AP Calculus BC

ANSWER KEY

TECHNIQUES OF INTEGRATION & DIFFERENTIAL EQUATIONS

1.

a) $\frac{dP}{dt} = kP \cdot \left(1 - \frac{P}{10{,}000}\right)$

$P = \frac{10{,}000}{1 + Ae^{-kt}}$

$$\left.\begin{array}{l} t = 0 \Rightarrow P = 400 \Rightarrow A = 24 \\ t = 1 \Rightarrow P = 1{,}200 \Rightarrow k = \ln\left(\frac{36}{11}\right) \approx 1.186 \end{array}\right\} \Rightarrow P = \frac{10{,}000}{1 + 24e^{-1.186t}}$$

b) $P = 5{,}000$. The point of inflection in the graph of a logistic model occurs when the population reaches half the value of the carrying capacity.

c) $P = 5{,}000 \Rightarrow \frac{dP}{dt} = 1.186 \cdot (5{,}000) \cdot \left(1 - \frac{(5{,}000)}{10{,}000}\right) \approx 2964.059 \frac{\text{trout}}{\text{year}}$

d) $P = 5{,}000 \Rightarrow 5{,}000 = \frac{10{,}000}{1 + 24e^{-1.186t}} \Rightarrow t \approx 2.680$ years

e) $\lim_{t \to \infty} P(t) = 10{,}000$

f) $P = 8{,}000 \Rightarrow 8{,}000 = \frac{10{,}000}{1 + 24e^{-1.186t}} \Rightarrow t \approx 3.850$ years

g) $\lim_{t \to \infty} P(t) = 10{,}000$. The carrying capacity is still equal to $\lim_{t \to \infty} P(t)$. However, when the initial population is smaller than the carrying capacity, the population continually increases. When the initial population is larger than the carrying capacity, the population continually decreases.

h) $P = \frac{10{,}000}{1 + Ae^{-kt}}$

$$\left.\begin{array}{l} t = 0 \Rightarrow P = 12{,}000 \Rightarrow A = -\frac{1}{6} \\ k = \ln\left(\frac{36}{11}\right) \approx 1.186 \end{array}\right\} \Rightarrow P = \frac{10{,}000}{1 - \frac{1}{6}e^{-1.186t}} = \frac{60{,}000}{6 - e^{-1.186t}}$$

2.

a) $k = 1$; the carrying capacity is 2,100 (the value of A is $e^{4.3} \approx 73.700$.)

b) $P(0) \approx 28.112$. In the first day of the spread of the disease, about 28 students came to school with the flue. These students spread the disease to their school mates!

c) $P = 400 \Rightarrow 400 = \frac{2{,}100}{1 + e^{4.3-t}} \Rightarrow t \approx 2.853$ days

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3.

a) $\frac{dy}{dt} = \frac{t}{20} \cdot \left(1 - \frac{t}{5{,}000}\right) \Rightarrow$

Not a logistic model

b) $\frac{100}{P} \cdot \frac{dP}{dt} = \left(1 - \frac{P}{350}\right) \Rightarrow$

$\frac{dP}{dt} = \frac{P}{100} \cdot \left(1 - \frac{P}{350}\right) \Rightarrow$ Logistic model:

$k = \frac{1}{100}$ and $M = 350$

c) $\frac{dy}{dt} = 0.003 \cdot \left(y - 0.001y^2\right) \Rightarrow$

$\frac{dy}{dt} = 0.003y \cdot \left(1 - \frac{y}{1000}\right) \Rightarrow$

Logistic model:

$k = 0.003$ and $M = 1{,}000$

d) $\frac{dP}{dt} = \frac{P}{45}\left(3 - \frac{P}{1{,}000}\right) \Rightarrow$

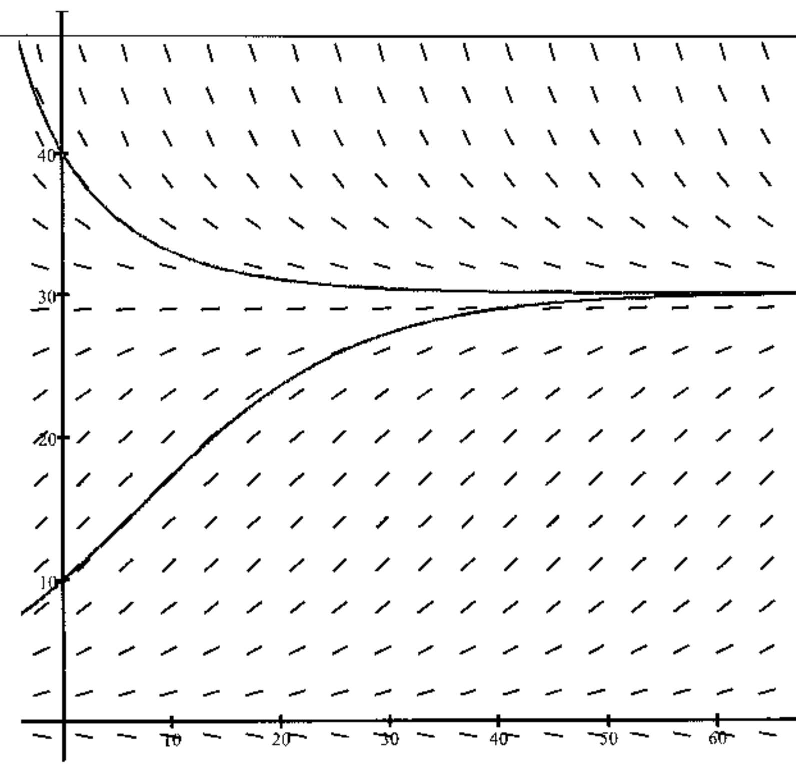
$\frac{dP}{dt} = \frac{3P}{45}\left(1 - \frac{P}{3{,}000}\right) = \frac{P}{15}\left(1 - \frac{P}{3{,}000}\right) \Rightarrow$

Logistic model:

$k = \frac{1}{15}$ and $M = 3{,}000$

4.

a) 30 wolves



b) and c) see above

d) From the graph, we can estimate that after 20 years there will be about 22 wolves.

$$P = \frac{30}{1 + Ae^{-kt}}$$

$$\left.\begin{aligned} &t = 0 \Rightarrow P = 10 \Rightarrow A = 2 \\ &t = 20 \Rightarrow P = 22 \Rightarrow k = \frac{1}{20}\ln\left(\frac{11}{2}\right) \approx 0.085 \end{aligned}\right\} \Rightarrow P = \frac{30}{1 + 2e^{-0.085t}}$$