AP Calculus Exam Prep Assignment #2 Name ________________________

Multiple Choice

1)

$$V = \pi\int_0^{2\pi}(1-\cos\phi)^2\,d\phi = \pi\int_0^{2\pi}\left(1-2\cos\phi+\cos^2\phi\right)d\phi = \pi\int_0^{2\pi}\left(1-2\cos\phi+\frac{1+\cos 2\phi}{2}\right)d\phi$$

$$= \pi\left[\frac{3}{2}\phi + 2\sin\phi + \frac{1}{4}\sin 2\phi\right]_0^{2\pi} = \pi\left[(3\pi+0+0)-(0+0+0)\right] = 3\pi^2$$

D)

2) $A = \int_0^{2\pi}(1-\cos x)\,dx = \left[x - \sin x\right]_0^{2\pi} = (2\pi - 0) - (0-0) = 2\pi$ **E)**

3)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-t^2+3}{3}}{e^t} = \frac{-t^2+3}{3e^t} \qquad x-\text{int} \Rightarrow y = 0 \Rightarrow t\left(-\frac{t^2}{9}+1\right) = 0 \Rightarrow t = 0, \pm 3$$

A)

$$\frac{dy}{dx}(-3) = \frac{-9+3}{3e^{-3}} = -2e^3$$

4)

$$\frac{dy}{dt} = 4\cos t + 12\cos 12t \qquad \frac{dy}{dt}(1) = 4\cos 1 + 12\cos 12 \approx 12.287$$

E)

5) Find $\frac{d^2y}{dx^2}$ if $x = 2\cos\phi$ and $y = \sin\phi$.

$$\frac{dy}{dx} = \frac{\cos\phi}{-2\sin\phi} = -\frac{1}{2}\cot\phi \qquad \frac{d^2y}{dx^2} = \frac{\frac{1}{2}\csc^2\phi}{-2\sin\phi} = -\frac{1}{4}\csc^3\phi$$

A)

6) The length of $r = 3\csc\phi$ from $\phi = \frac{\pi}{4}$ to $\phi = \frac{3\pi}{4}$ is:

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{\pi/4}^{3\pi/4}\sqrt{(3\csc\phi)^2 + (3\csc\phi\cot\phi)^2}\,d\phi = \int_{\pi/4}^{3\pi/4}\sqrt{9\csc^2\phi + 9\csc^2\phi\cot^2\phi}\,d\phi$$

A)

$$= \int_{\pi/4}^{3\pi/4}\sqrt{9\csc^2\phi(1+\cot^2\phi)}\,d\theta = \int_{\pi/4}^{3\pi/4}\sqrt{9\csc^4\phi}\,d\phi = \int_{\pi/4}^{3\pi/4}3\csc^2 d\phi = 3\left[-\cot\phi\right]_{\pi/4}^{3\pi/4} = 3\left[1-(-1)\right] = 6$$

7) Find the slope of the curve $r = \cos 2\phi$ at $\phi = \frac{\pi}{6}$

$$x = \cos 2\phi(\cos\phi) \quad y = \cos 2\phi(\sin\phi) \quad \frac{dy}{dx} = \frac{\cos 2\phi(\cos\phi) - 2\sin 2\phi(\sin\phi)}{-\cos 2\phi(\sin\phi) - 2\sin 2\phi(\cos\phi)}$$

A)

$$\frac{dy}{dx}\left(\frac{\pi}{6}\right) = \frac{\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}{-\frac{1}{2}\left(\frac{1}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} = -\frac{\sqrt{3}}{4}\cdot\frac{4}{-7} = \frac{\sqrt{3}}{7}$$

8) Find the first quadrant area inside the rose, $r = 3\sin 2\phi$, but outside the circle, $r = 2$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \quad 3\sin 2\phi = 2 \Rightarrow \sin 2\phi = \frac{2}{3} \Rightarrow \phi \approx 0.365, 1.206$$

D)

$$A = \frac{1}{2}\int_{0.365}^{1.206}\left[(3\sin 2\phi)^2 - (2)^2\right]d\phi \approx 1.328$$

9) The common area inside $r = 2\sin\phi$ and $r = 2\cos\phi$ is:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \quad A = 2\left(\frac{1}{2}\int_{\pi/4}^{\pi/2}(2\cos\phi)^2\, d\phi\right) = \int_{\pi/4}^{\pi/2} 4\cos^2\phi\, d\phi = 4\int_{\pi/4}^{\pi/2}\frac{1+\cos 2\phi}{2}\, d\phi = 2\int_{\pi/4}^{\pi/2}(1+\cos 2\phi)\, d\phi$$

B)

$$2\left[\phi + \frac{1}{2}\sin 2\phi\right]_{\pi/4}^{\pi/2} = 2\left[\left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{4}+\frac{1}{2}\right)\right] = \frac{\pi}{2} - 1$$

10) The area enclosed by the rose, $r = \cos 2\phi$ is:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta \quad A = \frac{1}{2}\int_{0}^{2\pi}(\cos 2\phi)^2\, d\phi = \frac{1}{2}\int_{0}^{2\pi}\frac{1+\cos 4\phi}{2}\, d\phi = \frac{1}{4}\int_{0}^{2\pi} d\phi + \frac{1}{4}\int_{0}^{2\pi}\cos 4\phi\, d\phi$$

B)

$$\frac{\pi}{2} + \frac{1}{16}\left[\sin 4\phi\right]_{0}^{2\pi} = \frac{\pi}{2}$$

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11)

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0 \Rightarrow \frac{dy}{dt}=-\frac{3}{4}\frac{dx}{dt} \text{ at } (3,4) \qquad \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=2 \Rightarrow \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=4$$

$$\left(\frac{dx}{dt}\right)^2+\left(-\frac{3}{4}\frac{dx}{dt}\right)^2=4 \Rightarrow \frac{25}{16}\left(\frac{dx}{dt}\right)^2=4 \Rightarrow \frac{dx}{dt}=\pm\frac{8}{5}, \text{ but counterclockwise } \Rightarrow \frac{dx}{dt}=-\frac{8}{5} \qquad \textbf{A)}$$

$$\frac{dy}{dt}=\frac{6}{5} \qquad \mathbf{v}=-\frac{8}{5}\mathbf{i}+\frac{6}{5}\mathbf{j}$$

12) **C) -1 ≤ *t* ≤ 1**

For problems 13-16, use the following:

$\mathbf{r}=\left(3\cos\left(\frac{\pi}{3}t\right)\right)\mathbf{i}+\left(2\sin\left(\frac{\pi}{3}t\right)\right)\mathbf{j}$ is the position vector from the origin to a moving point $P(x,y)$ at time t.

13) $x=3\cos\left(\frac{\pi}{3}t\right),\ y=2\sin\left(\frac{\pi}{3}t\right) \qquad \left(\frac{x}{3}\right)^2+\left(\frac{y}{2}\right)^2=1 \Rightarrow 4x^2+9y^2=36$ **E)**

14) $v=\left\langle -\pi\sin\left(\frac{\pi}{3}t\right), \frac{2\pi}{3}\cos\left(\frac{\pi}{3}t\right)\right\rangle$ speed at $t=3 \Rightarrow \|v\|=\sqrt{0^2+\left(-\frac{2\pi}{3}\right)^2}=\frac{2\pi}{3}$ **A)**

15) $a=\left\langle -\frac{\pi^2}{3}\cos\left(\frac{\pi}{3}t\right), -\frac{2\pi^2}{9}\sin\left(\frac{\pi}{3}t\right)\right\rangle$ When $t=3,\ \|a\|=\sqrt{\left(-\frac{\pi^2}{3}\right)^2+0}=\frac{\pi^2}{3}$ **B)**

16) At the point where $t=\frac{1}{2}\left[\left(\frac{3\sqrt{3}}{2},1\right)\right]$ the slope of the curve along which the particle moves is:

$$\frac{dy}{dx}=\frac{\frac{2\pi}{3}\cos\left(\frac{\pi}{3}t\right)}{-\pi\sin\left(\frac{\pi}{3}t\right)}=-\frac{2}{3}\cot\left(\frac{\pi}{3}t\right) \qquad \text{At } t=\frac{1}{2},\ \frac{dy}{dx}=-\frac{2}{3}\cot\left(\frac{\pi}{6}\right)=-\frac{2\sqrt{3}}{3} \qquad \textbf{D)}$$

17) **D) Its velocity and acceleration vectors must be perpendicular.**

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Problems

18) During the time period from $t = 0$ to $t = 6$ seconds, a particle moves along the path given by $x(t) = 3\cos(\pi t)$ and $y(t) = 5\sin(\pi t)$.

A) Find the position of the particle when $t = 2.5$ seconds. $x(2.5) = 3\cos(2.5\pi)$, $y(2.5) = 5\sin(2.5\pi)$

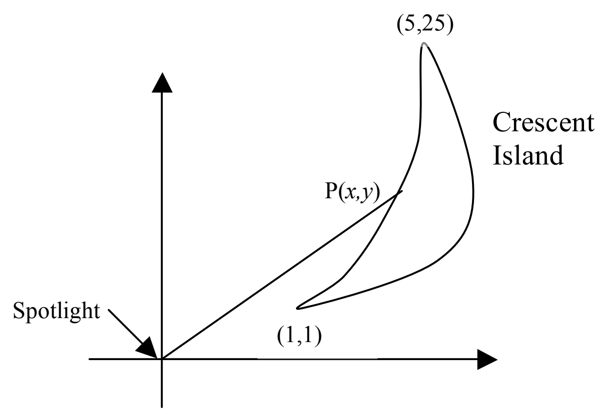
B) Sketch the graph of the path of the particle from $t = 0$ to $t = 6$. Indicate the direction of the particle along the path and the scale.

C) How many times does the particle pass through the point found in part (A)? At what time(s)?

D) Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance the particle travels from time $t = 1.25$ to $t = 1.75$. $\int_{1.25}^{1.75}\sqrt{1+\left(\frac{5\pi\cos t}{3\pi\sin t}\right)^2}\,dt \approx$

E) Find the velocity vector for the particle at any time t.

19) [1996 BC 6] The figure to the right shows a spotlight shining on a point $P(x,y)$ on the shoreline of Crescent Island. The spotlight is located at the origin and is rotating. The portion of the shoreline on which the spotlight shines is in the shape of the parabola $y = x^2$ from the point (1,1) to the point (5,25). Let ϕ be the angle between the beam of light and the positive x-axis.



A) For what values of ϕ on $[0,2\pi]$ does the spotlight shine on the shoreline?

B) Find the x- and y-coordinates of point P in terms of $\tan\phi$.

C) If the spotlight is rotating at the rate of one revolution per minute, how fast is the point P traveling along the shoreline at the instant it is at the point (3,9)?

20) A moving particle has position $(x(t), y(t))$ at time t. The position of the particle at time $t = 1$ is (2,6), and the velocity vector at any time $t > 0$ is given by $\left\langle 1-\frac{1}{t^2}, 2+\frac{1}{t^2}\right\rangle$.

A) Find the acceleration vector at time $t = 3$. $a(t) = \left\langle \frac{2}{t^3}, -\frac{2}{t^3}\right\rangle$ $a(3) = \left\langle \frac{2}{27}, -\frac{2}{27}\right\rangle$

B) Find the position of the particle at time $t = 3$.

$$s(t) = \left\langle t+\frac{1}{t}+C_x, 2t-\frac{1}{t}+C_y\right\rangle \quad \begin{matrix} 2+C_x = 2 \Rightarrow C_x = 0 \\ 1+C_y = 6 \Rightarrow C_y = 5 \end{matrix} \quad s(3) = \left\langle 3\frac{1}{3}, 10\frac{2}{3}\right\rangle$$

C) For what time $t > 0$ does the line tangent to the path of the particle at $(x(t), y(t))$ have a slope of 8?

$$\frac{dy}{dx} = \frac{2+\frac{1}{t^2}}{1-\frac{1}{t^2}} = \frac{2t^2+1}{t^2-1} = 8 \Rightarrow 2t^2+1 = 8t^2-8 \Rightarrow t^2 = \frac{9}{6} \Rightarrow t = \sqrt{\frac{9}{6}}$$

D) The particle approaches a line as $t \to \infty$. Find the slope of this line. Show the work that leads to your conclusion.

$$\lim_{t \to \infty} \frac{2t^2 + 1}{t^2 - 1} = 2$$

21) [2001 BC 1] An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = \cos(t^3)$ and $\frac{dy}{dt} = 3\sin(t^2)$ for $0 \le t \le 3$. At time $t = 2$, the object is at (4,5).

A) Write an equation for the line tangent to the curve at (4,5).
B) Find the speed of the object at time $t = 2$.
C) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.
D) Find the position of the object at time $t = 3$.