Sample Problems

1. Compute each of the following limits.

a) $\lim\limits_{x\to\infty} 3x^4$ c) $\lim\limits_{x\to\infty} \left(-2x^5\right)$ e) $\lim\limits_{x\to\infty} \left(-\frac{2}{3}x^6\right)$ g) $\lim\limits_{x\to\infty} 4x^3$

b) $\lim\limits_{x\to-\infty} 3x^4$ d) $\lim\limits_{x\to-\infty} \left(-2x^5\right)$ f) $\lim\limits_{x\to-\infty} \left(-\frac{2}{3}x^6\right)$ h) $\lim\limits_{x\to-\infty} 4x^3$

2. Compute each of the following limits.

a) $\lim\limits_{x\to\infty} \frac{1}{x}$ d) $\lim\limits_{x\to-\infty} \left(\frac{-5}{2x^3} - 7 + \frac{8}{x}\right)$ g) $\lim\limits_{x\to\infty} \frac{-5x^3 - 2x + 4}{x^2}$

b) $\lim\limits_{x\to-\infty} \frac{1}{x}$ e) $\lim\limits_{x\to\infty} \left(-2x^3 + 1 - \frac{5}{x} + \frac{12}{x^4}\right)$ h) $\lim\limits_{x\to\infty} \frac{-5x^3 - 2x + 4}{x^3}$

c) $\lim\limits_{x\to\infty} \frac{-5}{2x^3}$ f) $\lim\limits_{x\to-\infty} \frac{3x - 2}{x}$ i) $\lim\limits_{x\to\infty} \frac{-5x^3 - 2x + 4}{x^4}$

3. Compute each of the following limits.

a) $\lim\limits_{x\to-\infty} \left(-2x^5 - 8x^4 + 7x^3 - 10\right)$ c) $\lim\limits_{x\to-\infty} \left(-2x^5 + 8x^6\right)$

b) $\lim\limits_{x\to\infty} \left(-2x^5 - 8x^4 + 7x^3 - 10\right)$ d) $\lim\limits_{x\to\infty} \left(-2x^5 + 8x^6\right)$

4. Compute each of the following limits.

a) $\lim\limits_{x\to-\infty} \frac{x + x^2 - 6}{6x + 5x^2 + 2x^3}$ b) $\lim\limits_{x\to\infty} \frac{x^2 + 9}{5x + 2x^2 - 3}$ c) $\lim\limits_{x\to-\infty} \frac{x^3 - 9x + 1}{3x^2 - 2x - 15}$

Practice Problems

1. Compute each of the following limits.

a) $\lim\limits_{x\to\infty} \left(-\frac{3}{8}x^{15}\right)$ c) $\lim\limits_{x\to\infty} \frac{1}{3}x^8$ e) $\lim\limits_{x\to\infty} 4x^9$ g) $\lim\limits_{x\to\infty} \left(-7x^{10}\right)$

b) $\lim\limits_{x\to-\infty} \left(-\frac{3}{8}x^{15}\right)$ d) $\lim\limits_{x\to-\infty} \frac{1}{3}x^8$ f) $\lim\limits_{x\to-\infty} 4x^9$ h) $\lim\limits_{x\to-\infty} \left(-7x^{10}\right)$

2. Compute each of the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{3}{x^5}$

b) $\displaystyle\lim_{x\to-\infty} \frac{3}{x^5}$

c) $\displaystyle\lim_{x\to\infty} \left(1 - \frac{2}{x} + \frac{5}{3x^4}\right)$

d) $\displaystyle\lim_{x\to-\infty} \left(1 - \frac{2}{x} + \frac{5}{3x^4}\right)$

e) $\displaystyle\lim_{x\to\infty} \left(3 + \frac{5}{x^3} - \frac{7}{6x}\right)$

f) $\displaystyle\lim_{x\to-\infty} \left(3 + \frac{5}{x^3} - \frac{7}{6x}\right)$

g) $\displaystyle\lim_{x\to\infty} \left(5x - \frac{2}{x+3}\right)$

h) $\displaystyle\lim_{x\to-\infty} \left(5x - \frac{2}{x+3}\right)$

i) $\displaystyle\lim_{x\to\infty} \frac{5x-3}{x}$

j) $\displaystyle\lim_{x\to-\infty} \frac{5x-3}{x}$

k) $\displaystyle\lim_{x\to\infty} \frac{1-3x}{2x}$

l) $\displaystyle\lim_{x\to-\infty} \frac{1-3x}{2x}$

m) $\displaystyle\lim_{x\to\infty} \frac{-3x^5+2x-5}{x^2}$

n) $\displaystyle\lim_{x\to-\infty} \frac{-3x^5+2x-5}{x^2}$

o) $\displaystyle\lim_{x\to\infty} \frac{-4x^8+x^3-x+7}{x^4}$

p) $\displaystyle\lim_{x\to-\infty} \frac{-4x^8+x^3-x+7}{x^4}$

3. Compute each of the following limits.

a) $\displaystyle\lim_{x\to-\infty} \left(-7x^5+x^3\right)$

b) $\displaystyle\lim_{x\to\infty} \left(-7x^5+x^3\right)$

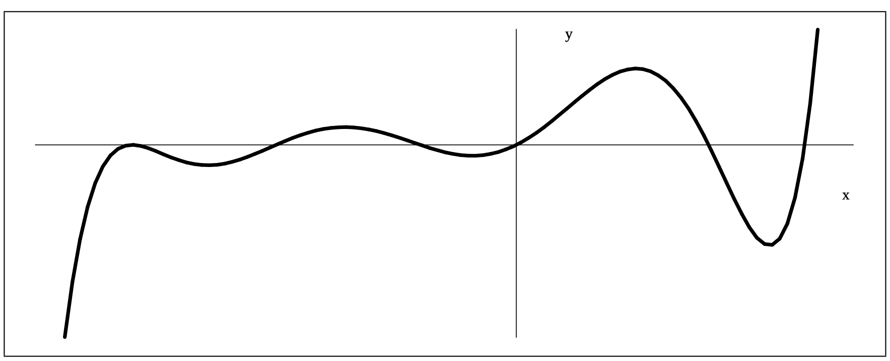
c) $\displaystyle\lim_{x\to-\infty} \left(120x^5 - \frac{1}{4}x^6\right)$

d) $\displaystyle\lim_{x\to\infty} \left(120x^5 - \frac{1}{4}x^6\right)$

e) $\displaystyle\lim_{x\to-\infty} \left(8x^4 - 3x^3 - \frac{1}{5}x + 2\right)$

f) $\displaystyle\lim_{x\to-\infty} \left(8x^4 - 3x^3 - \frac{1}{5}x + 2\right)$

4. The graph of a polynomial function is shown on the picture below. What can we state about this polynomial based on its end-behavior?



5. Compute each of the following limits.

a) $\displaystyle\lim_{x\to-\infty} \frac{1}{x}$

b) $\displaystyle\lim_{x\to-\infty} \frac{-5}{2x^3}$

c) $\displaystyle\lim_{x\to-\infty} \left(2 - \frac{5}{x^3}\right)$

d) $\displaystyle\lim_{x\to\infty} \left(3 - \frac{2}{x} + \frac{11}{x^4}\right)$

e) $\displaystyle\lim_{x\to\infty} \frac{2x^2+3x+1}{3x^2-5x+2}$

f) $\displaystyle\lim_{x\to-\infty} \frac{-3x^3+2x+1}{5x-3}$

g) $\displaystyle\lim_{x\to-\infty} \frac{3x^2-1}{5x^2-3x+2}$

h) $\displaystyle\lim_{x\to-\infty} \frac{20x-2x^2-42}{5x^3-20x^2-105x}$

Sample Problems - Answers

1. a) ∞ b) ∞ c) $-\infty$ d) ∞ e) $-\infty$ f) $-\infty$ g) ∞ h) $-\infty$

2. a) 0 b) 0 c) 0 d) -7 e) $-\infty$ f) 3 g) $-\infty$ h) -5 i) 0

3. a) ∞ b) $-\infty$ c) ∞ d) ∞

4. a) 0 b) $\frac{1}{2}$ c) $-\infty$

Practice Problems - Answers

1. a) $-\infty$ b) ∞ c) ∞ d) ∞ e) ∞ f) $-\infty$ g) $-\infty$ h) $-\infty$

2. a) 0 b) 0 c) 1 d) 1 e) 3 f) 3 g) ∞ h) $-\infty$ i) 5 j) 5 k) $-\frac{3}{2}$ l) $-\frac{3}{2}$

   m) $-\infty$ n) ∞ o) $-\infty$ p) $-\infty$

3. a) ∞ b) $-\infty$ c) $-\infty$ d) $-\infty$ e) ∞ f) ∞

4. Since $\lim\limits_{x\to-\infty} f(x) = -\infty$ and $\lim\limits_{x\to\infty} f(x) = \infty$, the polynomial is of odd degree and has a positive leading coefficient.

5. a) 0 b) 0 c) 2 d) 3 e) $\frac{2}{3}$ f) $-\infty$ g) $\frac{3}{5}$ h) 0

Sample Problems - Solutions

1. Compute each of the following limits.

a) $\lim\limits_{x\to\infty} 3x^4$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $3x^4$ is very large, and also positive because it is the product of five positive numbers.

$$3x^4 = \underset{\text{positive}}{3} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is ∞. We state the answer: $\lim\limits_{x\to\infty} 3x^4 = \infty$.

b) $\lim\limits_{x\to-\infty} 3x^4$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $3x^4$ is very large, and also positive because it is the product of one positive and four negative numbers.

$$3x^4 = \underset{\text{positive}}{3} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

So the answer is ∞. We state the answer: $\lim\limits_{x\to-\infty} 3x^4 = \infty$

c) $\lim\limits_{x\to\infty} (-2x^5)$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $-2x^5$ is very large, and also negative because it is the product of one negative and five positive numbers.

$$-2x^5 = \underset{\text{negative}}{-2} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is $-\infty$. We state the answer: $\lim\limits_{x\to\infty} (-2x^5) = -\infty$

d) $\lim\limits_{x\to-\infty} (-2x^5)$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $-2x^5$ is very large, and also positive because it is the product of six negative numbers.

$$-2x^5 = \underset{\text{negative}}{-2} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

So the answer is ∞. We state the answer: $\lim\limits_{x\to-\infty} (-2x^5) = \infty$

e) $\lim_{x\to\infty}\left(-\frac{2}{3}x^6\right)$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $-\frac{2}{3}x^6$ is very large, and also negative because it is the product of one negative and six positive numbers.

$$-\frac{2}{3}x^6 = \underbrace{-\frac{2}{3}}_{\text{negative}} \cdot \underbrace{x}_{\text{positive}} \cdot \underbrace{x}_{\text{positive}} \cdot \underbrace{x}_{\text{positive}} \cdot \underbrace{x}_{\text{positive}} \cdot \underbrace{x}_{\text{positive}} \cdot \underbrace{x}_{\text{positive}}$$

So the answer is $-\infty$. We state the answer: $\lim_{x\to\infty}\left(-\frac{2}{3}x^6\right) = -\infty$

f) $\lim_{x\to-\infty}\left(-\frac{2}{3}x^6\right)$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $-\frac{2}{3}x^6$ is very large, and also negative because it is the product of seven negative numbers.

$$-\frac{2}{3}x^6 = \underbrace{-\frac{2}{3}}_{\text{negative}} \cdot \underbrace{x}_{\text{negative}} \cdot \underbrace{x}_{\text{negative}} \cdot \underbrace{x}_{\text{negative}} \cdot \underbrace{x}_{\text{negative}} \cdot \underbrace{x}_{\text{negative}} \cdot \underbrace{x}_{\text{negative}}$$

So the answer is $-\infty$. We state the answer: $\lim_{x\to-\infty} -\frac{2}{3}x^6 = -\infty$

g) $\lim_{x\to\infty} 4x^3$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $4x^3$ is very large, and also positive because it is the product of four positive numbers.

$$4x^3 = \underbrace{4}_{\text{positive}} \cdot \underbrace{x}_{\text{positive}} \cdot \underbrace{x}_{\text{positive}} \cdot \underbrace{x}_{\text{positive}}$$

So the answer is ∞. We state the answer: $\lim_{x\to\infty} 4x^3 = \infty$

h) $\lim_{x\to-\infty} 4x^3$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $4x^3$ is very large, and also negative because it is the product of one positive and three negative numbers.

$$4x^3 = \underbrace{4}_{\text{positive}} \cdot \underbrace{x}_{\text{negative}} \cdot \underbrace{x}_{\text{negative}} \cdot \underbrace{x}_{\text{negative}}$$

So the answer is $-\infty$. We state the answer: $\lim_{x\to-\infty} 4x^3 = -\infty$

2. Compute each of the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{1}{x}$

Solution: This is a very important limit. Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. The reciprocal of a very large positive number is a very small positive number. This limit is 0.

b) $\displaystyle\lim_{x\to-\infty} \frac{1}{x}$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. The reciprocal of a very large negative number is a very small negative number. This limit is 0.

c) $\displaystyle\lim_{x\to\infty} \frac{-5}{2x^3}$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. We divide -5 by a very large positive number. This limit is 0.

d) $\displaystyle\lim_{x\to-\infty} \left(\frac{-5}{2x^3} - 7 + \frac{8}{x}\right)$

Solution: This limit is -7 since the other two terms aproach zero as x approaches negative infinity. Using mathematical notation,

$$\lim_{x\to-\infty} \frac{-5}{2x^3} - 7 + \frac{8}{x} = \lim_{x\to-\infty} \frac{-5}{2x^3} + \lim_{x\to-\infty} -7 + \lim_{x\to-\infty} \frac{8}{x} = 0 - 7 + 0 = -7$$

e) $\displaystyle\lim_{x\to\infty} \left(-2x^3 + 1 - \frac{5}{x} + \frac{12}{x^4}\right)$

Solution: This limit is $-\infty$ since the first term approaches negative infinity, the second term approaches 1 and the other two terms aproach zero as x approaches infinity. Using mathematical notation,

$$\lim_{x\to\infty} \left(-2x^3 + 1 - \frac{5}{x} + \frac{12}{x^4}\right) = \lim_{x\to\infty} \left(-2x^3\right) + \lim_{x\to\infty} 1 + \lim_{x\to\infty} \left(-\frac{5}{x}\right) + \lim_{x\to\infty} \left(\frac{12}{x^4}\right) = -\infty + 1 + 0 + 0 = -\infty$$

f) $\displaystyle\lim_{x\to-\infty} \frac{3x-2}{x}$

Solution: This problem is similar to the previous problems after a bit of algebra. We simply divide by x and then the limit becomes familiar.

$$\lim_{x\to-\infty} \frac{3x-2}{x} = \lim_{x\to-\infty} \left(\frac{3x}{x} - \frac{2}{x}\right) = \lim_{x\to-\infty} \left(3 - \frac{2}{x}\right) = 3$$

g) $\displaystyle\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^2}$

Solution:

$$\begin{aligned}\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^2} &= \lim_{x\to\infty}\left(\frac{-5x^3}{x^2}+\frac{-2x}{x^2}+\frac{4}{x^2}\right)=\lim_{x\to\infty}\left(-5x-\frac{2}{x}+\frac{4}{x^2}\right)\\ &= \lim_{x\to\infty}(-5x)+\lim_{x\to\infty}\left(-\frac{2}{x}\right)+\lim_{x\to\infty}\left(\frac{4}{x^2}\right)=-\infty+0+0=-\infty\end{aligned}$$

h) $\displaystyle\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^3}$

Solution:

$$\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^3}=\lim_{x\to\infty}\left(\frac{-5x^3}{x^3}+\frac{-2x}{x^3}+\frac{4}{x^3}\right)=\lim_{x\to\infty}\left(-5-\frac{2}{x^2}+\frac{4}{x^3}\right)=-5$$

i) $\displaystyle\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^4}$

Solution:

$$\lim_{x\to\infty}\frac{-5x^3-2x+4}{x^4}=\lim_{x\to\infty}\left(\frac{-5x^3}{x^4}+\frac{-2x}{x^4}+\frac{4}{x^4}\right)=\lim_{x\to\infty}\left(\frac{-5}{x}-\frac{2}{x^3}+\frac{4}{x^4}\right)=0$$

3. Compute each of the following limits.

a) $\displaystyle\lim_{x\to-\infty}\left(-2x^5-8x^4+7x^3-10\right)$

Solution: The first term, $-2x^5$ approaches infinity and the secomd term, $-8x^4$ approaches negative infinity. This does not give us enough information about the entire polynomial. A limit like this is called an **indeterminate**. We will bring this expression to a form that is not an indeterminate. In this case, factoring out the first term does the trick.

In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term. Recall that the leading term is the highest degree term.

$$\lim_{x\to-\infty}\left(-2x^5-8x^4+7x^3-10\right)=\lim_{x\to-\infty}\left(-2x^5\right)$$

Here is the computation showing why this is true. We first factor out the entire leading term.

$$\begin{aligned}\lim_{x\to-\infty}\left(-2x^5-8x^4+7x^3-10\right) &= \lim_{x\to-\infty}\left(-2x^5\right)\left(1+\frac{4}{x}-\frac{7}{x^2}+\frac{5}{x^5}\right)\\ &= \lim_{x\to-\infty}\left(-2x^5\right)\cdot\lim_{x\to-\infty}\left(1+\frac{4}{x}-\frac{7}{x^2}+\frac{5}{x^5}\right)\\ &= \lim_{x\to-\infty}\left(-2x^5\right)\cdot 1=\lim_{x\to-\infty}\left(-2x^5\right)\end{aligned}$$

We can now easily determine that this limit is ∞. (See problem number 1.)

b) $\lim_{x\to\infty}\left(-2x^5-8x^4+7x^3-10\right)$

Solution: **In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term.** Recall that the leading term is the highest degree term.

$$\lim_{x\to\infty}\left(-2x^5-8x^4+7x^3-10\right)=\lim_{x\to\infty}\left(-2x^5\right)$$

Here is the computation showing why this is true. We first factor out the entire leading term.

$$\begin{aligned}\lim_{x\to\infty}\left(-2x^5-8x^4+7x^3-10\right) &= \lim_{x\to\infty}\left(-2x^5\right)\left(1+\frac{4}{x}-\frac{7}{x^2}+\frac{5}{x^5}\right)\\ &= \lim_{x\to\infty}\left(-2x^5\right)\cdot\lim_{x\to\infty}\left(1+\frac{4}{x}-\frac{7}{x^2}+\frac{5}{x^5}\right)\\ &= \lim_{x\to\infty}\left(-2x^5\right)\cdot 1=\lim_{x\to\infty}\left(-2x^5\right)\end{aligned}$$

We can now easily determine that this limit is $-\infty$. (See problem number 1.)

c) $\lim_{x\to-\infty}\left(-2x^5+8x^6\right)$

Solution: **In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term.**

$$\begin{aligned}\lim_{x\to-\infty}\left(-2x^5+8x^6\right) &= \lim_{x\to-\infty}8x^6=\infty \quad \text{because}\\ \lim_{x\to-\infty}\left(-2x^5+8x^6\right) &= \lim_{x\to-\infty}\left(8x^6-2x^5\right)=\lim_{x\to-\infty}\left(8x^6\right)\left(1-\frac{1}{4x}\right)=\lim_{x\to-\infty}\left(8x^6\right)\cdot\lim_{x\to-\infty}\left(1-\frac{1}{4x}\right)\\ &= \lim_{x\to-\infty}\left(8x^6\right)\cdot 1=\lim_{x\to-\infty}8x^6\end{aligned}$$

We can now easily determine that this limit is ∞. (See problem number 1.)

d) $\lim_{x\to\infty}\left(-2x^5+8x^6\right)$

Solution: **In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term.**

$$\begin{aligned}\lim_{x\to\infty}\left(-2x^5+8x^6\right) &= \lim_{x\to\infty}8x^6=\infty \quad \text{because}\\ \lim_{x\to\infty}\left(-2x^5+8x^6\right) &= \lim_{x\to\infty}\left(8x^6-2x^5\right)=\lim_{x\to\infty}\left(8x^6\right)\left(1-\frac{1}{4x}\right)=\lim_{x\to\infty}\left(8x^6\right)\cdot\lim_{x\to\infty}\left(1-\frac{1}{4x}\right)\\ &= \lim_{x\to\infty}\left(8x^6\right)\cdot 1=\infty\end{aligned}$$

We can now easily determine that this limit is ∞. (See problem number 1.)

4. Compute each of the following limits.

a) $\displaystyle\lim_{x\to-\infty}\frac{x+x^2-6}{6x+5x^2+2x^3}=0$

Solution: The numerator approaches infinity and the denominator approaches negative infinity. This does not give us enough information about the quotient. A limit like this is called an **indeterminate**. We will bring this expression to a form that is not an indeterminate. Let us rearrange the polynomials in the rational function given. Then we will factor out the leading term in the numerator and denominator.

$$\lim_{x\to-\infty}\frac{x^2+x-6}{2x^3+5x^2+6x}=\lim_{x\to-\infty}\frac{x^2\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{2x^3\left(1+5+\dfrac{6}{x}\right)}$$

We now express the limit of the product as the product of two limits

$$\lim_{x\to-\infty}\frac{x^2\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{2x^3\left(1+5+\dfrac{6}{x}\right)}=\lim_{x\to-\infty}\frac{x^2}{2x^3}\cdot\lim_{x\to-\infty}\frac{\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{\left(1+5+\dfrac{6}{x}\right)}$$

The first expression can be simplified and thus has a limit we can easily determine its limit. The second expression, although looks unfriendly, is always going to approach 1.

$$\lim_{x\to-\infty}\frac{x^2}{2x^3}\cdot\lim_{x\to-\infty}\frac{\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{\left(1+5+\dfrac{6}{x}\right)}=\lim_{x\to-\infty}\frac{1}{2x}\cdot 1=0\cdot 1=0$$

The entire computation should look like this:

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^2+x-6}{2x^3+5x^2+6x} &= \lim_{x\to-\infty}\frac{x^2\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{2x^3\left(1+5+\dfrac{6}{x}\right)}=\lim_{x\to-\infty}\frac{x^2}{2x^3}\cdot\lim_{x\to-\infty}\frac{\left(1+\dfrac{1}{x}-\dfrac{6}{x^2}\right)}{\left(1+5+\dfrac{6}{x}\right)}\\ &= \lim_{x\to-\infty}\frac{1}{2x}\cdot 1=0\cdot 1=0\end{aligned}$$

b) $\displaystyle\lim_{x\to\infty}\frac{x^2+9}{5x+2x^2-3}=\frac{1}{2}$

Solution: Both numerator and denominator approach infinity. This does not give us enough information about the quotient. A limit like this is called an **indeterminate**. We will bring this expression to a form that is not an indeterminate. Let us rearrange the polynomials in the rational function given. Then we will factor out the leading term in the numerator and denominator.

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2+9}{2x^2+5x-3} &= \lim_{x\to\infty}\frac{x^2\left(1+\dfrac{9}{x^2}\right)}{2x^2\left(1+\dfrac{5}{2x}-\dfrac{3}{2x^2}\right)}=\lim_{x\to\infty}\frac{x^2}{2x^2}\cdot\lim_{x\to\infty}\frac{1+\dfrac{9}{x^2}}{1+\dfrac{5}{2x}-\dfrac{3}{2x^2}}\\ &= \lim_{x\to\infty}\frac{1}{2}\cdot\lim_{x\to\infty}\frac{1+\dfrac{9}{x^2}}{1+\dfrac{5}{2x}-\dfrac{3}{2x^2}}=\frac{1}{2}\cdot 1=\frac{1}{2}\end{aligned}$$

c) $\displaystyle\lim_{x\to-\infty}\frac{x^3-9x+1}{3x^2-2x-15}=-\infty$

Solution:

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^3-9x+1}{3x^2-2x-15} &= \lim_{x\to-\infty}\frac{x^3\left(1-\dfrac{9}{x^2}+\dfrac{1}{x^3}\right)}{3x^2\left(1-\dfrac{2}{3x}-\dfrac{5}{x^2}\right)}=\lim_{x\to-\infty}\frac{x^3}{3x^2}\cdot\lim_{x\to-\infty}\frac{1-\dfrac{9}{x^2}+\dfrac{1}{x^3}}{\left(1-\dfrac{2}{3x}-\dfrac{5}{x^2}\right)}\\ &= \left(\lim_{x\to-\infty}\frac{x^3}{3x^2}\right)\cdot 1=\left(\lim_{x\to-\infty}\frac{x}{3}\right)\cdot 1=-\infty\cdot 1=-\infty\end{aligned}$$

Sample Problems

Compute each of the following limits. Show all steps, using correct notation.

1.) $\lim\limits_{x\to\infty} 2^x$

2.) $\lim\limits_{x\to-\infty} 2^x$

3.) $\lim\limits_{x\to\infty} \left(\frac{2}{3}\right)^x$

4.) $\lim\limits_{x\to-\infty} \left(\frac{2}{3}\right)^x$

5.) $\lim\limits_{x\to\infty} \frac{2^{x+3}}{3^{x+1}}$

6.) $\lim\limits_{x\to-\infty} \frac{2^{x+3}}{3^{x+1}}$

7.) $\lim\limits_{x\to\infty} \frac{2^{2x+1}}{3^{x-1}}$

8.) $\lim\limits_{x\to-\infty} \frac{2^{2x+1}}{3^{x-1}}$

9.) $\lim\limits_{x\to\infty} \left(3^{x+1}-3^x\right)$

10.) $\lim\limits_{x\to\infty} \frac{3\sqrt{x}+2}{5\sqrt{x}+1}$

11.) $\lim\limits_{x\to\infty} \left(\sqrt{2x-1}-\sqrt{2x}\right)$

12.) $\lim\limits_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

13.) $\lim\limits_{x\to\infty} x\left(\frac{1}{5}-\frac{1}{5-\frac{1}{x}}\right)$

14.) $\lim\limits_{x\to\infty} \frac{\cos x}{x}$

15.) $\lim\limits_{x\to\infty} \frac{x+\frac{1}{x}}{x-\frac{1}{x}}$

16.) $\lim\limits_{x\to\infty} \frac{2^x+2^{-x}}{2^x-2^{-x}}$

17.) $\lim\limits_{x\to-\infty} \left(\sqrt{3x-1}-\sqrt{3x+1}\right)$

Practice Problems

Compute each of the following limits. Show all steps, using correct notation.

1.) $\lim\limits_{x\to\infty} \frac{2^{3x-1}}{5^{x-1}}$

2.) $\lim\limits_{x\to-\infty} \frac{2^{3x-1}}{5^{x-1}}$

3.) $\lim\limits_{x\to\infty} \frac{2^{2x+3}}{5^{x-1}}$

4.) $\lim\limits_{x\to-\infty} \frac{2^{2x+3}}{5^{x-1}}$

5.) $\lim\limits_{x\to\infty} \frac{2^{2x+3}}{4^{x-1}}$

6.) $\lim\limits_{x\to-\infty} \frac{2^{2x+3}}{4^{x-1}}$

7.) $\lim\limits_{x\to\infty} \frac{2^{x+3}\cdot 3^{x-1}}{7^{x-2}}$

8.) $\lim\limits_{x\to\infty} \frac{2^{2x+3}\cdot 3^{x-1}}{7^{x-2}}$

9.) $\lim\limits_{x\to\infty} \left(\sqrt{x+1}-\sqrt{x}\right)$

10.) $\lim\limits_{x\to\infty} \left(\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{x}}\right)$

11.) $\lim\limits_{x\to\infty} \frac{x}{\sqrt{x-1}+\sqrt{x+1}}$

12.) $\lim\limits_{x\to-\infty} \frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

13.) $\lim\limits_{x\to\infty} x\left(\frac{1}{a}-\frac{1}{a-\frac{1}{x}}\right)$

14.) $\lim\limits_{x\to\infty} \frac{0.5^x+0.5^{-x}}{0.5^x-0.5^{-x}}$

15.) $\lim\limits_{x\to\infty} \frac{\sin x-\cos x}{\sqrt{x^2+1}}$

Sample Problems - Answers

1.) ∞ 2.) 0 3.) 0 4.) ∞ 5.) 0 6.) ∞ 7.) ∞ 8.) 0 9.) ∞

10.) $\dfrac{3}{5}$ 11.) 0 12.) $\dfrac{1}{1-\sqrt{2}} = -1-\sqrt{2}$ 13.) $-\dfrac{1}{25}$ 14.) 0 15.) 1 16.) 1

17.) undefined

Practice Problems - Answers

1.) ∞ 2.) 0 3.) 0 4.) ∞ 5.) 32 6.) 32 7.) 0 8.) ∞ 9.) 0

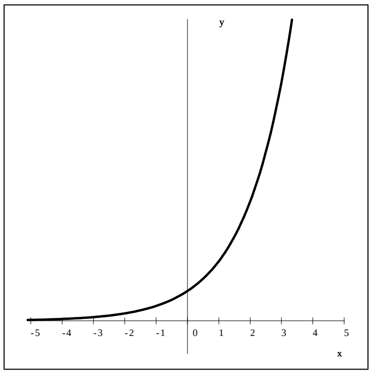
10.) 0 11.) ∞ 12.) undefined 13.) $-\dfrac{1}{a^2}$ 14.) -1 15.) 0

Sample Problems - Solutions

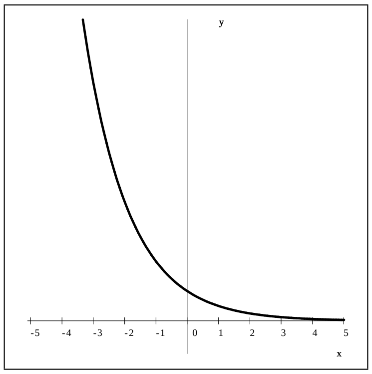
Let $a > 0$. Then the limit of the exponential function $f(x) = a^x$ is as follows.

Case 1. If $a > 1$, then $\lim\limits_{x\to\infty} a^x = \infty$ and $\lim\limits_{x\to-\infty} a^x = 0$

Case 2. If $0 < a < 1$, then $\lim\limits_{x\to\infty} a^x = 0$ and $\lim\limits_{x\to-\infty} a^x = \infty$



$a > 1$



$0 < a < 1$

1.) $\lim\limits_{x\to\infty} 2^x$ and 2.) $\lim\limits_{x\to-\infty} 2^x$

Solution: Since $2 > 1$, these limits are ∞ and 0, i.e. $\lim\limits_{x\to\infty} 2^x = \infty$ and $\lim\limits_{x\to-\infty} 2^x = 0$.

3.) $\lim_{x\to\infty}\left(\frac{2}{3}\right)^x$ and 4.) $\lim_{x\to-\infty}\left(\frac{2}{3}\right)^x$

Solution: Since $\frac{2}{3}<1$, these limits are 0 and ∞, i.e. $\lim_{x\to\infty}\left(\frac{2}{3}\right)^x=0$ and $\lim_{x\to-\infty}\left(\frac{2}{3}\right)^x=\infty$.

5.) $\lim_{x\to\infty}\frac{2^{x+3}}{3^{x+1}}$

Solution: We start by re-writing the exponential expressions. The goal is to bring it into a form where there is only one exponential expression involving x.

$$\frac{2^{x+3}}{3^{x+1}}=\frac{2^x\cdot 2^3}{3^x\cdot 3^1}=\frac{2^x\cdot 8}{3^x\cdot 3}=\frac{8}{3}\left(\frac{2}{3}\right)^x$$

$$\text{Thus}\quad \lim_{x\to\infty}\frac{2^{x+3}}{3^{x+1}}=\lim_{x\to\infty}\frac{8}{3}\left(\frac{2}{3}\right)^x=\frac{8}{3}\lim_{x\to\infty}\left(\frac{2}{3}\right)^x=0\qquad\text{since }\frac{2}{3}<1$$

6.) $\lim_{x\to-\infty}\frac{2^{x+3}}{3^{x+1}}$

Solution: $\lim_{x\to-\infty}\frac{2^{x+3}}{3^{x+1}}=\lim_{x\to-\infty}\frac{2^{x+3}}{3^{x+1}}=\lim_{x\to-\infty}\frac{8}{3}\left(\frac{2}{3}\right)^x=\frac{8}{3}\lim_{x\to-\infty}\left(\frac{2}{3}\right)^x=\infty$

7.) $\lim_{x\to\infty}\frac{2^{2x+1}}{3^{x-1}}$

Solution: We start by re-writing the exponential expressions. The goal is to bring it into a form where there is only one exponential expression involving x.

$$\frac{2^{2x+1}}{3^{x-1}}=\frac{2^{2x}\cdot 2^1}{\frac{3^x}{3^1}}=\frac{\left(2^2\right)^x\cdot 2}{3^x\cdot\frac{1}{3}}=\frac{4^x\cdot 6}{3^x}=6\left(\frac{4}{3}\right)^x$$

$$\text{Thus}\quad \lim_{x\to\infty}\frac{2^{2x+1}}{3^{x-1}}=\lim_{x\to\infty}6\left(\frac{4}{3}\right)^x=6\lim_{x\to\infty}\left(\frac{4}{3}\right)^x=\infty\quad\text{since }\frac{4}{3}>1$$

8.) $\lim_{x\to-\infty}\frac{2^{2x+1}}{3^{x-1}}$

Solution: $\lim_{x\to-\infty}\frac{2^{2x+1}}{3^{x-1}}=\lim_{x\to-\infty}6\left(\frac{4}{3}\right)^x=6\lim_{x\to-\infty}\left(\frac{4}{3}\right)^x=0$

9.) $\lim_{x\to\infty}\left(3^{x+1}-3^x\right)$

Solution: $\lim_{x\to\infty}\left(3^{x+1}-3^x\right)=\lim_{x\to\infty}\left(3^x\cdot 3-3^x\right)=\lim_{x\to\infty}\left(3\cdot 3^x-3^x\right)=\lim_{x\to\infty}\left(2\cdot 3^x\right)=2\lim_{x\to\infty}3^x=\infty$

10.) $\lim_{x\to\infty}\frac{3\sqrt{x}+2}{5\sqrt{x}+1}$

Solution: Since $\lim_{x\to\infty}\sqrt{x}=\infty$, clearly $\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0$. We will use this fact; we factor out $\sqrt{x}$ from both numerator and denominator.

$$\lim_{x\to\infty}\frac{3\sqrt{x}+2}{5\sqrt{x}+1}=\lim_{x\to\infty}\frac{\sqrt{x}\left(3+\frac{2}{\sqrt{x}}\right)}{\sqrt{x}\left(5+\frac{1}{\sqrt{x}}\right)}=\lim_{x\to\infty}\frac{3+\frac{2}{\sqrt{x}}}{5+\frac{1}{\sqrt{x}}}=\frac{3}{5}$$

11.) $\lim_{x\to\infty}\left(\sqrt{2x-1}-\sqrt{2x}\right)$

Solution: We will transform this expression by multiplying it by 1, written as a fraction with numerator and denominator both being the conjugate of the expression.

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{2x-1}-\sqrt{2x}\right) &= \lim_{x\to\infty}\frac{\sqrt{2x-1}-\sqrt{2x}}{1}\cdot\frac{\sqrt{2x-1}+\sqrt{2x}}{\sqrt{2x-1}+\sqrt{2x}}\\ &= \lim_{x\to\infty}\frac{(2x-1)-(2x)}{\sqrt{2x-1}+\sqrt{2x}} = \lim_{x\to\infty}\frac{-1}{\sqrt{2x-1}+\sqrt{2x}} = 0\end{aligned}$$

12.) $\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}}$

Solution: We factor out $\sqrt{x}$ from both numerator and denominator.

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{2x}} &= \lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x}\left(\frac{\sqrt{x+1}}{\sqrt{x}}-\sqrt{2}\right)} = \lim_{x\to\infty}\frac{1}{\sqrt{\frac{x+1}{x}}-\sqrt{2}}\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}-\sqrt{2}} = \frac{1}{1-\sqrt{2}} = \frac{1}{1-\sqrt{2}}\cdot\frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{1+\sqrt{2}}{-1}\\ &= -1-\sqrt{2}\end{aligned}$$

13.) $\lim_{x\to\infty} x\left(\frac{1}{5}-\frac{1}{5-\frac{1}{x}}\right)$

Solution: We just need to simplify the complex fraction. As it turns out, this problem boils down to a type we have already seen.

$$\begin{aligned}x\left(\frac{1}{5}-\frac{1}{5-\frac{1}{x}}\right) &= x\left(\frac{1}{5}-\frac{1}{\frac{5x-1}{x}}\right) = x\left(\frac{1}{5}-\frac{x}{5x-1}\right) = x\left(\frac{(5x-1)-5x}{5(5x-1)}\right)\\ &= x\left(\frac{5x-1-5x}{5(5x-1)}\right) = x\frac{-1}{25x-5} = \frac{-x}{25x-5}\end{aligned}$$

Thus

$$\lim_{x\to\infty} x\left(\frac{1}{5}-\frac{1}{5-\frac{1}{x}}\right) = \lim_{x\to\infty}\frac{-x}{25x-5} = \lim_{x\to\infty}\frac{x(-1)}{x\left(25-\frac{5}{x}\right)} = -\frac{1}{25}$$

14.) $\lim_{x\to\infty} \dfrac{\cos x}{x}$

Solution: This problem can be solved by the sandwich principle. Consider the limits $\lim_{x\to\infty} \dfrac{1}{x}$ and $\lim_{x\to\infty} \left(-\dfrac{1}{x}\right)$. These limits are both zero. Furthermore, since

$$\begin{aligned} -1 &\leq \cos x \leq 1 && \text{for all } x\text{, we also have} \\ -\frac{1}{x} &\leq \frac{\cos x}{x} \leq \frac{1}{x} && \text{for all positive } x \end{aligned}$$

Our function $f(x) = \dfrac{\cos x}{x}$ is 'locked' between $g(x) = \dfrac{1}{x}$ and $h(x) = -\dfrac{1}{x}$. Since these both approach zero, so must the function $f(x) = \dfrac{\cos x}{x}$. Thus $\lim_{x\to\infty} \dfrac{\cos x}{x} = 0$.

15.) $\lim_{x\to\infty} \dfrac{x+\frac{1}{x}}{x-\frac{1}{x}}$

Solution: We will factor out x from both numerator and denominator, and use the fact that $\lim_{x\to\infty} \dfrac{1}{x^k} = 0$ for all positive integers k..

$$\lim_{x\to\infty} \frac{x+\frac{1}{x}}{x-\frac{1}{x}} = \lim_{x\to\infty} \frac{x\left(1+\frac{1}{x^2}\right)}{x\left(1-\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}} = 1$$

16.) $\lim_{x\to\infty} \dfrac{2^x+2^{-x}}{2^x-2^{-x}}$

Solution: First, $\lim_{x\to\infty} 2^x = \infty$ (and so 2^x is large) and $\lim_{x\to\infty} 2^{-x} = 0$ (and so 2^{-x} is small). With that in mind, this limit is similar to $\lim_{x\to\infty} \dfrac{x+\frac{1}{x}}{x-\frac{1}{x}}$. The solution also will be similar. We will factor out 2^x from both numerator and denominator.

$$\lim_{x\to\infty} \frac{2^x+2^{-x}}{2^x-2^{-x}} = \lim_{x\to\infty} \frac{2^x+\frac{1}{2^x}}{2^x-\frac{1}{2^x}} = \lim_{x\to\infty} \frac{2^x\left(1+\frac{1}{(2^x)^2}\right)}{2^x\left(1-\frac{1}{(2^x)^2}\right)} = \lim_{x\to\infty} \frac{1+\frac{1}{2^{2x}}}{1-\frac{1}{2^{2x}}} = 1$$

17.) $\lim_{x\to-\infty} \left(\sqrt{3x-1}-\sqrt{3x+1}\right)$

Solution: When $x \to -\infty$, then we may assume it is negative. Then the expressions under the square root are negative and the function is not defined. Thus, there is no limit at negative infinity. The answer is: undefined.

Sample Problems

Compute each of the following limits.

1.) $\displaystyle\lim_{x\to 2}\left(3x^2-5x+2\right)$

2.) $\displaystyle\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}$

3.) $\displaystyle\lim_{x\to 6}\frac{\frac{1}{x}-\frac{1}{6}}{x-6}$

4.) $\displaystyle\lim_{x\to 7}\left(2x-|x-7|\right)$

5.) $\displaystyle\lim_{x\to 2^-}\frac{1}{x^2-9}$

6.) $\displaystyle\lim_{x\to -3^-}\frac{1}{x^2-9}$

7.) $\displaystyle\lim_{x\to -3^+}\frac{1}{x^2-9}$

8.) $\displaystyle\lim_{x\to -3}\frac{1}{x^2-9}$

9.) $\displaystyle\lim_{x\to 5^-}\frac{x^2-4x-5}{x^2-25}$

10.) $\displaystyle\lim_{x\to 5^+}\frac{x^2-4x-5}{x^2-25}$

11.) $\displaystyle\lim_{x\to 5}\frac{x^2-4x-5}{x^2-25}$

12.) $\displaystyle\lim_{x\to -5}\frac{x^2-4x-5}{x^2-25}$

13.) $\displaystyle\lim_{x\to 1}2\sqrt{x-1}$

14.) $\displaystyle\lim_{x\to 0}\frac{x+2}{x^3-5x^2}$

15.) $\displaystyle\lim_{x\to 0}\frac{x}{5x^3-x^4}$

Practice Problems

1.) $\displaystyle\lim_{x\to 8^-}\frac{1}{\sqrt{x+1}-3}$

2.) $\displaystyle\lim_{x\to 8^+}\frac{1}{\sqrt{x+1}-3}$

3.) $\displaystyle\lim_{x\to 8}\frac{1}{\sqrt{x+1}-3}$

4.) $\displaystyle\lim_{x\to 2^+}\frac{x^2+3x}{9-x^2}$

5.) $\displaystyle\lim_{x\to 3^-}\frac{x^2+3x}{9-x^2}$

6.) $\displaystyle\lim_{x\to 3^+}\frac{x^2+3x}{9-x^2}$

7.) $\displaystyle\lim_{x\to 3}\frac{x^2+3x}{9-x^2}$

8.) $\displaystyle\lim_{x\to -3^-}\frac{x^2+3x}{9-x^2}$

9.) $\displaystyle\lim_{x\to -3^+}\frac{x^2+3x}{9-x^2}$

10.) $\displaystyle\lim_{x\to -3^+}\frac{x^2+3x}{9-x^2}$

11.) $\displaystyle\lim_{x\to 4^+}\frac{\frac{1}{x}-\frac{1}{4}}{x^2-16}$

12.) $\displaystyle\lim_{x\to 4^-}\frac{\sqrt{x}-2}{x-4}$

13.) $\displaystyle\lim_{x\to 3^+}\frac{\sqrt{x+1}-2}{3-x}$

14.) $\displaystyle\lim_{x\to 3^-}\frac{\sqrt{x+1}-2}{3-x}$

15.) $\displaystyle\lim_{x\to 15^+}\frac{1}{\sqrt{x-15}}$

16.) $\displaystyle\lim_{x\to 15^-}\frac{1}{\sqrt{x-15}}$

Sample Problems - Answers

Compute each of the following limits.

1.) 4 2.) $\frac{1}{6}$ 3.) $-\frac{1}{36}$ 4.) 14 5.) $-\frac{1}{5}$ 6.) ∞ 7.) $-\infty$ 8.) undefined

9.) $\frac{3}{5}$ 10.) $\frac{3}{5}$ 11.) $\frac{3}{5}$ 12.) undefined 13.) undefined 14.) $-\infty$ 15.) ∞

Practice Problems - Answers

1.) $-\infty$ 2.) ∞ 3.) undefined 4.) 2 5.) ∞ 6.) $-\infty$ 7.) undefined 8.) $-\frac{1}{2}$

9.) $-\frac{1}{2}$ 10.) $-\frac{1}{2}$ 11.) $-\frac{1}{128}$ 12.) $\frac{1}{4}$ 13.) $-\frac{1}{4}$ 14.) $-\frac{1}{4}$ 15.) ∞ 16.) undefined

Sample Problems - Solutions

1.) $\lim\limits_{x\to 2}(3x^2-5x+2)$

Solution: Let us first substitute $x=2$ into the expression.

$$3\cdot 2^2-5\cdot 2+2=12-10+2=4$$

Since the result is a well-defined number, we have a two-sided limit:

$$\begin{aligned}
\lim_{x\to 2^-}\left(3x^2-5x+2\right) &= 4 \quad \text{is the left-hand side limit, and}\\
\lim_{x\to 2^+}\left(3x^2-5x+2\right) &= 4 \quad \text{is the right-hand side limit, and thus}\\
\lim_{x\to 2}\left(3x^2-5x+2\right) &= 4 \quad \text{is the two-sided limit.}
\end{aligned}$$

2.) $\lim\limits_{x\to 5}\dfrac{\sqrt{x+4}-3}{x-5}$

Solution: Let us first substitute $x=5$ into the expression.

$$\frac{\sqrt{5+4}-3}{5-5}=\frac{3-3}{0}=\frac{0}{0} \quad \text{undefined}$$

This is not the end of the problem. Our result does not indicate that the limit doesn't exist. This expression is an **indeterminate**. In case of an indeterminate, we have to manipulate the expression until it is in a form so that we can evaluate the limit. The methods of manipulation depends on the given problem. In this case, we will use the conjugate of the numerator. We will multiply both numerator and denominator by $\sqrt{x+4}+3$.

$$\begin{aligned}\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5} &= \lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}\cdot\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = \lim_{x\to 5}\frac{\left(\sqrt{x+4}-3\right)\left(\sqrt{x+4}+3\right)}{(x-5)\left(\sqrt{x+4}+3\right)}\\ &= \lim_{x\to 5}\frac{\left(\sqrt{x+4}\right)^2-3^2}{(x-5)\left(\sqrt{x+4}+3\right)} = \lim_{x\to 5}\frac{x+4-9}{(x-5)\left(\sqrt{x+4}+3\right)} = \lim_{x\to 5}\frac{x-5}{(x-5)\left(\sqrt{x+4}+3\right)}\\ &= \lim_{x\to 5}\frac{1}{\sqrt{x+4}+3}\end{aligned}$$

Although we did not change the value of this expression (after all we only multiplied it by 1), this is no longer an indeterminate. We substitute $x=5$ into this new expression:

$$\frac{1}{\sqrt{5+4}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

Since the result is a well-defined number, we have a two-sided limit:
$\lim_{x\to 5^-}\frac{1}{\sqrt{x+4}+3}=\frac{1}{\sqrt{5+4}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$ is the left-hand side limit, $\lim_{x\to 5^+}\frac{1}{\sqrt{x+4}+3}=\frac{1}{6}$ is the right-hand side limit, and thus $\lim_{x\to 5}\frac{1}{\sqrt{x+4}+3}=\frac{1}{6}$ is the two-sided limit. Thus $\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}=\frac{1}{6}$.

3.) $\lim_{x\to 6}\frac{\frac{1}{x}-\frac{1}{6}}{x-6}$

Solution: Let us first substitute $x=6$ into the expression.

$$\frac{\frac{1}{6}-\frac{1}{6}}{6-6}=\frac{0}{0}=\text{undefined}$$

This expression is an **indeterminate**. We only simplify this complex fraction.

$$\lim_{x\to 6}\frac{\frac{1}{x}-\frac{1}{6}}{x-6}=\lim_{x\to 6}\frac{\frac{6}{6x}-\frac{x}{6x}}{x-6}=\lim_{x\to 6}\frac{\frac{6-x}{6x}}{x-6}=\lim_{x\to 6}\frac{6-x}{6x}\cdot\frac{1}{x-6}=\lim_{x\to 6}\frac{-(x-6)}{6x}\cdot\frac{1}{x-6}=\lim_{x\to 6}\frac{-1}{6x}$$

We substitute $x=6$ into this new expression:

$$\lim_{x\to 6}\frac{-1}{6x}=\frac{-1}{6\cdot 6}=-\frac{1}{36}$$

Thus we have a two-sided limit:

$$\lim_{x\to 6^-}\frac{\frac{1}{x}-\frac{1}{6}}{x-6}=-\frac{1}{36},\quad \lim_{x\to 6^+}\frac{\frac{1}{x}-\frac{1}{6}}{x-6}=-\frac{1}{36},\quad\text{and so}\quad \lim_{x\to 6}\frac{\frac{1}{x}-\frac{1}{6}}{x-6}=-\frac{1}{36}$$

4.) $\lim_{x\to 7}(2x-|x-7|)$

Solution: Recall the definition of $f(x)=|x|$ first.

$$|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0\end{cases}$$

We will compute the left limit, $\lim_{x\to 7^-}(2x-|x-7|)$ first. As x approaches 7 from the left, x is less than 7. Thus $x-7$ is negative and so $|x-7|=-(x-7)$.

$$\lim_{x\to 7^-}(2x-|x-7|)=\lim_{x\to 7^-}(2x-[-(x-7)])=\lim_{x\to 7^-}(2x+x-7)=\lim_{x\to 7^-}(3x-7)=3\cdot 7-7=14$$

We will now compute the right limit, $\lim_{x\to 7^+}(2x-|x-7|)$. As x approaches 7 from the right, x is greater than 7. Thus $x-7$ is positive and so $|x-7|=x-7$.

$$\lim_{x\to 7^+}(2x-|x-7|)=\lim_{x\to 7^+}(2x-(x-7))=\lim_{x\to 7^+}(2x-x+7)=\lim_{x\to 7^+}(x+7)=7+7=14$$

Because the left-hand side limit equals to the right-hand side limit, there is a two-sided limit and it is 14.

$$\lim_{x\to 7^-}(2x-|x-7|)=\lim_{x\to 7^+}(2x-|x-7|)=14 \quad\Longrightarrow\quad \lim_{x\to 7}(2x-|x-7|)=14$$

5.) $\lim_{x\to 2^-}\dfrac{1}{x^2-9}$

Solution: Let us first substitute $x=2$ into the expression.

$$\frac{1}{2^2-9}=\frac{1}{4-9}=\frac{1}{-5}=-\frac{1}{5}$$

Thus $\lim_{x\to 2^-}\dfrac{1}{x^2-9}=-\dfrac{1}{5}$. (although the problem did not ask us to find them, our computation shows that the right-hand side limit is also $-\dfrac{1}{5}$, and thus the two-sided limit also exists and is $-\dfrac{1}{5}$.)

6.) $\lim_{x\to -3^-}\dfrac{1}{x^2-9}$

Solution: Let us first substitute $x=3$ into the expression.

$$\frac{1}{3^2-9}=\frac{1}{9-9}=\frac{1}{0}\quad \text{undefined}$$

This is not a case of an indeterminate. If we divide 1 by a very small number, the result is a very large number. Thus, the answer is either $-\infty$ or ∞. The ambiguity results from the fact that if a number is very close to zero, it may be a small negative or a small positive number. We just have to find out which one it is. $x\to -3^-$ means that x is very close to -3, and that x is less than -3.

$$x^2-9=(x+3)(x-3)$$

$x<-3$	add 3	$x<-3$	subtract 3
$x+3<0$		$x-3<-6$	
the factor $x+3$ is negative		the factor $x-3$ is negative (since less than -6)	

As x approaches -3 from the left, $x^2-9=(x+3)(x-3)$ is positive since both factors are negative. Thus the limit is ∞.

7.) $\lim\limits_{x \to -3^+} \dfrac{1}{x^2-9}$

Solution: When we substitute $x=3$ into the expression (see previous problem) we get $\dfrac{1}{0}$. This means that the answer is either $-\infty$ or ∞; and we only have to figure out which one. Keep in mind that $x \to -3^+$ means that x is very close to -3, and that x is greater than -3.

$$x^2-9=(x+3)(x-3)$$

$x>-3$ add 3	$x>-3$ subtract 3
$x+3>0$	$x-3>-6$
the factor $x+3$ is positive	the factor $x-3$ is negative. Although $x-3$ is greater than -6, but is also close to -6 and so must be negative.

As x approaches -3 from the right, $x^2-9=(x+3)(x-3)$ is negative since one factor is positive and the other is negative. Thus the limit is $-\infty$.

8.) $\lim\limits_{x \to -3} \dfrac{1}{x^2-9}$

Solution: We have worked out the one-sided limits in the previous problems. Our results were

$$\lim_{x \to -3^-} \frac{1}{x^2-9}=\infty \quad \text{and} \quad \lim_{x \to -3^+} \frac{1}{x^2-9}=-\infty$$

Since these limits are not equal, the two-sided limit does not exist. $\lim\limits_{x \to -3} \dfrac{1}{x^2-9}=$ undefined

9.) $\lim\limits_{x \to 5^-} \dfrac{x^2-4x-5}{x^2-25}$

Solution: Let us first substitute $x=5$ into the expression.

$$\frac{5^2-4\cdot 5-5}{5^2-25}=\frac{25-20-5}{25-25}=\frac{0}{0} \quad \text{undefined}$$

This expression is an **indeterminate**. In case of a $\dfrac{0}{0}$ indeterminate in a rational function, we must factor both numerator and denominator and cancel common factors. If a polynomial takes a zero at $x=5$, it is then divisible by $x-5$. Thus both numerator and denominator are divisible by $x-5$. Once we cancelled all common factors, the expression will no longer be an indeterminate.

$$\lim_{x \to 5^-} \frac{x^2-4x-5}{x^2-25}=\lim_{x \to 5^-} \frac{(x-5)(x+1)}{(x-5)(x+5)}=\lim_{x \to 5^-} \frac{x+1}{x+5}$$

We can now substitute $x=5$ into this new expression

$$\frac{5+1}{5+5}=\frac{6}{10}=\frac{3}{5}$$

10.) $\displaystyle\lim_{x\to5^+}\frac{x^2-4x-5}{x^2-25}$

Solution: We actually have the answer to this question. In the previous problem, the expression was first an indetermiate. After cancellation, we were able to substitute $x=5$ and that indicates a two-sided limit. Thus $\displaystyle\lim_{x\to5^+}\frac{x^2-4x-5}{x^2-25}=\frac{3}{5}$.

11.) $\displaystyle\lim_{x\to5}\frac{x^2-4x-5}{x^2-25}$

Solution: Since the left-hand side limit and right-hand side limit (worked out in the previous problems) both exist and they are equal, we have a two-sided limit

$$\lim_{x\to5^-}\frac{x^2-4x-5}{x^2-25}=\frac{3}{5}\quad\text{and}\quad\lim_{x\to5^+}\frac{x^2-4x-5}{x^2-25}=\frac{3}{5}\quad\Longrightarrow\quad\lim_{x\to5}\frac{x^2-4x-5}{x^2-25}=\frac{3}{5}$$

12.) $\displaystyle\lim_{x\to-5}\frac{x^2-4x-5}{x^2-25}$

Solution: Let us first substitute $x=-5$ into the expression.

$$\frac{(-5)^2-4(-5)-5}{(-5)^2-25}=\frac{25+20-5}{25-25}=\frac{40}{0}\qquad\text{undefined}$$

This is not a case of an indeterminate. If we divide 1 by a very small number, the result is a very large number. Thus, the answer is either $-\infty$ or ∞ on either sides of -5. We have to compute the left-hand and right-hand limits separately. Let us first simplify this expression:

$$\lim_{x\to-5}\frac{x^2-4x-5}{x^2-25}=\lim_{x\to-5}\frac{(x-5)(x+1)}{(x-5)(x+5)}=\lim_{x\to-5}\frac{x+1}{x+5}$$

Let us now compute the left-hand side limit. $x\to-5^-$ means that x is very close to -5, and that x is less than -5.

$x<-5$ add 5	$x<-5$ add 1
$x+5<0$	$x+1<-4$
the factor $x+5$ is negative	the factor $x+1$ is negative

As x approaches -5 from the left, $\dfrac{x+1}{x+5}$ is positive since both numerator denominator are negative. Thus the left-hand side limit is ∞, i.e. $\displaystyle\lim_{x\to-5^-}\frac{x^2-4x-5}{x^2-25}=\infty$.

Now for the right-hand limit: $x\to-5^+$ means that x is very close to -5, and that x is greater than -5.

$x>-5$ add 5	$x>-5$ add 1
$x+5>0$	$x+1>-4$
the factor $x+5$ is positive	the factor $x+1$ is negative. Although $x+1$ is greater than -4, but is also close to -4 and so must be negative.

As x approaches -5 from the right, $\dfrac{x+1}{x+5}$ is negative since the numerator is negative and the denominator is positive. Thus the right-hand side limit is $-\infty$, i.e. $\lim\limits_{x \to -5^+} \dfrac{x^2-4x-5}{x^2-25} = -\infty$. Because of the two one-sided limits are not equal, the two-sided limit does not exist. $\lim\limits_{x \to -5} \dfrac{x^2-4x-5}{x^2-25} = \text{undefined}$.

13.) $\lim\limits_{x \to 1} 2\sqrt{x-1}$

Solution: When we substitute $x=1$, the result is zero. However, we do not have a two-sided limit. As x approaches 1 from the left, $\sqrt{x-1}$ is undefined since $x-1$ is negative. Thus

$$\lim_{x \to 1^-} 2\sqrt{x-1} = \text{undefined} \quad \text{and} \quad \lim_{x \to 1^+} 2\sqrt{x-1} = 0$$

thus $\lim\limits_{x \to 1} 2\sqrt{x-1} = \text{undefined}$.

14.) $\lim\limits_{x \to 0} \dfrac{x+2}{x^3-5x^2}$

Solution: Let us first substitute $x=0$ into the expression.

$$\frac{0+2}{0^3-5\cdot 0} = \frac{2}{0} \quad \text{undefined}$$

The one-sided limits are ∞ or $-\infty$. Let us bring the expression first in a more convenient form, where both numerator and denominator are factored. Also, if there is cancellation, that makes the problem easier.

$$\lim_{x \to 0} \frac{x+2}{x^3-5x^2} = \lim_{x \to 0} \frac{x+2}{x^2(x-5)}$$

We compute first the left-hand side limit. As x approaches 0 from the left, it is very close to zero, and it is also less than zero. Then $x+2$ is positive, $x-5$ is negative, and x^2 is positive. Thus $\lim\limits_{x \to 0^-} \dfrac{x+2}{x^2(x-5)} = -\infty$. As x approaches 0 from the right, it is very close to zero, and it is also greater than zero. Then $x+2$ is positive, $x-5$ is negative, and x^2 is positive. Thus $\lim\limits_{x \to 0^+} \dfrac{x+2}{x^2(x-5)} = -\infty$. Thus the two-sided limit exists and $\lim\limits_{x \to 0} \dfrac{x+2}{x^2(x-5)} = -\infty$

15.) $\lim\limits_{x \to 0} \dfrac{x}{5x^3-x^4}$

Solution: Let us first substitute $x=0$ into the expression.

$$\frac{0}{5\cdot 0^3-0^4} = \frac{0}{0} \quad \text{undefined}$$

This expression is an **indeterminate**. In case of a $\dfrac{0}{0}$ indeterminate in a rational function, we must factor both numerator and denominator and cancel common factors. Both numerator and denominator are clearly divisible by x. Once we cancelled all common factors, the expression will no longer be an indeterminate.

$$\lim_{x\to 0}\frac{x}{5x^3-x^4}=\lim_{x\to 0}\frac{x}{-x^4+5x^3}=\lim_{x\to 0}\frac{x}{-x^3\left(x+5\right)}=\lim_{x\to 0}\frac{-1}{x^2\left(5x-1\right)}$$

We substitute $x=0$ into this new expression:

$$\frac{-1}{0^2\left(5\cdot 0-1\right)}=\frac{-1}{0}\quad\text{undefined}$$

This is no longer an indeterminate, the one-sided limits are ∞ or $-\infty$. We separately compute these limits. First, the left-hand side limit: If x is less than zero and close to zero, then -1 is negative, x^2 is positive, and $5x-1$ is negative. Thus $\dfrac{-1}{x^2\left(5x-1\right)}$ is positive, and so $\lim\limits_{x\to 0^-}\dfrac{x}{5x^3-x^4}=\infty$. For the right-hand side limit: If x is greater than zero and close to zero, then -1 is negative, x^2 is positive, and $5x-1$ is negative. Thus $\dfrac{-1}{x^2\left(5x-1\right)}$ is positive, and so $\lim\limits_{x\to 0^+}\dfrac{x}{5x^3-x^4}=\infty$. Then the two-sided limit is also ∞.

