Calculus AB Take Home

#3 Solve for tangent to curve

$\frac{d}{dx}[y^3x + y^2x^2 = 6]$ at $(2,1)$

$3y^2\frac{dy}{dx}\cdot x + 1\cdot y^3 + 2y\frac{dy}{dx}\cdot x^2 + 2x\cdot y^2 = 0$

$3(1)^2\frac{dy}{dx}(2) + (1)(1^3) \cdot 2(1)(\frac{dy}{dx})\cdot(2)^2 + 2(2)1 = 0$

$6\,dy/dx + 1 + 8\frac{dy}{dx} + 4 = 0$

$\frac{14\,dy/dx}{14} = \frac{-5}{14}$ C

#6 If $f(x) = \sin^2(3-x)$ then $f'(0) =$

$u = \sin(3-x)$

$du/dx = -1\cos(3-x)$

$y = u^2$

$y' = 2u\frac{du}{dx}$

$f'(x) = 2(\sin(3-x))\cdot(-1\cos(3-x))$

$f'(0) = -2\sin(3-0)\cdot(\cos 3-0)$

$= -2\sin 3\cos 3$ B

7. $\frac{dy}{dx} = \frac{4x}{y}$

$y\,dy = 4x\,dx$

$\int y\,dy = \int 4x\,dx$

$\frac{y^2}{2} = \frac{4x^2}{2} + C$

GIVEN $y(2) = -2$

So $\frac{(-2)^2}{2} = \frac{4(2)^2}{2} + C$

$2 = 8 + C$

$-6 = C$

Now $2\left(\frac{y^2}{2}\right) = 2\left(\frac{4x^2}{2} - 6\right)$

$\sqrt{y^2} = \sqrt{4x^2 - 12}$

$y = \pm\sqrt{4x^2-12}$

Domain $4x^2 - 12 > 0$ $\frac{\sqrt{4x^2}}{4} > \frac{\sqrt{12}}{\sqrt{4}}$

$x^2 > 3 \Rightarrow x > 3$ or $x < -3$

chose $x > 3$ because given

chose − because given $y(2) = -2$

C $y = -\sqrt{4x^2-12}$, $x > \sqrt{3}$

9. $s(t) = 2t^3 - 24t^2 + 90t + 7$ for $t \geq 0$

So $s'(t) = v(t) = 6t^2 - 48t + 90 = 6(t^2 - 8t + 15) = 6(t-5)(t-3)$ $t = 5, 3$

$v'(t) = a(t) = 12t - 48 = 12(t-4) = 0$ $t = 4$

Now make sign line $v(t)$ $+ \ + \ 0 \ - \ - \ 0 \ + \ +$ $v(t)$ (0 1 2 3 4 5 6)

So speed is increasing when $v(t)$ & $a(t)$ is same sign at $3 < t < 4$ and $t > 5$ E.

$a(t)$ $- \ - \ - \ - \ 0 \ + \ + \ +$ (0 1 2 3 4 5 6)

11. $\lim_{x \to \infty} \frac{x^2 - 4}{2 + x - 4x^2} = \frac{\infty}{\infty}$ Can use L'Hopitals Rule $\lim_{x \to \infty} \frac{2x}{1 - 8x} = \lim_{x \to \infty} \frac{2}{-8}$

B $= -1/4$

OR if $x \to \infty$ can keep highest power in numerator and denominator and evaluate at $x \to \infty$ $\lim_{x \to \infty} \frac{\cancel{x^2}\,1}{-4\cancel{x^2}} =$ B. $-\frac{1}{4}$

13. If $y = 5 + \int_2^{2x} e^{-t^2}\,dt$

$\frac{dy}{dx} = 0 + \frac{d}{dx}\int_2^{2x} e^{-t^2}\,dt$

$u = 2x$ $du = 2dx$

So $\frac{dy}{dx} = \frac{d}{dx}\int_2^{u} e^{-t^2}\,dt = e^{-u^2} \cdot \frac{du}{dx} = 2e^{-4x^2}$

$y(1) = 5$ so $y = 5 + \int_2^{2x} e^{-t^2}\,dt$

$5 = 5 + \underbrace{\int_2^{2(1)} e^{-t^2}\,dt}$

$5 = 5 + 0$

E. $\frac{dy}{dx} = 2e^{-4x^2}$ and $y(1) = 5$

18.

x	1.1	1.2	1.3	1.4
f(x)	4.18	4.38	4.56	4.73

$f''(x) < 0$ is concave ↓ from $[1, 2]$

$f'(1.2)$ is slope of tangent line at $x = 1.2$

$\frac{\Delta y}{\Delta x} = \frac{4.38 - 4.18}{1.2 - 1.1} = \frac{.2}{.1} = 2$ Left

$\frac{4.56 - 4.18}{1.3 - 1.1} = \frac{.38}{.2} = 1.9$ Middle

$\frac{4.56 - 4.38}{1.3 - 1.2} = \frac{.18}{.1} = 1.8$ Right

D. $1.8 < f'(1.2) < 2.0$

19. particle 0 → 10

$x_1 = \sin t$

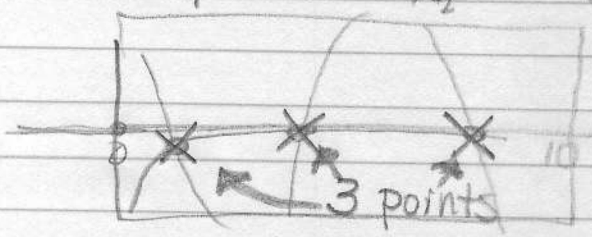
$x_2 = e^{-2t} - 1$

Same velocity when derivatives are =

$x_1' = \cos t \quad x_2' = -e^{-2t}$

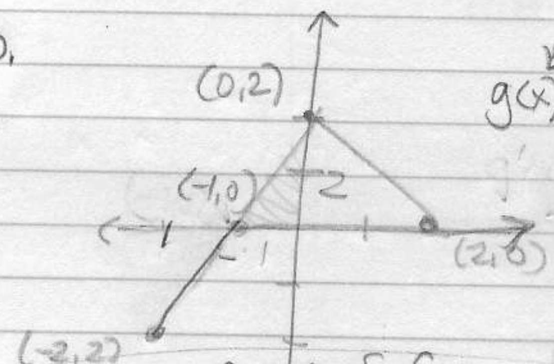
So $\cos t = -e^{-2t}$

Put $y_1 = \cos t\, x$, $y_2 = -e^{-2t}$ in calc + see how many times cross $0 \le t \le 10$



3 places D.

20.



graph of f

$g(x) = \int_0^x f(t)\,dt$ then $g(-1) =$

$g(-1) = \int_0^{-1} f(t)\,dt$

$= -\int_{-1}^{0} f(t)\,dt$

So area under $f(x) = g(x)$ $g(-1)$

$g(-1) = \int_0^{-1} f(t)\,dt = -\int_{-1}^{0} f(t)\,dt \quad A = \frac{1}{2}(1)(2)$

$= -1$ B

22. $f(5) = 3$ is point $(5,3)$

$f'(5) = 4$ is the slope of 4 at $x = 5$

Now use local linear approximation

$L(x) = f(a) + f'(a)(x - a) \quad a = 5$

$L(x) = f(5) + f'(5)(x - 5) \quad x = 4.8$

$= 3 + 4(x - 5)$

$= 3 + 4(4.8 - 5)$

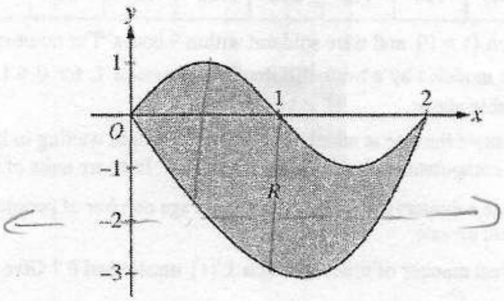
$= 3 + 4(-.2)$

A $= 2.2$

* ★ CALCULATOR

Calculus AB Sample Free-Response Questions

Question 1



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

$\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$

(a) $A = \int_0^2 (\sin\pi x - (x^3 - 4x))\,dx = -\frac{1}{\pi}\cos\pi x - \frac{x^4}{4} + 2x^2 \Big|_0^2$

$= \left[-\frac{1}{\pi}(1) - 4 + 8\right] - \left[-\frac{1}{\pi}(1) - 0 + 0\right]$

$= -\frac{1}{\pi} + 4 + \frac{1}{\pi} = 4$ units^3

(b) $\int_r^s (-2 - (x^3 - 4x))\,dy$

to find limits $x^3 - 4x = -2$
at $r = 0.5391889$
$s = 1.6751309$

(c) $\int_0^2 ((\sin\pi x - (x^3 - 4x))^2\,dx \approx 9.978$

(d) $\int_0^2 x(3-x)(\sin(\pi x) - (x^3 - 4x))\,dx$

$= 8.369$

or 8.370

CALC.

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Question 3

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

$\frac{dV}{dt} = 2000 \frac{cm^3}{min}$

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

a. When $r = 100$ cm and $h = .5$ cm, $\frac{dr}{dt} = 2.5$ cm/min and $\frac{dV}{dt} = 2000 \frac{cm^3}{min}$

$V = \pi r^2 h$

$\frac{dV}{dt} = \pi\left(r^2 \cdot \frac{dh}{dt} + 2r \cdot \frac{dr}{dt} \cdot h\right)$

$2000 = \pi\left(100^2 \cdot \frac{dh}{dt} + 2(100)(2.5)(.5)\right)$

$\frac{2000 - 250\pi}{10000\pi} = \frac{10000\pi \frac{dh}{dt} + 250\pi - 250\pi}{10000\pi}$

$\frac{dh}{dt} = 0.038$ or 0.039 cm/min

b. $\frac{dV}{dt} = 2000 - 400\sqrt{t}$ so $\frac{dV}{dt} = 0$ when $\frac{400\sqrt{t}}{400} = \frac{2000}{400}$

$\sqrt{t} = 5$ $t = 25$ min

$\frac{dV}{dt} > 0$ for $0 < t < 25$ and

$\frac{dV}{dt} < 0$ for $t > 25$ the max. Volume occurs when the recovery device has been working 25 min.

c. $\int_0^{25} (2000 - 400\sqrt{t})\,dt + 6000$

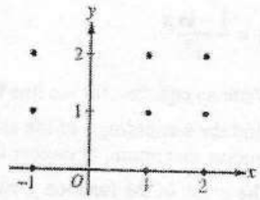
Sample Questions for **Calculus AB: Section II**

NO CALC.

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

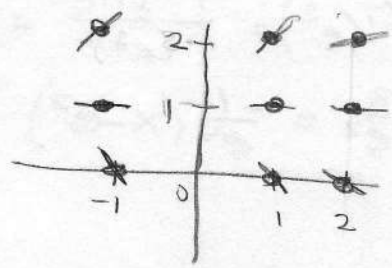
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x\to\infty} f(x)$.

a.



c. $\lim_{x\to\infty} 1 - e^{(1/2 - 1/x)} = 1 - e^{1/2}$

b. $\frac{dy}{dx} = \frac{y-1}{x^2}$

$\frac{x^2\,dy}{x^2(y-1)} = \frac{(y-1)\,dx}{(y-1)\,x^2}$

$\int \frac{dy}{(y-1)} = \int x^{-2}\,dx$

$\ln|y-1| = -x^{-1} + C$

$|y-1| = e^{-1/x + C}$

$|y-1| = e^{C} e^{-1/x}$

$y - 1 = Ke^{-1/x}$ where $K = \pm e^{C}$

Sub in $(2,0)$

$0 - 1 = Ke^{-1/2}$

$K = \frac{-1}{e^{-1/2}} = -e^{\frac{1}{2}}$

So, $y - 1 = -e^{1/2} \cdot e^{-1/x}$

$y = 1 - e^{(1/2 - 1/x)}$, $x > 0$

NO CALC

Question 6

Let f be the function given by $f(x) = \frac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by $f'(x) = \frac{1 - \ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of f at $x = e^2$.

(b) Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

(c) The graph of the function f has exactly one point of inflection. Find the x-coordinate of this point.

(d) Find $\lim_{x \to 0^+} f(x)$.

(a) $f(e^2) = \frac{\ln e^2}{e^2} = \frac{2}{e^2}$ $\quad f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2} = \frac{-1}{e^4}$

Equation tangent $\quad y - \frac{2}{e^2} = \frac{-1}{e^4}(x - e^2)$

(b) $0 = \frac{1 - \ln x}{x^2}$

$1 - \ln x = 0$

$1 = \ln x$

$x = e$

$\underset{e}{\overset{+ \quad -}{\longrightarrow}}\ f'(x)$

$f'(x)$ changes from neg to pos at $x = e$ ∴ its a relative max.

(c) $f'(x) = \frac{1 - \ln x}{x^2}$ $\quad f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)2x}{x^4}$

$= \frac{-x - 2x + 2x \ln x}{x^4} = \frac{-3 + 2\ln x}{x^3}$ for all $x > 0$

$f''(x) = 0$ when $-3 + 2\ln x = 0$

$\frac{2 \ln x}{2} = \frac{3}{2}$ $\quad x = e^{3/2}$

(d) $\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$ or Does Not exist