Logistic Growth – Notes and Problems
BC Calculus

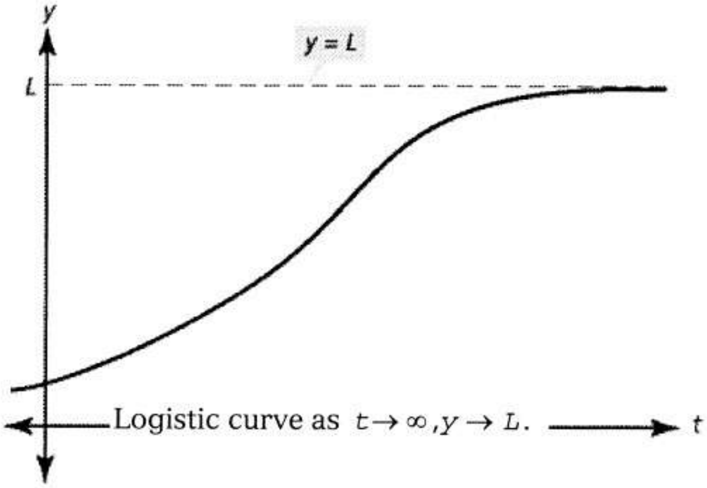
Exponential growth is unlimited. There are instances, however, when exponential growth can be used to model the first portion of a population cycle which levels off to a finite upper limit L. This maximum population L or $y(t)$ that can be sustained or supported as time t increases is called the carrying capacity.

A logistic differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$ is a model that is often used for this type of growth, where k and L are positive constants. If a population satisfies this equation, it approaches the carrying capacity, L, as t increases; it does not grow without bound.

If y is between 0 and L, then $\frac{dy}{dt} > 0$, and the population increases.

If $k > L$, then $\frac{dy}{dt} < 0$, and the population decreases.

After applying the separation of variables' techniques to the logistic differential equation and using partial fractions to integrate, the general solution is of the form $y = \frac{L}{1 + be^{-kt}}$, $b = \frac{L - y(0)}{y(0)}$ by letting $t = 0$ and solving for b.



Also, note that the maximum rate of growth occurs at $\frac{L}{2}$.

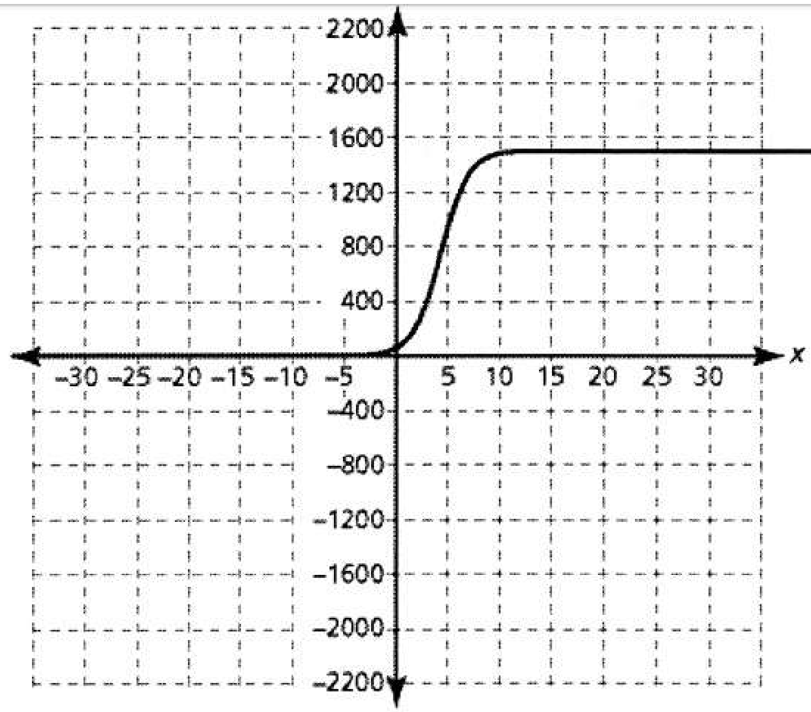
EXAMPLE 1: Try to interpret $P(t) = \frac{1500}{1 + 24e^{-0.75t}}$.

- L (carrying capacity) = 1500 units (this comes directly from the form $y = \frac{L}{1 + be^{-kt}}$).
- $k = 0.75$ (constant part of the exponent of e).
- Initial population is when $t = 0$ (t in years); therefore = $\frac{1500}{1 + 24e^{0}} = 1500/25 = 60$ units.

y

To determine when the population will reach 50% of its carrying capacity, let $P(t) = \frac{1500}{1+24e^{-0.75t}} = 750$ and solve for t. Therefore $2 = 1 + 24e^{-0.75t}$, or $1/24 = e^{-0.75t}$.

Taking the natural logarithm of both sides produces $\ln(1/24) = -0.75t$, and therefore $t \approx 4.24$. Thus, after about 4.24 years, the population is at one-half of its carrying capacity.



If you wanted to know how long it would take to get to 100% of its carrying capacity, you would set $P(t) = 1500$. However, this won't work because you would get $0 = e^{-0.75t}$. Therefore, let's take the limit as t approaches infinity to see what happens.

As $t \to \infty$ in $P(t) = \frac{1500}{1+24e^{-0.75t}}$, then $\lim_{t\to\infty} = \frac{1500}{1+24e^{-0.75t}} = 1500$ (the carrying capacity) because $e^{-0.75t}$ approaches 0.

And finally, solve for the logistic differential equation that has a solution of $P(t) = \frac{1500}{1+24e^{-0.75t}}$.

Start with the growth rate equation $\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$, where P is the population at a given time, k is the constant, and L is the carrying capacity.

Then substitute in the known values, and the solution is $\frac{dP}{dt} = 0.75P\left(1-\frac{P}{1500}\right)$.

EXAMPLE 2: Now let's start with the logistic differential equation and, given an initial condition, solve for the logistic equation.

$\frac{dy}{dt} = y\left(1-\frac{y}{40}\right)$, initial condition is (0, 8).

Therefore, at time $t = 0$, the population is $8y(0)$.

We know that $L = 40$ and $k = 1$ from the form of the equation $\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$. (*Note:* y is equivalent to P.) Solving for b in $y = \frac{L}{1+be^{-kt}}$, we know that $b = \frac{L-y(0)}{y(0)} = \frac{40-8}{8} = 4$.

Therefore, $y = \frac{40}{1+4e^{-t}}$ is the final solution, by substitution.

Try some problems!

1) A population of rabbits in a certain habitat grows according to the differential equation $\frac{dy}{dt} = y\left(1 - \frac{1}{10}y\right)$ where t is measured in months ($t \geq 0$) and y is measured in hundreds of rabbits. There were initially 100 rabbits in this habitat; that is, $y(0) = 1$.

*8. What is the fastest growth rate, in rabbits per month, that this population exhibits?
(A) 50
(B) 100
(C) 200
(D) 250
(E) 500

2) At what population, T, will the rate be greatest given that $T'(x) = 3T(1 - \frac{T}{4000})$ and $0 \leq T \leq holding\ capacity$

3) Find the limit of the function $P(t)$ as $t \to \infty$ if $P' = 7.2P(3200 - P)$.

4) Find the carrying capacity and initial population if the population fits the following model.

$$P(t) = \frac{108{,}000}{1 + 17e^{-0.3t}}$$

4b) At what time is the population growing the fastest?

Logistic problems are almost always multiple choice….Here's a rare part 2 logistic question.

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Question 5

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t \to \infty} Y(t)$?

Scoring Rubric:

(a) For this logistic differential equation, the carrying capacity is 12. If $P(0) = 3, \ \lim_{t\to\infty} P(t) = 12.$ If $P(0) = 20, \ \lim_{t\to\infty} P(t) = 12.$	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$
(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.	1 : answer
(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$ $\ln\lvert Y\rvert = \frac{t}{5} - \frac{t^2}{120} + C$ $Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$ $K = 3$ $Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$	$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables
(d) $\lim_{t\to\infty} Y(t) = 0$	1 : answer $0/1$ if Y is not exponential