C.$$

Thus,

$$y = \tan(\sin^{-1} x + C),$$

where $C$ is an arbitrary constant. The initial condition $y(0) = 0$ allows us to determine the value of $C$.

$$0 = \tan\left(\sin^{-1} 0 + C\right) = \tan C \quad \text{so} \quad 0 = C.$$

Finally,

$$y = \tan\left(\sin^{-1} x\right).$$

**41.** $y' = \tan y, \quad y(\ln 2) = \dfrac{\pi}{2}$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = \tan y \qquad \text{as} \qquad \frac{dy}{\tan y} = dx,$$

and then integrate to obtain

$$\ln|\sin y| = x + C.$$

Thus,

$$y = \sin^{-1}(Ae^x),$$

where $A = \pm e^C$ is an arbitrary constant. The initial condition $y(\ln 2) = \frac{\pi}{2}$ allows us to determine the value of $A$.

$$\frac{\pi}{2} = \sin^{-1}(2A); \quad 1 = 2A \quad \text{so} \quad A = \frac{1}{2}.$$

Finally,

$$y = \sin^{-1}\left(\frac{1}{2}e^x\right).$$

**42.** $y' = y^2 \sin x, \quad y(\pi) = 2$

**SOLUTION** First, we find the general solution of the differential equation. Rewrite

$$\frac{dy}{dx} = y^2 \sin x \qquad \text{as} \qquad y^{-2}\,dy = \sin x\,dx,$$

and then integrate to obtain

$$-y^{-1} = -\cos x + C.$$

Thus,

$$y = \frac{1}{A + \cos x},$$

where $A = -C$ is an arbitrary constant. The initial condition $y(\pi) = 2$ allows us to determine the value of $A$.

$$2 = \frac{1}{A-1}; \quad A - 1 = \frac{1}{2} \quad \text{so} \quad A = \frac{1}{2} + 1 = \frac{3}{2}.$$

Finally,

$$y = \frac{1}{\cos x + (3/2)} = \frac{2}{3 + 2\cos x}.$$

**43.** Find all values of $a$ such that $y = x^a$ is a solution of

$$y'' - 12x^{-2}y = 0$$

**SOLUTION** Let $y = x^a$. Then

$$y' = ax^{a-1} \quad \text{and} \quad y'' = a(a-1)x^{a-2}.$$

Substituting into the differential equation, we find

$$y'' - 12x^{-2}y = a(a-1)x^{a-2} - 12x^{a-2} = x^{a-2}(a^2 - a - 12).$$

Thus, $y'' - 12x^{-2}y = 0$ if and only if

$$a^2 - a - 12 = (a-4)(a+3) = 0.$$

Hence, $y = x^a$ is a solution of the differential equation $y'' - 12x^{-2}y = 0$ provided $a = 4$ or $a = -3$.

**44.** Find all values of $a$ such that $y = e^{ax}$ is a solution of

$$y'' + 4y' - 12y = 0$$

**SOLUTION** Let $y = e^{ax}$. Then

$$y' = ae^{ax} \quad \text{and} \quad y'' = a^2 e^{ax}.$$

Substituting into the differential equation, we find

$$y'' + 4y' - 12y = e^{ax}(a^2 + 4a - 12).$$

Because $e^{ax}$ is never zero, $y'' + 4y' - 12y = 0$ if only if $a^2 + 4a - 12 = (a+6)(a-2) = 0$. Hence, $y = e^{ax}$ is a solution of the differential equation $y'' + 4y' - 12y = 0$ provided $a = -6$ or $a = 2$.

*In Exercises 45 and 46, let $y(t)$ be a solution of $(\cos y + 1)\dfrac{dy}{dt} = 2t$ such that $y(2) = 0$.*

**45.** Show that $\sin y + y = t^2 + C$. We cannot solve for $y$ as a function of $t$, but, assuming that $y(2) = 0$, find the values of $t$ at which $y(t) = \pi$.

**SOLUTION** Rewrite

$$(\cos y + 1)\frac{dy}{dt} = 2t \quad \text{as} \quad (\cos y + 1)\,dy = 2t\,dt$$

and integrate to obtain

$$\sin y + y = t^2 + C$$

where $C$ is an arbitrary constant. Since $y(2) = 0$, we have $\sin 0 + 0 = 4 + C$ so that $C = -4$ and the particular solution we seek is $\sin y + y = t^2 - 4$. To find values of $t$ at which $y(t) = \pi$, we must solve $\sin \pi + \pi = t^2 - 4$, or $t^2 - 4 = \pi$; thus $t = \pm\sqrt{\pi + 4}$.

**46.** Assuming that $y(6) = \pi/3$, find an equation of the tangent line to the graph of $y(t)$ at $(6, \pi/3)$.

**SOLUTION** At $(6, \pi/3)$, we have

$$\left(\cos\frac{\pi}{3} + 1\right)\frac{dy}{dt} = 2(6) = 12 \quad \Rightarrow \quad \frac{3}{2}y' = 12$$

and hence $y' = 8$. The tangent line has equation

$$(y - \pi/3) = 8(x - 6)$$

*In Exercises 47–52, use Eq. (4) and Torricelli's Law [Eq. (5)].*

**47.** Water leaks through a hole of area 0.002 $m^2$ at the bottom of a cylindrical tank that is filled with water and has height 3 m and a base of area 10 $m^2$. How long does it take (a) for half of the water to leak out and (b) for the tank to empty?

**SOLUTION** Because the tank has a constant cross-sectional area of 10 $m^2$ and the hole has an area of 0.002 $m^2$, the differential equation for the height of the water in the tank is

$$\frac{dy}{dt} = \frac{0.002v}{10} = 0.0002v.$$

By Torricelli's Law,

$$v = -\sqrt{2gy} = -\sqrt{19.6y},$$

using $g = 9.8$ m/s$^2$. Thus,

$$\frac{dy}{dt} = -0.0002\sqrt{19.6y} = -0.0002\sqrt{19.6}\cdot\sqrt{y}.$$

Separating variables and then integrating yields

$$y^{-1/2}\,dy = -0.0002\sqrt{19.6}\,dt$$
$$2y^{1/2} = -0.0002\sqrt{19.6}t + C$$

Solving for $y$, we find

$$y(t) = \left(C - 0.0001\sqrt{19.6}t\right)^2.$$

Since the tank is originally full, we have the initial condition $y(0) = 10$, whence $\sqrt{10} = C$. Therefore,

$$y(t) = \left(\sqrt{10} - 0.0001\sqrt{19.6}t\right)^2.$$

When half of the water is out of the tank, $y = 1.5$, so we solve:

$$1.5 = \left(\sqrt{10} - 0.0001\sqrt{19.6}t\right)^2$$

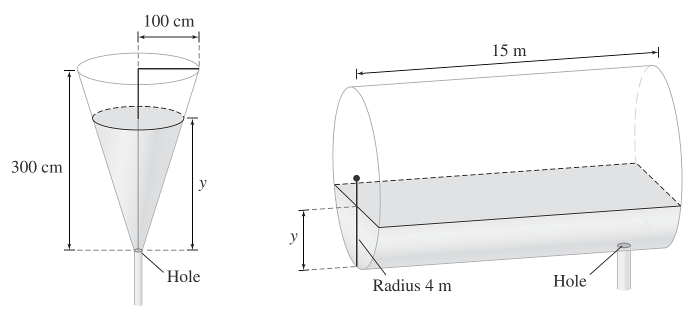
for $t$, finding

$$t = \frac{1}{0.0002\sqrt{19.6}}(2\sqrt{10} - \sqrt{6}) \approx 4376.44 \text{ sec.}$$

When all of the water is out of the tank, $y = 0$, so

$$\sqrt{10} - 0.0001\sqrt{19.6}t = 0 \quad \text{and} \quad t = \frac{\sqrt{10}}{0.0001\sqrt{19.6}} \approx 7142.86 \text{ sec.}$$

**48.** At $t = 0$, a conical tank of height 300 cm and top radius 100 cm [Figure 1(A)] is filled with water. Water leaks through a hole in the bottom of area 3 $\text{cm}^2$. Let $y(t)$ be the water level at time $t$.
**(a)** Show that the tank's cross-sectional area at height $y$ is $A(y) = \frac{\pi}{9}y^2$.
**(b)** Find and solve the differential equation satisfied by $y(t)$
**(c)** How long does it take for the tank to empty?




(A) Conical tank (B) Horizontal tank

**FIGURE 1**

**SOLUTION**
**(a)** By similar triangles, the radius $r$ at height $y$ satisfies

$$\frac{r}{y} = \frac{100}{300} = \frac{1}{3},$$

so $r = y/3$ and

$$A(y) = \pi r^2 = \frac{\pi}{9}y^2.$$

**(b)** The area of the hole is $B = 3$ $\text{cm}^2$, so the differential equation for the height of the water in the tank becomes:

$$\frac{dy}{dt} = -\frac{3\sqrt{19.6}\sqrt{y}}{A(y)} = -\frac{27\sqrt{19.6}}{\pi}y^{-3/2}.$$

Separating variables and integrating then yields

$$y^{3/2}\,dy = -\frac{27\sqrt{19.6}}{\pi}\,dt$$

$$\frac{2}{5}y^{5/2} = C - \frac{27\sqrt{19.6}}{\pi}t$$

When $t = 0$, $y = 300$, so we find $C = \frac{2}{5}(300)^{5/2}$. Therefore,

$$y(t) = \left(300^{5/2} - \frac{135\sqrt{19.6}}{2\pi}t\right)^{2/5}.$$

**(c)** The tank is empty when $y = 0$. Using the result from part (b), $y = 0$ when

$$t = \frac{4000\pi\sqrt{300}}{3\sqrt{19.6}} \approx 16{,}387.82 \text{ seconds.}$$

Thus, it takes roughly 4 hours, 33 minutes for the tank to empty.

**49.** The tank in Figure 1(B) is a cylinder of radius 4 m and height 15 m. Assume that the tank is half-filled with water and that water leaks through a hole in the bottom of area $B = 0.001$ m$^2$. Determine the water level $y(t)$ and the time $t_e$ when the tank is empty.

**SOLUTION** When the water is at height $y$ over the bottom, the top cross section is a rectangle with length 15 m, and with width $x$ satisfying the equation:

$$(x/2)^2 + (y-4)^2 = 16.$$

Thus, $x = 2\sqrt{8y - y^2}$, and

$$A(y) = 15x = 30\sqrt{8y - y^2}.$$

With $B = 0.001$ m$^2$ and $v = -\sqrt{2gy} = -\sqrt{19.6}\sqrt{y}$, it follows that

$$\frac{dy}{dt} = -\frac{0.001\sqrt{19.6}\sqrt{y}}{30\sqrt{8y-y^2}} = -\frac{0.001\sqrt{19.6}}{30\sqrt{8-y}}.$$

Separating variables and integrating then yields:

$$\sqrt{8-y}\,dy = -\frac{0.001\sqrt{19.6}}{30}\,dt = -\frac{0.0001\sqrt{19.6}}{3}\,dt$$

$$-\frac{2}{3}(8-y)^{3/2} = -\frac{0.0001\sqrt{19.6}}{3}t + C$$

When $t = 0$, $y = 4$, so $C = -\frac{2}{3}4^{3/2} = -\frac{16}{3}$, and

$$-\frac{2}{3}(8-y)^{3/2} = -\frac{0.0001\sqrt{19.6}}{3}t - \frac{16}{3}$$

$$y(t) = 8 - \left(\frac{0.0001\sqrt{19.6}}{2}t + 8\right)^{2/3}.$$

The tank is empty when $y = 0$. Thus, $t_e$ satisfies the equation

$$8 - \left(\frac{0.0001\sqrt{19.6}}{2}t + 8\right)^{2/3} = 0.$$

It follows that

$$t_e = \frac{2(8^{3/2} - 8)}{0.0001\sqrt{19.6}} \approx 66{,}079.9 \text{ seconds}.$$

**50.** A tank has the shape of the parabola $y = x^2$, revolved around the $y$-axis. Water leaks from a hole of area $B = 0.0005$ m$^2$ at the bottom of the tank. Let $y(t)$ be the water level at time $t$. How long does it take for the tank to empty if it is initially filled to height $y_0 = 1$ m.

**SOLUTION** When the water is at height $y$, the surface of the water is a circle with radius $\sqrt{y}$, so the cross-sectional area is $A(y) = \pi y$. With $B = 0.0005$ m and $v = -\sqrt{2gy} = -\sqrt{19.6}\sqrt{y}$, it follows that

$$\frac{dy}{dt} = -\frac{0.0005\sqrt{19.6}\sqrt{y}}{A(y)} = -\frac{0.0005\sqrt{19.6}\sqrt{y}}{\pi y} = -\frac{0.0005\sqrt{19.6}}{\pi\sqrt{y}}$$

Separating variables and integrating yields

$$\pi y^{1/2}\,dy = -0.0005\sqrt{19.6}\,dt$$

$$\frac{2}{3}\pi y^{3/2} = -0.0005\sqrt{19.6}t + C$$

$$y^{3/2} = -\frac{0.00075\sqrt{19.6}}{\pi}t + C$$

Since $y(0) = 1$, we have $C = 1$, so that

$$y = \left(1 - \frac{0.00075\sqrt{19.6}}{\pi}t\right)^{2/3}$$

The tank is empty when $y = 0$, so when $1 - \frac{0.00075\sqrt{19.6}}{\pi}t = 0$ and thus

$$t = \frac{\pi}{0.00075\sqrt{19.6}} \approx 946.15 \text{ s}$$

**51.** A tank has the shape of the parabola $y = ax^2$ (where $a$ is a constant) revolved around the $y$-axis. Water drains from a hole of area $B$ m$^2$ at the bottom of the tank.

**(a)** Show that the water level at time $t$ is

$$y(t) = \left(y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t\right)^{2/3}$$

where $y_0$ is the water level at time $t = 0$.

**(b)** Show that if the total volume of water in the tank has volume $V$ at time $t = 0$, then $y_0 = \sqrt{2aV/\pi}$. *Hint:* Compute the volume of the tank as a volume of rotation.

**(c)** Show that the tank is empty at time

$$t_e = \left(\frac{2}{3B\sqrt{g}}\right)\left(\frac{2\pi V^3}{a}\right)^{1/4}$$

We see that for fixed initial water volume $V$, the time $t_e$ is proportional to $a^{-1/4}$. A large value of $a$ corresponds to a tall thin tank. Such a tank drains more quickly than a short wide tank of the same initial volume.

**SOLUTION**

**(a)** When the water is at height $y$, the surface of the water is a circle of radius $\sqrt{y/a}$, so that the cross-sectional area is $A(y) = \pi y/a$. With $v = -\sqrt{2gy} = -\sqrt{2g}\sqrt{y}$, we have

$$\frac{dy}{dt} = -\frac{B\sqrt{2g}\sqrt{y}}{A} = -\frac{aB\sqrt{2g}\sqrt{y}}{\pi y} = -\frac{aB\sqrt{2g}}{\pi}y^{-1/2}$$

Separating variables and integrating gives

$$\sqrt{y}\,dy = -\frac{aB\sqrt{2g}}{\pi}\,dt$$

$$\frac{2}{3}y^{3/2} = -\frac{aB\sqrt{2g}}{\pi}t + C_1$$

$$y^{3/2} = -\frac{3aB\sqrt{2g}}{2\pi}t + C$$

Since $y(0) = y_0$, we have $C = y_0^{3/2}$; solving for $y$ gives

$$y = \left(y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t\right)^{2/3}$$

**(b)** The volume of the tank can be computed as a volume of rotation. Using the disk method and applying it to the function $x = \sqrt{y/a}$, we have

$$V = \int_0^{y_0} \pi\sqrt{\frac{y}{a}}^2\,dy = \frac{\pi}{a}\int_0^{y_0} y\,dy = \frac{\pi}{2a}y^2\bigg|_0^{y_0} = \frac{\pi}{2a}y_0^2$$

Solving for $y_0$ gives

$$y_0 = \sqrt{2aV/\pi}$$

**(c)** The tank is empty when $y = 0$; this occurs when

$$y_0^{3/2} - \frac{3aB\sqrt{2g}}{2\pi}t = 0$$

From part (b), we have

$$y_0^{3/2} = \sqrt{2aV/\pi}^{3/2} = ((2aV/\pi)^{1/2})^{3/2} = (2aV/\pi)^{3/4}$$

so that

$$t_e = \frac{2\pi y_0^{3/2}}{3aB\sqrt{2g}} = \frac{2\pi\sqrt[4]{8a^3V^3}}{3\pi^{3/4}B\sqrt[4]{a^4}\sqrt[4]{4}\sqrt{g}} = \frac{2\pi^{1/4}\sqrt[4]{2V^3a^{-1}}}{3B\sqrt{g}} = \left(\frac{2}{3B\sqrt{g}}\right)\left(\frac{2\pi V^3}{a}\right)^{1/4}$$

**52.** A cylindrical tank filled with water has height $h$ and a base of area $A$. Water leaks through a hole in the bottom of area $B$.

**(a)** Show that the time required for the tank to empty is proportional to $A\sqrt{h}/B$.
**(b)** Show that the emptying time is proportional to $Vh^{-1/2}$, where $V$ is the volume of the tank.
**(c)** Two tanks have the same volume and a hole of the same size, but they have different heights and bases. Which tank empties first: the taller or the shorter tank?

**SOLUTION** Torricelli's law gives the differential equation for the height of the water in the tank as

$$\frac{dy}{dt} = -\sqrt{2g}\frac{B\sqrt{y}}{A}$$

Separating variables and integrating then yields:

$$y^{-1/2}\,dy = -\sqrt{2g}\frac{B}{A}\,dt$$
$$2y^{1/2} = -\sqrt{2g}\frac{Bt}{A} + C$$
$$y^{1/2} = -\sqrt{g/2}\frac{Bt}{A} + C$$

When $t = 0$, $y = h$, so $C = h^{1/2}$ and

$$y^{1/2} = \sqrt{h} - \sqrt{g/2}\frac{Bt}{A}.$$

**(a)** When the tank is empty, $y = 0$. Thus, the time required for the tank to empty, $t_e$, satisfies the equation

$$0 = \sqrt{h} - \sqrt{g/2}\frac{Bt_e}{A}.$$

It follows that

$$t_e = \frac{A}{B}\sqrt{2h/g} = \sqrt{2/g}\left(\frac{A\sqrt{h}}{B}\right);$$

that is, the time required for the tank to empty is proportional to $A\sqrt{h}/B$.
**(b)** The volume of the tank is $V = Ah$; therefore

$$\frac{A\sqrt{h}}{B} = \frac{1}{B}\frac{V}{\sqrt{h}},$$

and

$$t_e = \sqrt{2/g}\left(\frac{A\sqrt{h}}{B}\right) = \frac{\sqrt{2/g}}{B}\left(\frac{V}{\sqrt{h}}\right);$$

that is, the time required for the tank to empty is proportional to $Vh^{-1/2}$.
**(c)** By part (b), with $V$ and $B$ held constant, the emptying time decreases with height. The taller tank therefore empties first.

**53.** Figure 2 shows a circuit consisting of a resistor of $R$ ohms, a capacitor of $C$ farads, and a battery of voltage $V$. When the circuit is completed, the amount of charge $q(t)$ (in coulombs) on the plates of the capacitor varies according to the differential equation ($t$ in seconds)

$$R\frac{dq}{dt} + \frac{1}{C}q = V$$

where $R$, $C$, and $V$ are constants.
**(a)** Solve for $q(t)$, assuming that $q(0) = 0$.
**(b)** Show that $\lim_{t\to\infty} q(t) = CV$.
**(c)** Show that the capacitor charges to approximately 63% of its final value $CV$ after a time period of length $\tau = RC$ ($\tau$ is called the time constant of the capacitor).

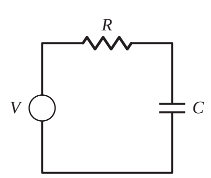



**FIGURE 2** An $RC$ circuit.

**SOLUTION**
**(a)** Upon rearranging the terms of the differential equation, we have

$$\frac{dq}{dt} = -\frac{q - CV}{RC}.$$

Separating the variables and integrating both sides, we obtain

$$\frac{dq}{q - CV} = -\frac{dt}{RC}$$

$$\int \frac{dq}{q - CV} = -\int \frac{dt}{RC}$$

and

$$\ln|q - CV| = -\frac{t}{RC} + k,$$

where $k$ is an arbitrary constant. Solving for $q(t)$ yields

$$q(t) = CV + Ke^{-\frac{1}{RC}t},$$

where $K = \pm e^{k}$. We use the initial condition $q(0) = 0$ to solve for $K$:

$$0 = CV + K \quad \Rightarrow \quad K = -CV$$

so that the particular solution is

$$q(t) = CV(1 - e^{-\frac{1}{RC}t})$$

**(b)** Using the result from part (a), we calculate

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} CV(1 - e^{-\frac{1}{RC}t}) = CV(1 - \lim_{t\to\infty} 1 - e^{-\frac{1}{RC}t}) = CV.$$

**(c)** We have

$$q(\tau) = q(RC) = CV(1 - e^{-\frac{1}{RC}RC}) = CV(1 - e^{-1}) \approx 0.632CV.$$

**54.** Assume in the circuit of Figure 2 that $R = 200\ \Omega$, $C = 0.02$ F, and $V = 12$ V. How many seconds does it take for the charge on the capacitor plates to reach half of its limiting value?

**SOLUTION** From Exercise 53, we know that

$$q(t) = CV\left(1 - e^{-t/(RC)}\right) = 0.24(1 - e^{-t/4}),$$

and the limiting value of $q(t)$ is $CV = 0.24$. If the charge on the capacitor plates has reached half its limiting value, then

$$\frac{0.24}{2} = 0.24(1 - e^{-t/4})$$
$$1 - e^{-t/4} = 1/2$$
$$e^{-t/4} = 1/2$$
$$t = 4\ln 2$$

Therefore, the charge on the capacitor plates reaches half of its limiting value after $4\ln 2 \approx 2.773$ seconds.

**55.** According to one hypothesis, the growth rate $dV/dt$ of a cell's volume $V$ is proportional to its surface area $A$. Since $V$ has cubic units such as $\text{cm}^3$ and $A$ has square units such as $\text{cm}^2$, we may assume roughly that $A \propto V^{2/3}$, and hence $dV/dt = kV^{2/3}$ for some constant $k$. If this hypothesis is correct, which dependence of volume on time would we expect to see (again, roughly speaking) in the laboratory?

**(a)** Linear **(b)** Quadratic **(c)** Cubic

**SOLUTION** Rewrite

$$\frac{dV}{dt} = kV^{2/3} \qquad \text{as} \qquad V^{-2/3}\,dv = k\,dt,$$

and then integrate both sides to obtain

$$3V^{1/3} = kt + C$$
$$V = (kt/3 + C)^3.$$

Thus, we expect to see $V$ increasing roughly like the cube of time.

**56.** We might also guess that the volume $V$ of a melting snowball decreases at a rate proportional to its surface area. Argue as in Exercise 55 to find a differential equation satisfied by $V$. Suppose the snowball has volume 1000 $\text{cm}^3$ and that it loses half of its volume after 5 min. According to this model, when will the snowball disappear?

**SOLUTION** Since the volume is decreasing, we write (as in Exercise 55) $V' = -kV^{2/3}$ where $k$ is positive, so $V(t) = (C - kt/3)^3$. $V(0) = 1000$ implies that $C = 10$ so $V(t) = (10 - kt/3)^3$. Since it loses half of its volume after 5 minutes, we have $V(5) = \frac{1}{2}V(0)$, so that

$$(10 - 5k/3)^3 = 500 \qquad \text{so that} \quad k = 6 - 3 \cdot 2^{2/3} \approx 1.2378$$

and finally the equation is

$$V(t) = \left(10 - \frac{1.2378t}{3}\right)^3$$

The snowball is melted when its volume is zero, so when

$$10 - \frac{1.2378t}{3} = 0 \quad \Rightarrow \quad t = \frac{30}{1.2378} \approx 24.24 \text{ minutes}$$

**57.** In general, $(fg)'$ is not equal to $f'g'$, but let $f(x) = e^{3x}$ and find a function $g(x)$ such that $(fg)' = f'g'$. Do the same for $f(x) = x$.

**SOLUTION** If $(fg)' = f'g'$, we have

$$\begin{aligned} f'(x)g(x) + g'(x)f(x) &= f'(x)g'(x) \\ g'(x)(f(x) - f'(x)) &= -g(x)f'(x) \\ \frac{g'(x)}{g(x)} &= \frac{f'(x)}{f'(x) - f(x)} \end{aligned}$$

Now, let $f(x) = e^{3x}$. Then $f'(x) = 3e^{3x}$ and

$$\frac{g'(x)}{g(x)} = \frac{3e^{3x}}{3e^{3x} - e^{3x}} = \frac{3}{2}.$$

Integrating and solving for $g(x)$, we find

$$\begin{aligned} \frac{dg}{g} &= \frac{3}{2}\,dx \\ \ln|g| &= \frac{3}{2}x + C \\ g(x) &= Ae^{(3/2)x}, \end{aligned}$$

where $A = \pm e^C$ is an arbitrary constant.

If $f(x) = x$, then $f'(x) = 1$, and

$$\frac{g'(x)}{g(x)} = \frac{1}{1 - x}.$$

Thus,

$$\begin{aligned} \frac{dg}{g} &= \frac{1}{1 - x}\,dx \\ \ln|g| &= -\ln|1 - x| + C \\ g(x) &= \frac{A}{1 - x}, \end{aligned}$$

where $A = \pm e^C$ is an arbitrary constant.

**58.** A boy standing at point $B$ on a dock holds a rope of length $\ell$ attached to a boat at point $A$ [Figure 3(A)]. As the boy walks along the dock, holding the rope taut, the boat moves along a curve called a **tractrix** (from the Latin *tractus*, meaning "to pull"). The segment from a point $P$ on the curve to the $x$-axis along the tangent line has constant length $\ell$. Let $y = f(x)$ be the equation of the tractrix.

**(a)** Show that $y^2 + (y/y')^2 = \ell^2$ and conclude $y' = -\dfrac{y}{\sqrt{\ell^2 - y^2}}$. Why must we choose the negative square root?

**(b)** Prove that the tractrix is the graph of

$$x = \ell \ln\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2}$$

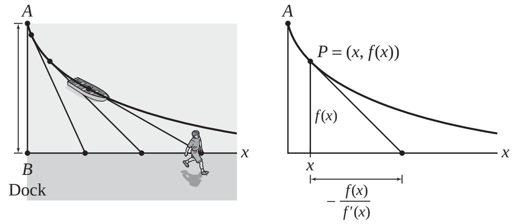



**FIGURE 3**

**SOLUTION**

**(a)** From the diagram on the right in Figure 3, we see that

$$f(x)^2 + \left(-\frac{f(x)}{f'(x)}\right)^2 = \ell^2.$$

If we let $y = f(x)$, this last equation reduces to $y^2 + (y/y')^2 = \ell^2$. Solving for $y'$, we find

$$y' = -\frac{y}{\sqrt{\ell^2 - y^2}},$$

where we must choose the negative sign because $y$ is a decreasing function of $x$.

**(b)** Rewrite

$$\frac{dy}{dx} = -\frac{y}{\sqrt{\ell^2 - y^2}} \qquad \text{as} \qquad \frac{\sqrt{\ell^2 - y^2}}{y}\,dy = -dx,$$

and then integrate both sides to obtain

$$-x + C = \int \frac{\sqrt{\ell^2 - y^2}}{y}\,dy.$$

For the remaining integral, we use the trigonometric substitution $y = \ell\sin\theta$, $dy = \ell\cos\theta\,d\theta$. Then

$$\int \frac{\sqrt{\ell^2 - y^2}}{y}\,dy = \ell\int \frac{\cos^2\theta}{\sin\theta}\,d\theta = \ell\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = \ell\int(\csc\theta - \sin\theta)\,d\theta$$

$$= \ell\,[\ln|\csc\theta - \cot\theta| + \cos\theta] + C = \ell\ln\left(\frac{\ell}{y} - \frac{\sqrt{\ell^2 - y^2}}{y}\right) + \sqrt{\ell^2 - y^2} + C$$

Therefore,

$$x = -\ell\ln\left(\frac{\ell - \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2} + C = \ell\ln\left(\frac{y}{\ell - \sqrt{\ell^2 - y^2}}\right) - \sqrt{\ell^2 - y^2} + C$$

$$= \ell\ln\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2} + C$$

Now, when $x = 0$, $y = \ell$, so we find $C = 0$. Finally, the equation for the tractrix is

$$x = \ell\ln\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{y}\right) - \sqrt{\ell^2 - y^2}.$$

**59.** Show that the differential equations $y' = 3y/x$ and $y' = -x/3y$ define **orthogonal families** of curves; that is, the graphs of solutions to the first equation intersect the graphs of the solutions to the second equation in right angles (Figure 4). Find these curves explicitly.

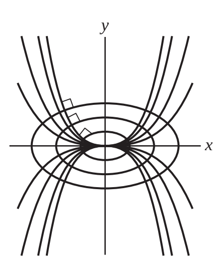



**FIGURE 4** Two orthogonal families of curves.

**SOLUTION** Let $y_1$ be a solution to $y' = \frac{3y}{x}$ and let $y_2$ be a solution to $y' = -\frac{x}{3y}$. Suppose these two curves intersect at a point $(x_0, y_0)$. The line tangent to the curve $y_1(x)$ at $(x_0, y_0)$ has a slope of $\frac{3y_0}{x_0}$ and the line tangent to the curve $y_2(x)$ has a slope of $-\frac{x_0}{3y_0}$. The slopes are negative reciprocals of one another; hence the tangent lines are perpendicular.

Separation of variables and integration applied to $y' = \frac{3y}{x}$ gives

$$\frac{dy}{y} = 3\frac{dx}{x}$$
$$\ln|y| = 3\ln|x| + C$$
$$y = Ax^3$$

On the other hand, separation of variables and integration applied to $y' = -\frac{x}{3y}$ gives

$$3y\,dy = -x\,dx$$
$$3y^2/2 = -x^2/2 + C$$
$$y = \pm\sqrt{C - x^2/3}$$

**60.** Find the family of curves satisfying $y' = x/y$ and sketch several members of the family. Then find the differential equation for the orthogonal family (see Exercise 59), find its general solution, and add some members of this orthogonal family to your plot.

**SOLUTION** Separation of variables and integration applied to $y' = x/y$ gives

$$y\,dy = x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$
$$y = \pm\sqrt{x^2 + C}$$

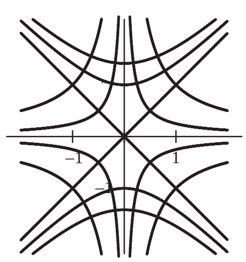
If $y(x)$ is a curve of the family orthogonal to these, it must have tangent lines of slope $-y/x$ at every point $(x, y)$. This gives

$$y' = -y/x$$

Separation of variables and integration give

$$\frac{dy}{y} = -\frac{dx}{x}$$
$$\ln|y| = -\ln|x| + C$$
$$y = \frac{A}{x}$$

Several solution curves of both differential equations appear below:




**61.** A 50-kg model rocket lifts off by expelling fuel downward at a rate of $k = 4.75$ kg/s for 10 s. The fuel leaves the end of the rocket with an exhaust velocity of $b = -100$ m/s. Let $m(t)$ be the mass of the rocket at time $t$. From the law of conservation of momentum, we find the following differential equation for the rocket's velocity $v(t)$ (in meters per second):

$$m(t)v'(t) = -9.8m(t) + b\frac{dm}{dt}$$

**(a)** Show that $m(t) = 50 - 4.75t$ kg.

**(b)** Solve for $v(t)$ and compute the rocket's velocity at rocket burnout (after 10 s).

**SOLUTION**

**(a)** For $0 \le t \le 10$, the rocket is expelling fuel at a constant rate of 4.75 kg/s, giving $m'(t) = -4.75$. Hence, $m(t) = -4.75t + C$. Initially, the rocket has a mass of 50 kg, so $C = 50$. Therefore, $m(t) = 50 - 4.75t$.

**(b)** With $m(t) = 50 - 4.75t$ and $\dfrac{dm}{dt} = -4.75$, the equation for $v$ becomes

$$\frac{dv}{dt} = -9.8 + \frac{b\frac{dm}{dt}}{50 - 4.75t} = -9.8 + \frac{(-100)(-4.75)}{50 - 4.75t}$$

and therefore

$$v(t) = -9.8t + 100\int \frac{4.75\,dt}{50 - 4.75t} = -9.8t - 100\ln(50 - 4.75t) + C$$

Because $v(0) = 0$, we find $C = 100\ln 50$ and

$$v(t) = -9.8t - 100\ln(50 - 4.75t) + 100\ln(50).$$

After 10 seconds the velocity is:

$$v(10) = -98 - 100\ln(2.5) + 100\ln(50) \approx 201.573 \text{ m/s}.$$

**62.** Let $v(t)$ be the velocity of an object of mass $m$ in free fall near the earth's surface. If we assume that air resistance is proportional to $v^2$, then $v$ satisfies the differential equation $m\frac{dv}{dt} = -g + kv^2$ for some constant $k > 0$.

**(a)** Set $\alpha = (g/k)^{1/2}$ and rewrite the differential equation as

$$\frac{dv}{dt} = -\frac{k}{m}(\alpha^2 - v^2)$$

Then solve using separation of variables with initial condition $v(0) = 0$.

**(b)** Show that the terminal velocity $\lim_{t\to\infty} v(t)$ is equal to $-\alpha$.

**SOLUTION**

**(a)** Let $\alpha = (g/k)^{1/2}$. Then

$$\frac{dv}{dt} = -\frac{g}{m} + \frac{k}{m}v^2 = -\frac{k}{m}\left(\frac{g}{k} - v^2\right) = -\frac{k}{m}\left(\alpha^2 - v^2\right)$$

Separating variables and integrating yields

$$\int \frac{dv}{\alpha^2 - v^2} = -\frac{k}{m}\int dt = -\frac{k}{m}t + C$$

We now use partial fraction decomposition for the remaining integral to obtain

$$\int \frac{dv}{\alpha^2 - v^2} = \frac{1}{2\alpha}\int \left(\frac{1}{\alpha + v} + \frac{1}{\alpha - v}\right) dv = \frac{1}{2\alpha}\ln\left|\frac{\alpha + v}{\alpha - v}\right|$$

Therefore,

$$\frac{1}{2\alpha}\ln\left|\frac{\alpha + v}{\alpha - v}\right| = -\frac{k}{m}t + C.$$

The initial condition $v(0) = 0$ allows us to determine the value of $C$:

$$\frac{1}{2\alpha}\ln\left|\frac{\alpha + 0}{\alpha - 0}\right| = -\frac{k}{m}(0) + C$$

$$C = \frac{1}{2\alpha}\ln 1 = 0.$$

Finally, solving for $v$, we find

$$v(t) = -\alpha\left(\frac{1 - e^{-2(\sqrt{gk}/m)t}}{1 + e^{-2(\sqrt{gk}/m)t}}\right).$$

**(b)** As $t \to \infty$, $e^{-2(\sqrt{gk}/m)t} \to 0$, so

$$v(t) \to -\alpha\left(\frac{1 - 0}{1 + 0}\right) = -\alpha.$$

**63.** If a bucket of water spins about a vertical axis with constant angular velocity $\omega$ (in radians per second), the water climbs up the side of the bucket until it reaches an equilibrium position (Figure 5). Two forces act on a particle located at a distance $x$ from the vertical axis: the gravitational force $-mg$ acting downward and the force of the bucket on the particle (transmitted indirectly through the liquid) in the direction perpendicular to the surface of the water. These two forces must combine to supply a centripetal force $m\omega^2 x$, and this occurs if the diagonal of the rectangle in Figure 5 is normal to the water's surface (that is, perpendicular to the tangent line). Prove that if $y = f(x)$ is the equation of the curve obtained by taking a vertical cross section through the axis, then $-1/y' = -g/(\omega^2 x)$. Show that $y = f(x)$ is a parabola.

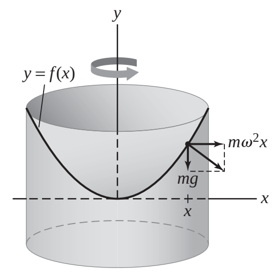



**FIGURE 5**

**SOLUTION** At any point along the surface of the water, the slope of the tangent line is given by the value of $y'$ at that point; hence, the slope of the line perpendicular to the surface of the water is given by $-1/y'$. The slope of the resultant force generated by the gravitational force and the centrifugal force is

$$\frac{-mg}{m\omega^2 x} = -\frac{g}{\omega^2 x}.$$

Therefore, the curve obtained by taking a vertical cross-section of the water surface is determined by the equation

$$-\frac{1}{y'} = -\frac{g}{\omega^2 x} \qquad \text{or} \qquad y' = \frac{\omega^2}{g}x.$$

Performing one integration yields

$$y = f(x) = \frac{\omega^2}{2g}x^2 + C,$$

where $C$ is a constant of integration. Thus, $y = f(x)$ is a parabola.

## Further Insights and Challenges

**64.** In Section 6.2, we computed the volume $V$ of a solid as the integral of cross-sectional area. Explain this formula in terms of differential equations. Let $V(y)$ be the volume of the solid up to height $y$, and let $A(y)$ be the cross-sectional area at height $y$ as in Figure 6.

**(a)** Explain the following approximation for small $\Delta y$:

$$V(y + \Delta y) - V(y) \approx A(y)\,\Delta y \tag{8}$$

**(b)** Use Eq. (8) to justify the differential equation $dV/dy = A(y)$. Then derive the formula

$$V = \int_a^b A(y)\,dy$$

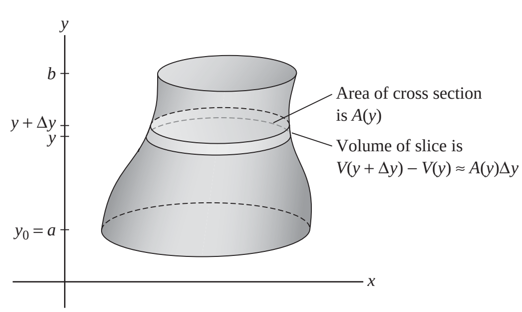



**FIGURE 6**

**SOLUTION**

**(a)** If $\Delta y$ is very small, then the slice between $y$ and $y + \Delta y$ is very similar to the *prism* formed by thickening the cross-sectional area $A(y)$ by a thickness of $\Delta y$. A prism with cross-sectional area $A$ and height $\Delta y$ has volume $A\Delta y$. This gives

$$V(y + \Delta y) - V(y) \approx A(y)\Delta y.$$

**(b)** Dividing Eq. (8) by $\Delta y$, we obtain

$$\frac{V(y + \Delta y) - V(y)}{\Delta y} \approx A(y).$$

In the limit as $\Delta y \to 0$, this becomes

$$\frac{dV}{dy} = A(y).$$

Integrating this last equation yields

$$V = \int_a^b A(y)\, dy.$$

**65.** A basic theorem states that a *linear* differential equation of order $n$ has a general solution that depends on $n$ arbitrary constants. There are, however, nonlinear exceptions.

**(a)** Show that $(y')^2 + y^2 = 0$ is a first-order equation with only one solution $y = 0$.

**(b)** Show that $(y')^2 + y^2 + 1 = 0$ is a first-order equation with no solutions.

**SOLUTION**

**(a)** $(y')^2 + y^2 \geq 0$ and equals zero if and only if $y' = 0$ and $y = 0$

**(b)** $(y')^2 + y^2 + 1 \geq 1 > 0$ for all $y'$ and $y$, so $(y')^2 + y^2 + 1 = 0$ has no solution

**66.** Show that $y = Ce^{rx}$ is a solution of $y'' + ay' + by = 0$ if and only if $r$ is a root of $P(r) = r^2 + ar + b$. Then verify directly that $y = C_1e^{3x} + C_2e^{-x}$ is a solution of $y'' - 2y' - 3y = 0$ for any constants $C_1, C_2$.

**SOLUTION** Let $y(x) = Ce^{rx}$. Then $y' = rCe^{rx}$, and $y'' = r^2Ce^{rx}$. Thus

$$y'' + ay' + by = r^2Ce^{rx} + arCe^{rx} + bCe^{rx} = Ce^{rx}\left(r^2 + ar + b\right) = Ce^{rx}P(r).$$

Hence, $Ce^{rx}$ is a solution of the differential equation $y'' + ay' + by = 0$ if and only if $P(r) = 0$. Now, let $y(x) = C_1e^{3x} + C_2e^{-x}$. Then

$$y'(x) = 3C_1e^{3x} - C_2e^{-x}$$
$$y''(x) = 9C_1e^{3x} + C_2e^{-x}$$

and

$$\begin{aligned} y'' - 2y' - 3y &= 9C_1e^{3x} + C_2e^{-x} - 6C_1e^{3x} + 2C_2e^{-x} - 3C_1e^{3x} - 3C_2e^{-x} \\ &= (9 - 6 - 3)\,C_1e^{3x} + (1 + 2 - 3)\,C_2e^{-x} = 0. \end{aligned}$$

**67.** A spherical tank of radius $R$ is half-filled with water. Suppose that water leaks through a hole in the bottom of area $B$. Let $y(t)$ be the water level at time $t$ (seconds).

**(a)** Show that $\dfrac{dy}{dt} = \dfrac{-\sqrt{2g}\,B\sqrt{y}}{\pi(2Ry - y^2)}$.

**(b)** Show that for some constant $C$,

$$\frac{2\pi}{15B\sqrt{2g}}\left(10Ry^{3/2} - 3y^{5/2}\right) = C - t$$

**(c)** Use the initial condition $y(0) = R$ to compute $C$, and show that $C = t_e$, the time at which the tank is empty.

**(d)** Show that $t_e$ is proportional to $R^{5/2}$ and inversely proportional to $B$.

**SOLUTION**

**(a)** At height $y$ above the bottom of the tank, the cross section is a circle of radius

$$r = \sqrt{R^2 - (R - y)^2} = \sqrt{2Ry - y^2}.$$

The cross-sectional area function is then $A(y) = \pi(2Ry - y^2)$. The differential equation for the height of the water in the tank is then

$$\frac{dy}{dt} = -\frac{\sqrt{2g}\,B\sqrt{y}}{\pi(2Ry - y^2)}$$

by Torricelli's law.

**(b)** Rewrite the differential equation as

$$\frac{\pi}{\sqrt{2g}B}\left(2Ry^{1/2} - y^{3/2}\right)\,dy = -\,dt,$$

and then integrate both sides to obtain

$$\frac{2\pi}{\sqrt{2g}B}\left(\frac{2}{3}Ry^{3/2} - \frac{1}{5}y^{5/2}\right) = C - t,$$

where $C$ is an arbitrary constant. Simplifying gives

$$\frac{2\pi}{15B\sqrt{2g}}(10Ry^{3/2} - 3y^{5/2}) = C - t \qquad (*)$$

**(c)** From Equation (*) we see that $y = 0$ when $t = C$. It follows that $C = t_e$, the time at which the tank is empty. Moreover, the initial condition $y(0) = R$ allows us to determine the value of $C$:

$$\frac{2\pi}{15B\sqrt{2g}}(10R^{5/2} - 3R^{5/2}) = \frac{14\pi}{15B\sqrt{2g}}R^{5/2} = C$$

**(d)** From part (c),

$$t_e = \frac{14\pi}{15\sqrt{2g}} \cdot \frac{R^{5/2}}{B},$$

from which it is clear that $t_e$ is proportional to $R^{5/2}$ and inversely proportional to $B$.

## 9.2 Models Involving $y' = k(y - b)$

### Preliminary Questions

**1.** Write down a solution to $y' = 4(y - 5)$ that tends to $-\infty$ as $t \to \infty$.

**SOLUTION** The general solution is $y(t) = 5 + Ce^{4t}$ for any constant $C$; thus the solution tends to $-\infty$ as $t \to \infty$ whenever $C < 0$. One specific example is $y(t) = 5 - e^{4t}$.

**2.** Does $y' = -4(y - 5)$ have a solution that tends to $\infty$ as $t \to \infty$?

**SOLUTION** The general solution is $y(t) = 5 + Ce^{-4t}$ for any constant $C$. As $t \to \infty$, $y(t) \to 5$. Thus, there is no solution of $y' = -4(y - 5)$ that tends to $\infty$ as $t \to \infty$.

**3.** True or false? If $k > 0$, then all solutions of $y' = -k(y - b)$ approach the same limit as $t \to \infty$.

**SOLUTION** True. The general solution of $y' = -k(y - b)$ is $y(t) = b + Ce^{-kt}$ for any constant $C$. If $k > 0$, then $y(t) \to b$ as $t \to \infty$.

**4.** As an object cools, its rate of cooling slows. Explain how this follows from Newton's Law of Cooling.

**SOLUTION** Newton's Law of Cooling states that $y' = -k(y - T_0)$ where $y(t)$ is the temperature and $T_0$ is the ambient temperature. Thus as $y(t)$ gets closer to $T_0$, $y'(t)$, the rate of cooling, gets smaller and the rate of cooling slows.

### Exercises

**1.** Find the general solution of $y' = 2(y - 10)$. Then find the two solutions satisfying $y(0) = 25$ and $y(0) = 5$, and sketch their graphs.

**SOLUTION** The general solution of $y' = 2(y - 10)$ is $y(t) = 10 + Ce^{2t}$ for any constant $C$. If $y(0) = 25$, then $10 + C = 25$, or $C = 15$; therefore, $y(t) = 10 + 15e^{2t}$. On the other hand, if $y(0) = 5$, then $10 + C = 5$, or $C = -5$; therefore, $y(t) = 10 - 5e^{2t}$. Graphs of these two functions are given below.