8-7 The Fundamental Theorem of Algebra

TEKS FOCUS

TEKS (7)(D) Determine the linear factors of a polynomial function of degree three and of degree four using algebraic methods.

TEKS (1)(C) Select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and **number sense** as appropriate, to solve problems.

Additional TEKS (7)(E)

VOCABULARY

- **Number sense** – the understanding of what numbers mean and how they are related

ESSENTIAL UNDERSTANDING

The degree of a polynomial equation tells you how many roots the equation has.

take note

Key Concept The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly n roots, including multiple and complex roots.

Here are equivalent ways to state the Fundamental Theorem of Algebra. You can use any one of these statements to prove the others.

- Every polynomial function of degree $n \geq 1$ has at least one complex zero.
- Every polynomial of degree $n \geq 1$ has n linear factors.

Problem 1

Using the Fundamental Theorem of Algebra

What are all the roots of $x^5 - x^4 - 3x^3 + 3x^2 - 4x + 4 = 0$?

Know
The polynomial equation has degree 5.
There are 5 roots.

Need
The zeros of the function

Plan
Use the Rational Root and Factor Theorems, synthetic division, and factoring.

Step 1 The polynomial is in standard form. The possible rational roots are $\pm 1, \pm 2, \pm 4$.

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Problem 1 *continued*

Step 2 Evaluate the related polynomial function for $x = 1$. Since $P(1) = 0$, 1 is a root and $x - 1$ is a factor. Use synthetic division to factor out $x - 1$:

1	1	−1	−3	3	−4	4
		1	0	−3	0	−4
	1	0	−3	0	−4	0

Think

How many linear factors will there be? If there are five roots, there must be five linear factors.

Step 3 Continue factoring until you have five linear factors.

$$x^5 - x^4 - 3x^3 + 3x^2 - 4x + 4 = (x - 1)(x^4 - 3x^2 - 4)$$
$$= (x - 1)(x^2 - 4)(x^2 + 1)$$
$$= (x - 1)(x - 2)(x + 2)(x - i)(x + i)$$

Step 4 The roots are 1, 2, −2, i, and $-i$.

By the Fundamental Theorem of Algebra, these are the only roots.

Problem 2

TEKS Process Standard (1)(C)

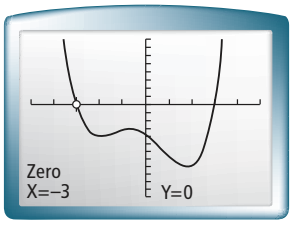
Finding All the Zeros of a Polynomial Function

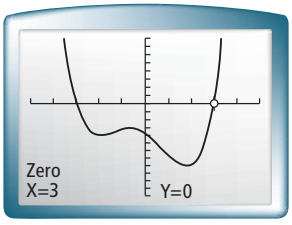
Find the factored form of $f(x) = x^4 + x^3 - 7x^2 - 9x - 18$. What are the zeros of $f(x)$?

Step 1 Use a graphing calculator to find any real roots. The graph of $y = x^4 + x^3 - 7x^2 - 9x - 18$ shows real zeros at $x = -3$ and $x = 3$.

Think

Does the graph show all of the real roots? Yes; the graphs of quartic functions have one or three turning points. Since the graph shows three turning points, it will not turn again to cross the x-axis a third time.





Step 2 Factor out the linear factors $x + 3$ and $x - 3$. Use synthetic division twice.

−3	1	1	−7	−9	−18
		−3	6	3	18
	1	−2	−1	−6	0

3	1	−2	−1	−6
		3	3	6
	1	1	2	0

$$x^4 + x^3 - 7x^2 - 9x - 18 = (x + 3)(x^3 - 2x^2 - x - 6)$$
$$= (x + 3)(x - 3)(x^2 + x + 2)$$

Linear factors: $(x + 3)$ and $(x - 3)$. Quadratic factor: $(x^2 + x + 2)$.

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Problem 2 *continued*

Step 3 Use the Quadratic Formula. Find the complex roots of $x^2 + x + 2 = 0$.

$a = 1, b = 1, c = 2$ Identify the values of *a*, *b*, and *c*.

$\frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$ Substitute.

$\frac{-1 \pm \sqrt{-7}}{2}$ Simplify.

The complex roots are $\frac{-1 + i\sqrt{7}}{2}$ and $\frac{-1 - i\sqrt{7}}{2}$.

Step 4 The four zeros of the function are $-3, 3, \frac{-1 + i\sqrt{7}}{2}$, and $\frac{-1 - i\sqrt{7}}{2}$.

By the Fundamental Theorem of Algebra, there can be no other zeros.

PRACTICE and APPLICATION EXERCISES

Scan page for a Virtual Nerd™ tutorial video.

For additional support when completing your homework, go to **PearsonTEXAS.com**.

Without using a calculator, find all the roots of each equation. Write the equation in factored form. Identify the linear factors of the polynomial.

1. $x^3 - 3x^2 + x - 3 = 0$

2. $x^3 + x^2 + 4x + 4 = 0$

3. $x^3 + 4x^2 + x - 6 = 0$

4. $x^3 - 5x^2 + 2x + 8 = 0$

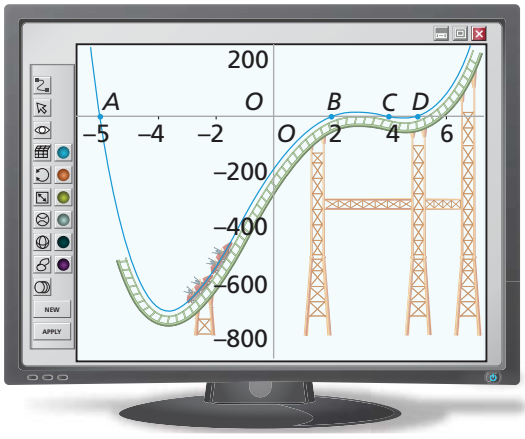
5. $x^4 + 4x^3 + 7x^2 + 16x + 12 = 0$

6. $x^4 - 4x^3 + x^2 + 12x - 12 = 0$

7. $x^5 + 3x^3 - 4x = 0$

8. $x^5 - 8x^3 - 9x = 0$

9. A polynomial function, $f(x) = x^4 - 5x^3 - 28x^2 + 188x - 240$ with one zero at 5, is used to model a new roller coaster section. The loading zone will be placed at one of the zeros. What are the possible locations for the loading zone?



10. Apply Mathematics (1)(A) A twist in a river can be modeled by the function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - x$, $-3 \le x \le 2$. A city wants to build a road that goes directly along the *x*-axis. How many bridges would it have to build?

For each equation, state the number of complex roots, the possible number of real roots, and the possible rational roots. Find the factors and identify the linear and quadratic factors of the polynomial.

11. $4x^4 + 8x^3 - 11x^2 + 2x - 3 = 0$

12. $3x^5 + 2x^4 + 15x^3 + 10x^2 + 12x + 8 = 0$

13. $2x^3 + 9x^2 + 3x - 4 = 0$

14. $x^6 + 5x^5 + 9x^4 + 45x^3 = 0$

15. Evaluate Reasonableness (1)(B) Maurice says: "Every linear function has exactly one zero. It follows from the Fundamental Theorem of Algebra." Cheryl disagrees. "What about the linear function $y = 2$?" she asks. "Its graph is a line, but it has no x-intercept." Whose reasoning is incorrect? Where is the flaw?

Find all the zeros of each function. Write the polynomial in factored form and identify the linear factors.

16. $y = x^3 - 4x^2 + 9x - 36$

17. $f(x) = x^3 + 2x^2 - 5x - 10$

18. $y = 2x^3 - 3x^2 - 18x - 8$

19. $y = 3x^3 - 7x^2 - 14x + 24$

20. $g(x) = x^3 - 4x^2 - x + 22$

21. $y = x^3 - x^2 - 3x - 9$

22. $y = x^4 - x^3 - 5x^2 - x - 6$

23. $y = 2x^4 + 3x^3 - 17x^2 - 27x - 9$

24. Explain Mathematical Ideas (1)(G) A 4th-degree polynomial function has zeros at 3 and $5 - i$. Can $4 + i$ also be a zero of the function? Explain.

25. Connect Mathematical Ideas (1)(F) Write a polynomial function that has four possible rational zeros but no actual rational zeros.

26. Justify Mathematical Arguments (1)(G) Show that the Fundamental Theorem of Algebra must be true for all quadratic polynomial functions.

27. Use the Fundamental Theorem of Algebra and the Conjugate Root Theorem to show that any odd degree polynomial equation with real coefficients has at least one real root.

28. Connect Mathematical Ideas (1)(F) What is the maximum number of points of intersection between the graphs of a quartic and a quintic polynomial function?

29. What is the least possible degree of a polynomial with rational coefficients, leading coefficient 1, constant term 5, and zeros at $\sqrt{2}$ and $\sqrt{3}$? Show that such a polynomial has a rational zero and indicate this zero.

TEXAS Test Practice

30. How many roots does $f(x) = x^4 + 5x^3 + 3x^2 + 2x + 6$ have?

A. 5 **B.** 4 **C.** 3 **D.** 2

31. Which translation takes $y = |x + 2| - 1$ to $y = |x| + 2$?

F. 2 units right, 3 units down **H.** 2 units left, 3 units up

G. 2 units right, 3 units up **J.** 2 units left, 3 units down

32. What is the factored form of the expression $x^4 - 3x^3 + 2x^2$?

A. $x^2(x - 1)(x + 2)$ **B.** $x^2(x + 1)(x + 2)$ **C.** $x^2(x + 1)(x - 2)$ **D.** $x^2(x - 1)(x - 2)$

33. How would you test whether $(2, -2)$ is a solution of the system? $\begin{cases} y < -2x + 3 \\ y \geq x - 4 \end{cases}$