Answers

Practice for Quiz #3 Existence Theorems, Related Rates, Extrema and Inflection Points
AP Calculus

1) Find the absolute extrema of the function f(x) on the closed interval $[-2, 3]$

$$f(x) = -\frac{1}{4}x^4 + x^3 - 6$$

$f'(x) = -x^3 + 3x^2$

$0 = -x^2(x-3)$

$x = 0,\ x = 3$

x	$f(x)$
-2	-18
0	-6
3	$.75$

The absolute max is .75 at $x=3$
The absolute min is -18 at $x=-2$

2) Find all extrema and inflection points of the function g(x) on the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$g(x) = e^{-x}\cos x$ (Use a separate piece of paper!)

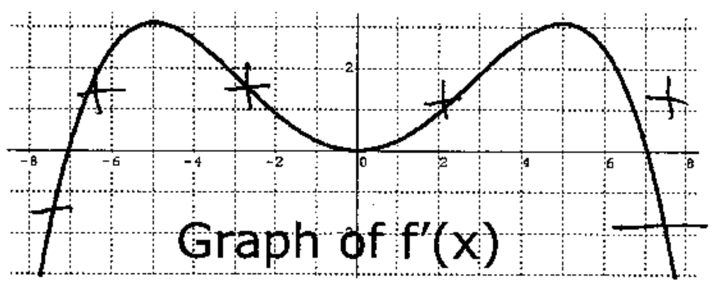
See last page

3) Use the graph of **f′(x)** shown to determine **at which x-values** the original function f(x) has...

Critical point(s) at x = $-7, 0, 7$

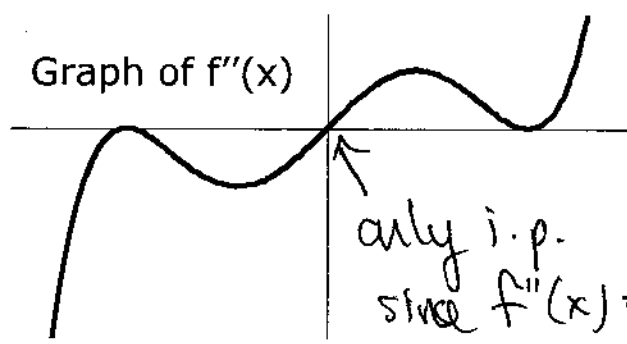
Relative maximum(s) at x = 7

Relative minimum(s) at x = -7



4) Given the graph of **f″(x) below**, how many inflection points does the original have?

(a) f(x) has no inflection points
(b) f(x) has 1 inflection point
(c) f(x) has 2 inflection points
(d) f(x) has 3 inflection points
(e) f(x) has 4 inflection points



only i.p. since $f''(x)=0$ & sign of $f''(x)$ changes

5) Given the **second** derivative of a function, $f''(x) = x^2(x^2-1)(x+3)^2$

determine the x-values of inflection points for the **original** function.

possible i.p → at $x = 0, \pm 1, -3$

$f''(x)$ + + − − +

−4 −3 −2 −1 $-\frac{1}{2}$ 0 $\frac{1}{2}$ 1 2

Inflection points at $x = -1$ and $x = 1$

6)

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

a) $C'(3.5) \approx \frac{f(4)-f(3)}{4-3} = \frac{12.8-11.2}{1} = 1.6$ ounces/min

b) $\frac{C(4)-C(2)}{4-2} = \frac{12.8-8.8}{2} = 2$ Since $C(t)$ is diff ∴ continuous, there must be a point where $C'(t)$ equals the average rate over an interval (MVT).

2) Let f be a function that is differentiable on the open interval (1, 10). If $f(1) = -6$, $f(5) = 2$, and $f(10) = -4$ which of the following MUST be true. (Choose all that apply.)

I. f has at least 2 zeros. ✓ IVT

II. The graph of f has at least one horizontal tangent. ✓ IVT with MVT

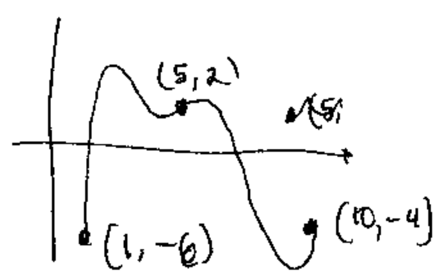
III. $f'(5) = 0$ Not necessarily

IV. For some c, $1 < c < 10$, $f(c) = 1$. ✓ IVT

V. $f(1)$ is the absolute minimum of the function in (1, 10) Not necessarily

VI. For some c, $1 < c < 10$, $f'(c) = \frac{2}{9}$. ✓ yes. MVT

$\frac{f(10)-f(1)}{10-1} = \frac{-4-(-6)}{9} = \frac{2}{9}$



8) Gravel is being dumped from a conveyor belt at a rate of 30 cubic feet per minute and its coarseness is such that it forms a pile in the shape of a cone whose height and diameter are approximately equal. How fast is the height of the pile increasing when the pile is 10 feet high? (Leave answer in exact form in terms of π.)

$h=2r$

$\frac{1}{2}h=r$

$V=\frac{1}{3}\pi r^2 h$

$V=\frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$V=\frac{1}{12}\pi h^3$

$\frac{dV}{dt}=\frac{3}{12}\pi h^2\frac{dh}{dt}$

$30=\frac{1}{4}\pi(10)^2\frac{dh}{dt}$

$30=25\pi\frac{dh}{dt}$

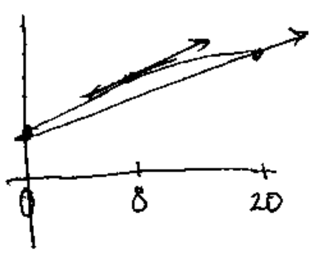
$\frac{30}{25\pi}=\frac{dh}{dt}$

Height is incr. by $\frac{6}{5\pi}$ ft/min when $h=10$ ft.

9) Set up an equation and solve it to find the c value(s) guaranteed by the Mean Value Theorem. Then use your calculator to draw a sketch of each function and sketch the secant and tangent lines that demonstrate the MVT. If the MVT is not applicable, state why. (Note: You won't have a calculator on the quiz, but use it here to check your work.)

b) $f(x)=\sqrt{9+2x}$ on [0, 20]

$f'(x)=\frac{f(0)-f(20)}{0-20}$



$\frac{1}{2}(9+2x)^{-1/2}(2)=\frac{\sqrt{9}-\sqrt{49}}{-20}$

$\frac{1}{\sqrt{9+2x}}=\frac{1}{5}$

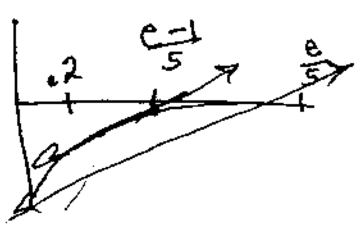
$9+2x=25$

$2x=16$

$\boxed{x=8}$

c) $f(x)=\ln(5x)$ on $[\frac{1}{5},\frac{e}{5}]$

$\frac{f\left(\frac{e}{5}\right)-f\left(\frac{1}{5}\right)}{\frac{e}{5}-\frac{1}{5}}=f'(x)$



$\frac{1-0}{\frac{e-1}{5}}=\frac{1}{5x}\cdot 5$

$\frac{1}{\frac{e-1}{5}}=\frac{1}{x}$

$\boxed{x=\frac{e-1}{5}}$

c) $f(x)=(x-4)^{\frac{2}{5}}+4$ on [3, 36]

f(x) is not differentiable at $x=4$ so MVT is not applicable.

2) . $f(x) = e^{-x}\cos x$, $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

SOLUTION Let $f(x) = e^{-x}\cos x$ on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

- Then, $f'(x) = -e^{-x}\sin x - e^{-x}\cos x = -e^{-x}(\sin x + \cos x) = 0$ gives $x = -\frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ as candidates for extrema.
- Moreover,

$$f''(x) = -e^{-x}(\cos x - \sin x) + e^{-x}(\sin x + \cos x) = 2e^{-x}\sin x = 0$$

gives $x = 0$ and $x = \pi$ as inflection point candidates.

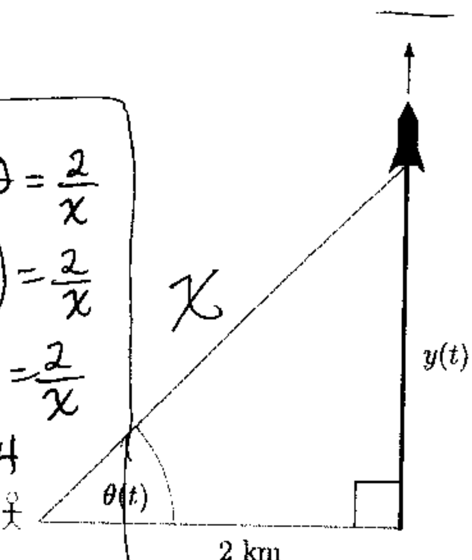
x	$\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$	$-\frac{\pi}{4}$	$\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$	$\frac{3\pi}{4}$	$\left(\frac{3\pi}{4}, \frac{3\pi}{2}\right)$
f'	+	0	−	0	+
f	↗	M	↘	m	↗

x	$\left(-\frac{\pi}{2}, 0\right)$	0	$(0, \pi)$	π	$\left(\pi, \frac{3\pi}{2}\right)$
f''	−	0	+	0	−
f	⌢	I	⌣	I	⌢

10) An observer watches a rocket launch from a distance of 2 kilometers. The angle of elevation θ is increasing at $\frac{\pi}{60}$ radians per second at the instant when $\theta = \frac{\pi}{4}$.

$$\frac{d\theta}{dt} = \frac{\pi}{60}$$

At what rate is the distance between the rocket and the observer increasing at that instant? (Leave answer in exact form in terms of π.)



$$\cos\theta = \frac{2}{x}$$
$$\cos\left(\frac{\pi}{4}\right) = \frac{2}{x}$$
$$\frac{\sqrt{2}}{2} = \frac{2}{x}$$
$$\sqrt{2}\,x = 4$$
$$x = \frac{4}{\sqrt{2}}$$

$$\cos\theta = \frac{2}{x} = 2x^{-1}$$

$$-\sin\theta \cdot \frac{d\theta}{dt} = -2x^{-2} \cdot \frac{dx}{dt}$$

$$\sin\theta \cdot \frac{d\theta}{dt} = \frac{2}{x^2} \cdot \frac{dx}{dt}$$

$$\sin\left(\frac{\pi}{4}\right)\left(\frac{\pi}{60}\right) = \frac{2}{\left(\frac{4}{\sqrt{2}}\right)^2}\frac{dx}{dt}$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\pi}{60} = \frac{2}{16}\frac{dx}{dt}$$

$$\frac{\sqrt{2}\pi}{120} = \frac{1}{4}\frac{dx}{dt}$$

$$\frac{4\sqrt{2}\pi}{120} = \frac{dx}{dt}$$

$$\boxed{\frac{dx}{dt} = \frac{\sqrt{2}\pi}{30}}$$