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LET'S CUT THE CAKE

Although implemented with elementary school teachers to encourage them to consider area and perimeter simultaneously, this task also offers insight into how students might approach it.

Zulfiye Zeybek and Dionne I. Cross Francis

Measurement is an important component of K–grade 12 mathematics curricula. The concepts of area and perimeter of polygons are first introduced in third grade and serve as the basis for teaching in the upper grades. Researchers (e.g., Battista 2012; Outhred and Mitchelmore 2000) have observed that students struggle with understanding the concept of area because of weaker knowledge of area as a quantitative attribute. Results of the National Assessment of Educational Progress (NAEP) support the findings of these studies by revealing elementary school students' struggles with concepts related to area and perimeter (e.g., NCES 2007). Without a strong understanding of measurement, students will struggle to meaningfully grasp three-dimensional measurement concepts (e.g., volume) and will later make connections between area and more complex concepts, such as integration, which requires finding areas under curved surfaces.

Elementary school teachers explored a task that encouraged them to consider area and perimeter simultaneously. Studies (e.g., Reinke 1997; Menon 1998) show that teachers, similarly to students, also struggle to develop robust understandings of measurement. So, although we implemented this task with teachers, our discussion offers insight into how upper elementary school students might approach this task and gives suggestions on effective ways to implement the task.

Area and perimeter concepts in the standards

The concepts of area and perimeter in the Common Core State Standards for Mathematics (CCSSM) include three main components (CCSSI 2010): (1) recognizing area and perimeter as an attribute of plane figures (3.MD.C.5), (2) understanding the concepts of area and perimeter measurements by using concrete knowledge (3.MD. C. 6), and (3) connecting this concrete knowledge of finding area and perimeter with abstract knowledge from generating and using formulas and applying this knowledge to solve real-life problems (3.MD.C.7, 3.MD.D.8, 4.MD.A.3, 6.G.A.1).

The standards show that as students transition through grades 3–6, they are expected to move from an intuitive approach that emphasizes the covering of the surface of polygons without gaps or overlaps to a more formal approach that relies on relating the area to the linear dimensions of the figure. To achieve this transition, students must have experience with (a) relating the unit of measurement and the attribute being measured (e.g., using line segments to measure length while using unit squares to measure area); (b) iterating a fixed unit without any gaps or overlap; and (c) developing the idea of counting the units systematically.

Implementing a nonroutine, real-life task could help both teachers and students make connections between abstract knowledge of formulas and concrete knowledge of measuring area and perimeter as well as apply this knowledge to solve a real-life problem. First we describe the problem, then we discuss a range of solutions. These solutions are included to show common errors and misconceptions regarding concepts of area and perimeter and also ways that teachers meaningfully applied knowledge of these constructs. We suggest strategies, too, for effectively implementing the task with upper elementary school students.

The Cake problem

Twenty-three elementary school teachers of kindergarten through grade 6 completed the Cake problem as part of their summer professional development program. They engaged in an hour-long problem-solving session during which they worked on the task (see **fig. 1**). This task is an adaptation of the Figure This! task (NCTM 2004), available at **figurethis.nctm.org**. We provided teachers with graph paper, dot paper, and geoboards to work on this problem. We also suggest including unit cubes when engaging elementary school students. Using unit cubes to construct the cake might help students better visualize a three-dimensional

FIGURE 1

Using unit cubes to construct the cake could help students visualize a three-dimensional cake, understand the task, and allow flexibility in partitioning the cake into three pieces.
(Adapted from the Figure This! Cut the Cake problem.)

The Cake problem

Mateo invited his best friends to his birthday party. Mateo and his two friends want to share a 9-inch square chocolate cake with butter cream frosting. The cake is frosted evenly on the four sides and the top. How can Mateo cut the cake so that each person receives an equal share of both cake and frosting? How can you justify that each person got the equal amount of cake and the frosting?

Mateo's three other friends come to the party right before cake cutting. How can Mateo cut the cake evenly among six people now? How do you know it is fair?

When Mateo attempts to cut the cake, three more friends come along. How can Mateo cut the cake now? How can you justify that each person gets the same amount of cake and frosting?

Modifications to the task

Previous research shows evidence that young children first employ a halving mechanism and then algorithmic application of the doubling process to produce parts that are powers of two in the process of sharing (e.g. Lamon 1996). Thus, we suggest teachers who want to implement this task in their classrooms start with asking students to partition the cake into two and then powers of two pieces before moving to odd number of partitions.

cake and, as a result, better understand the task, and it might also provide flexibility in partitioning the cake into three pieces. We recommend that readers stop here, work on the task, then examine the teachers' solutions below.

Exploring responses to the Cake problem

Researchers state that the concept of unit is a central idea underlying the concept of measurement (e.g. Izsak 2005). The task gave teachers opportunities to not only develop their knowledge about the basic properties of units and the attribute measured with a particular unit but also make connections between their concrete knowledge of measurement and their abstract knowledge of the formula.

Connecting concrete knowledge of perimeter with the formula

In the excerpt below, one of the teachers, Naomi, tried to connect her knowledge of the formula for perimeter with the actual length units in her diagram.

Naomi: I have a problem here. The problem says I need to have thirty-six units on the perimeter, but when I counted it, I have only thirty-two units. I guess I did not draw it right!

Facilitator: OK, let's count it. What does thirty-six units represent?

Naomi: Thirty-six units represents the perimeter of the cake.

Facilitator: What is *perimeter*?

Naomi: Perimeter is the distance around the shape.

Facilitator: So, what would be each side measure if the perimeter is thirty-six units?

Naomi: Each side should be nine units.

Facilitator: Let's count whether you have nine units for each side.

Naomi: [*Counting the units*] Nine, but I am counting this [*referring to one of the corner squares*) square again, right?

Facilitator: Are you supposed to count them twice?

Naomi: [*Pausing*] OK, now I know. I only counted the squares one time, but I needed to count the corner squares twice.

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We provided teachers with graph paper, dotted paper, and geoboards to work on this problem. We also suggest including unit cubes when engaging elementary school students.

Facilitator: Why do you need to count corner squares twice?

Naomi: Because I need to count this side of the square for this side [*pointing to one side of the original square*], and the other side for this side [*pointing to another side of the original square*].

Naomi knew what perimeter meant as well as how to calculate the perimeter of a square if one side of the square is known or how to find out one side length if the perimeter of a square is known. However, Naomi's knowledge of how to calculate perimeter of a square formulaically was not connected to her concrete knowledge of measurement when counting units to determine the perimeter. Thus, to make the connection, Naomi needed to figure out what attribute was being measured and how to apply the appropriate unit of measure.

Research findings suggest that teachers and students often do not use appropriate units when computing area and perimeter (e.g., Baturo and Nason 1996). This challenge results from a lack of understanding about how to derive the units for each measurement, specifically area. Thus, we expect that students could experience struggles similar to those that Naomi experienced above if they were asked to make connections between their understandings of a particular measurement and how that is represented in the formula. We encourage teachers to engage students in calculating perimeter and area by counting the appropriate units, which results in making clear distinctions between the one-dimensional concept of perimeter by tracking the iteration of a length unit and the two-dimensional concept of area by tracking the iteration of a square unit (a composite unit derived from a length unit and a width unit). We also encourage teachers to probe students' thinking using focused questions similar to the facilitator's in the excerpt above, as this encourages students to think deeply about the concepts and develop answers to their own questions.

FIGURE 2

In this incorrect representation, the frosting was not shared fairly among three people; the frosting on the top of the cake was partitioned equally, but the frosting on the sides was not.

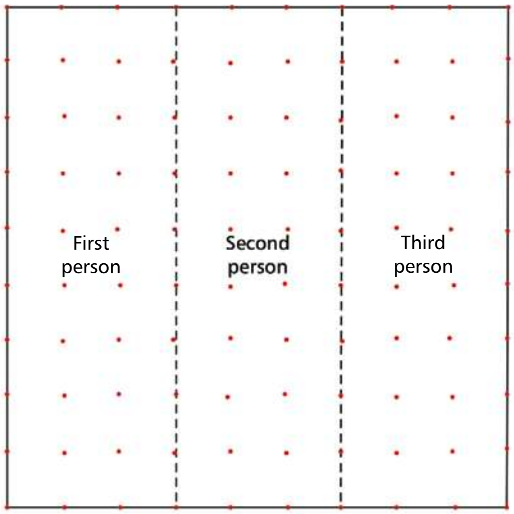
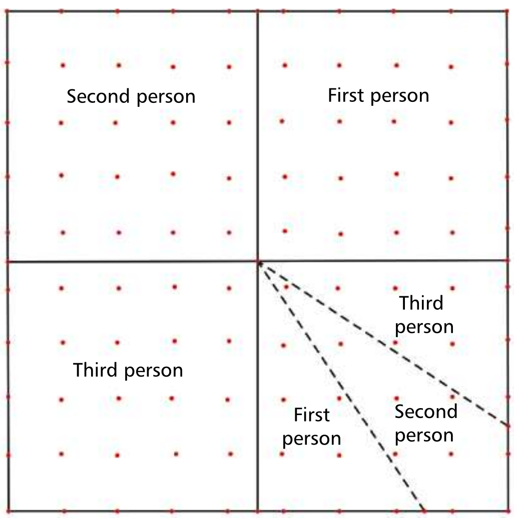


FIGURE 3

A real-world problem context helped teachers see why this computational representation with multiple pieces is a correct but impractical solution.



Making sense of the task: A common error

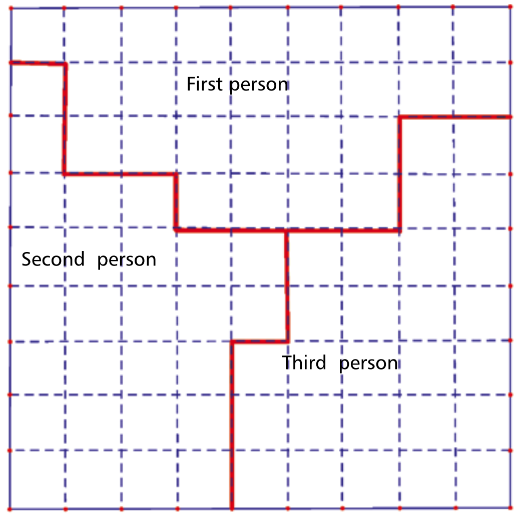
One of the more common responses that the teachers constructed was to equally partition the frosting on the top of the cake but not on the sides. This particular group partitioned the area (cake) into three equal-size pieces, each piece having an area of twenty-seven square units (see **fig. 2**). Given that the cake had the same density, partitioning the top surface area equally would ensure that each of three people received the same amount of cake. However, partitioning the cake in this way meant the frosting overall was not shared fairly among three people. The frosting on the top of the cake was partitioned equally, but the frosting on the sides was not. We found that having teachers share this solution first was helpful for all teachers to make sense of the task and for highlighting the importance of being critical about possible solutions. We expect that this solution might also arise if the task is implemented in an elementary school classroom. As it was helpful for teachers to share this solution to make sense of the task, we believe that it would be helpful for students to not only understand what the task is asking but to also revise prerequisite knowledge, such as the knowledge about perimeter and area, to engage in this task. Thus, we suggest that teachers encourage students to share and justify both correct and incorrect solutions. Although the task itself offers opportunities to think deeply about measurement, having meaningful conversations about the different strategies can greatly strengthen students' understanding.

Applying computational knowledge: Opportunities to draw on spatial reasoning

In the next solution we describe, teachers partitioned the cake into four squares initially, as they saw this as an easier task than partitioning the cake into three equal pieces (see **fig. 3**). Then they partitioned one of those pieces into three equal-size pieces. In this solution, the first person will receive one square piece (see **fig. 3, first person**) with the area of 20.25 square units and a small triangular piece with the area of 6.75 square units. These two pieces have sides of 12 units with frosting (the square piece has sides of 9 units with frosting; the triangular piece has sides of 3 units with frosting). The same idea applies to the pieces for the second and third person. The solution required teachers to mentally manipulate the area of the cake and consider alternative and creative ways of dividing the cake, given the conditions. However, although they applied spatial reasoning to identify possible partitions, teachers tended to resort to the formula and failed to consider other geometric approaches, for example, further partitioning the shape into square units to determine the area. As such, those who did not identify the portion for the second person as a kite shape struggled to prove that the three smaller pieces were of equal area because they were unable to quickly access a formula to find the area. In light of this, we suggest that students have opportunities to explore multiple ways of finding the area of a range of shapes (e.g., working on geoboard areas) before working on this task, so they have access to a range of approaches to finding the area of irregular shapes.

FIGURE 4

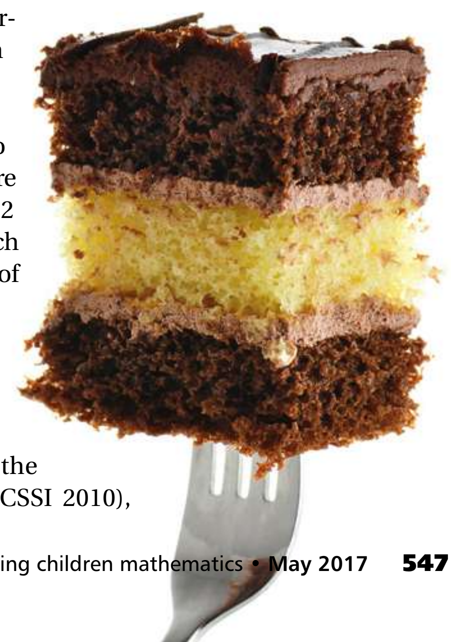
Although the pieces were not congruent in this practical representation, teachers could see each piece as a composite unit consisting of twenty-seven individual units.



Foregrounding practical solutions

Boaler (2008) argues the importance of using sensible, real-world problems to support students in making sense of problems as well as solutions. The real-world context of this problem helped teachers see why the solution in **figure 3** was correct but impractical. Teachers were encouraged to find a way to cut the cake into three pieces only, so that each person received one big piece instead of multiple pieces. Asking this question encouraged teachers to think deeply about the area and perimeter of the original cake and how it should be distributed among three people, thereby encouraging teachers to consider even more solutions.

Although the cake pieces were not congruent when partitioned into three pieces with an area of 27 square units and sides with frosting of 12 units, teachers were able to see each piece as a composite unit consisting of 27 individual units (see **fig. 4**).



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Summary

For teachers and students to use the concepts of area and perimeter to problem solve and build the core ideas outlined in CCSSM (CCSSI 2010),

they need opportunities to develop conceptual understanding of the concepts. In this article, we shared elementary school teachers' explorations of a nonroutine task, and we highlighted essential aspects of this task that teachers should foreground when implementing it with students in upper elementary grades.

The task was challenging and allowed teachers to make sense of the concepts, apply their prior knowledge of formulas to new situations, and acquire conceptual knowledge through discourse and interaction—all of which are essential characteristics of active learning as highlighted in NCTM's *Principles to Actions: Ensuring Mathematical Success for All* (2014). We believe that these features of the task will also promote students' mathematical reasoning and problem-solving skills as highlighted in *Principles and Standards for School Mathematics* (NCTM 2000).

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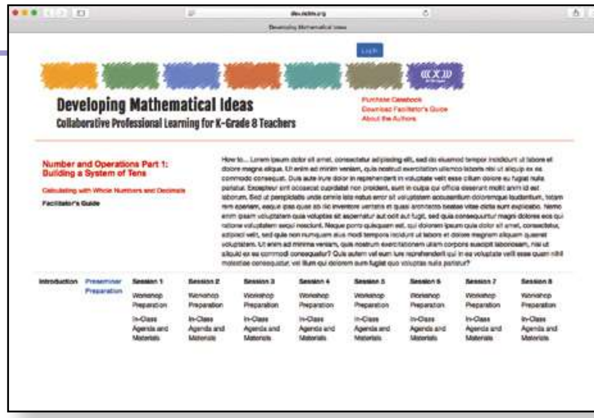

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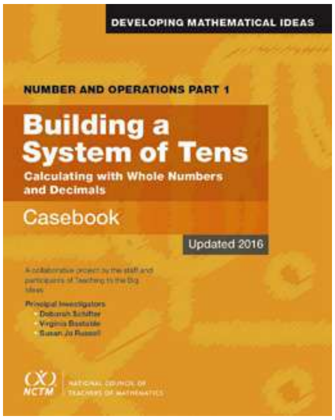

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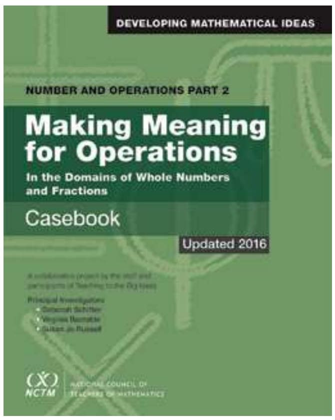

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