Evaluating the correctness of a solution

One of the most common questions I get in my physics course is “David, if you’re never going tell me if I’m right or not, how am I supposed to know if I got the problem right or not? Your job as the instructor is to tell me the correct answer.” My response is: “My job is to equip you with the skills you need to be successful in your future career. You should do what I do: When I’m preparing homework or exams and solve a problem, how do I know I got it right? Many times, I have **no idea** if I solved the problem correctly. The way I figure it out is by evaluating whether my answer is reasonable or not.”
Education is about preparing for real life: If you are working in your job and your boss asks you to calculate something or figure something out for her, how do you know if you got it right? Evaluation is a key life-skill. Evaluation is also probably the most difficult ability you are going to learn in this course. Keep working at it. You are not going to get it right on the first try.

How many different ways can we evaluate an answer to see if it is reasonable or not?

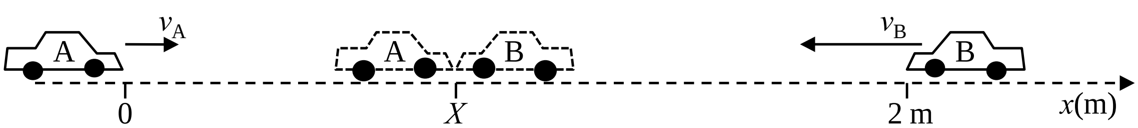
1. Physical sense and order of magnitude (is the final number physically reasonable, or reasonable to within a factor of 10?)
2. Limiting cases, special cases, and trend cases
3. Dimensional analysis (checking the units),
4. Compare with another independent solution method,
5. Reducing a complicated solution to a known case,
6. Cross substitution.

We will address each of these methods in more detail below.

1. Physical sense and order of magnitude

Physical sense: Suppose a problem asks you to calculate the volume of your bedroom (from some other given parameters) and you come up with an answer of $1.5m^3$. Is this answer reasonable? To answer this question, you should realize that this volume corresponds to a cube, each side is 1.14m long. Do you have some point of reference for this? How tall are you? Could you lie down in a cube whose sides are 1.14m? It is imperative that you have a sense for how big the number is. If you don't feel comfortable with meters, convert it to feet! 1.14 meters corresponds to 3.75 feet. You could curl up in a room that size, but you couldn't lie down straight. Conclusion: This is NOT a reasonable volume for human's bedroom, not even a very small one.
Notes: Using common sense knowledge usually requires some mathematical manipulation. In the example above I had to cube-root the answer, visualize the room as a cube AND convert meters into feet before I could have a good physical sense of whether the answer was reasonable or not. Manipulating your answer into a form that permits you to compare with common sense is the **key** part of the skill of this evaluation method!

Physical sense-making can take on many different guises and forms. A few more examples should help you to see how varied it can be:
Suppose you are solving the following problem. You are doing an experiment with two constant speed toy cars, car A and car B which move at speeds $v_A = 3$ m/s and $v_B = 5$ m/s respectively. You set them 2 meters apart and release them simultaneously. Predict at what X they will collide.



After doing some calculating you find a solution: $X = 3$ m. Is this right nor not? If you are thinking in terms of physical sense-making, you should conclude that this answer cannot be correct because the cars have to collide somewhere between 0 m and 2 m. An answer of 3 m is simply impossible.

Physical sense can involve deeper layers of understanding. Working from the same example as above, imagine you got a solution $X = 1.5$ m. Is this answer reasonable? Yes, it is less than 2 m. BUT, think about it a little deeper. Let's suppose that the velocities of cars A and B were equal. Then we'd expect them to collide at 1 m. Now let's take the above example where $v_B > v_A$. Would we expect them to collide to the left or the right of 1 m? We would expect them to collide to the left of 1 m because car B would cover more distance because it is faster than car A. Realizing this, 1.5 m cannot be correct because 1.5 m is a position to the right of 1 m.

Order of magnitude: Sometimes common sense may fail. In this case you can argue that an answer is or isn't reasonable to within a factor of 10 (generally referred to as "an order of magnitude" in physics). This is like common sense, but with a little more leeway. For example suppose a problem asks me to calculate the range of a tennis ball launcher (a projectile motion problem) and I come up with an answer of 3,000 meters. First I need to manipulate it into numbers that I find meaningful. 3,000 meters is 3 kilometers, which is

1.9 miles, or 30-40 football fields. (Look up the conversion on Google if you are stuck.). I don't have a good sense of how powerful tennis ball launchers can be. I could imagine a powerful tennis ball launcher might cover the length of a football field or two (100 – 200 meters), certainly better than the best human could throw, but within the bounds of possibility. 3,000 meters is roughly 10 times (an order of magnitude) bigger than what seems physically plausible. It doesn't seem like a likely or reasonable answer.

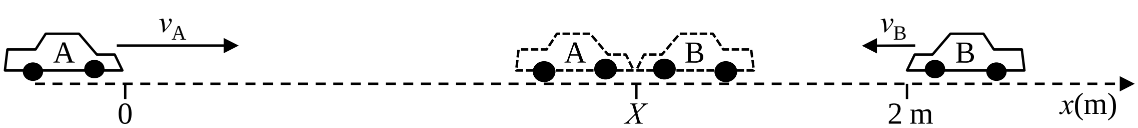
Final note: It is up to the instructor or textbook writer to make sure that the problems they pose have answers in the realm of physical reasonableness. If your instructor doesn't adhere to that rule, this evaluation method can be limited.

2. Limiting cases, special cases, and trend cases

Limiting/special cases and trend cases are like mental testing experiments. They all involve the following three-step reasoning process:
a) Identify a limiting/special case or trend whose answer you intuitively know from physical experience.
b) Plug the numerical/algebraic values of the limiting/special case or trend into the equation that is being evaluated.
c) Draw a conclusion about the validity of the equation by making a comparison between the behavior of the equation in part b) with your intuitive expectations from part a). If they are inconsistent the equation being evaluated is wrong. If consistent, it should boost your confidence that your answer is right. Note that checking with a single limiting/special/trend case DOES NOT PROVE that your equation is right, it is always possible that a wrong solution could still be consistent with the limiting/special/trend case you picked.

I am going to present one example of a problem with two constant velocity toy cars colliding to illustrate each of the examples of a limiting, special, and trend case:

You are solving the following problem. You are doing an experiment with two constant speed toy cars, car A and car B which move at speeds v_A and v_B respectively. You set them 2 meters apart and release them simultaneously. Predict at what X they will collide.



After doing some calculation you come up with the following answer: $X = \dfrac{v_A (2\text{ m})}{v_A + v_B}$

The question that arises is: Is this a reasonable answer to the problem? Did you get it right? In the following examples, I'll show you how to *test* the equation to see if it is reasonable or not, first with a limiting case, then a special case, then a trend case.

Limiting case:
a) Identify *limiting case* and *predicted* value for X:
Consider the limiting case $v_B = 0$: What if $v_B = 0$? Where do you think the cars would collide? If car B was not moving at all, it makes sense that the collision would happen at $X = 2$ m. In other words for the limiting case $v_B = 0$, we *predict* that $X = 2$ m, based on our intuitive understanding of motion.

b) *Test* the equation by plugging the limiting case $v_B = 0$ into the equation: $X = \dfrac{v_A (2\text{ m})}{v_A + v_B}$

$$X = \frac{v_A (2\text{ m})}{v_A + v_B} = \frac{v_A (2\text{ m})}{v_A + 0} = \frac{v_A (2\text{ m})}{v_A} = 2\text{ m}$$

c) *Draw a conclusion*: Is the answer in part b) consistent with your prediction in part a)? The answers in b) and a) make sense and are consistent. My expectation is that IF car A moves in a straight line at constant speed and car B remains stationary, car A will collide with car B at car B's initial position of 2 m. Since the equation does indeed give an answer of 2 m for X, I am a little more confident that $X = \frac{v_{\mathrm{A}}(2\text{ m})}{v_{\mathrm{A}} + v_B}$ is valid.

Note: If the answers in a) and b) are NOT consistent with each other, this immediately proves the equation is NOT valid. In physics, the expectation is that the equation should work for ALL cases. If you can identify just one case where the equation doesn't work, the equation is not correct.

Special case:

A special case is like a limiting case, but instead of taking the largest or smallest possible value for either v_{A} or v_{B}, and predicting what should happen in that limiting case, a special case is a case where you can identify unique values for the variables or a unique relationship between the variables in the problem for which you can predict the outcome X. Consider the following example: I am going to carefully lay out the three-step reasoning process:

a) *Identify special case and predict value of X*: In the problem described above with the two cars colliding, there is one obvious special case for which we already know the answer: What if $v_{\mathrm{A}} = v_{\mathrm{B}}$? In this case, no matter what the values of v_{A} and v_{B} are, we know that if they're equal, the cars will collide in the middle, i.e., at $X = 1$ m.

b) *Test* the equation by plugging the special case $v_{\mathrm{A}} = v_{\mathrm{B}} = v$ into the equation:

$$X = \frac{v_{\mathrm{A}}(2\text{ m})}{v_{\mathrm{A}} + v_B} = \frac{v(2\text{ m})}{v + v} = \frac{v(2\text{ m})}{2v} = \frac{(2\text{ m})}{2} = 1\text{ m}$$

c) *Draw a conclusion*: Is the answer in part b) consistent with your prediction in part a)? The answers in b) and a) make sense and are consistent. Once again it appears that the equation is valid. We cannot prove the equation is correct. It may be that we just got lucky with the cases so far. I mentioned this above, but it does bear repeating: If the answers in a) and b) are NOT consistent with each other, this immediately proves the equation is NOT valid. In physics, the expectation is that the equation should work for ALL cases. If you can identify just one case where the equation doesn't work, the equation is not correct

Trend case:

A trend case follows the same three-step reasoning process as limiting and special cases, but here we examine a trend in the equation, not a unique value based on a single case (limiting or special). As before, we are going to test the equation against our intuitive understanding of the behavior of the system. Take a look at this example:

a) *Identify trend case and predict behavior of X function:* In the example above of the two cars colliding, think about how would the collision position X depend on, say, the velocity of car A, v_{A}, while keeping v_{B} fixed? If we fix v_{B} at say one value (e.g., 2 m/s) and then let v_{A} get bigger and bigger, we would expect the collision point X to move closer and closer to car B. Specifically, let's fix $v_{\mathrm{B}} = 2$m/s and then try two different values for v_{A}, first $v_{\mathrm{A}} = 2$m/s and then $v_{\mathrm{A}} = 4$m/s, I predict that the value of X should be larger for $v_{\mathrm{A}} = 4$m/s than for $v_{\mathrm{A}} = 2$m/s because the faster A moves the more distance it will cover relative to the distance that B covers.

b) *Test the equation*: First calculate X for $v_{\mathrm{A}} = 2$m/s and $v_{\mathrm{B}} = 2$m/s:

$$X = \frac{v_A(2\text{ m})}{v_A + v_B} = \frac{(2\text{ m/s})(2\text{ m})}{2\text{ m/s} + 2\text{ m/s}} = \frac{(2\text{ m/s})(2\text{ m})}{(4\text{ m/s})} = 1\text{ m}$$

Second, calculate X for $v_A = 4$m/s and $v_B = 2$m/s:

$$X = \frac{v_A(2\text{ m})}{v_A + v_B} = \frac{(4\text{ m/s})(2\text{ m})}{4\text{ m/s} + 2\text{ m/s}} = \frac{(4\text{ m/s})(2\text{ m})}{(6\text{ m/s})} = 1.33\text{ m}$$

c) *Draw a conclusion*: We should compare the behavior of X we found in b) with the predicted behavior we discussed in a). What we see in b) is that X increased as v_A increased while keeping v_B constant. In other words when we tried numbers in the equation, X followed the trend we identified in a). Once again, consistency with expectations doesn't prove the equation is correct, but certainly gives us more confidence that we calculated it correctly.

General comment: The greatest difficulty I observe people having with doing limiting/special/trend cases is that they struggle to identify a "good" case. A good case is one in which you are able to predict the outcome without equations, just based on simple physical understanding of the behavior of the system. In the example above, let's suppose you decide to try a case like $v_A = 3$ m/s, and $v_B = 5$ m/s, how are you doing to be able to predict the collision position X without the equation? I can't. This is not a good case to use because it doesn't allow us to use any simply physical understanding of the system to predict X independently of the equation that we're testing.

3. Dimensional Analysis

This is probably the easiest evaluation method to understand. Dimensional analysis means that you check to see if your units are consistent. This follows the physics rule that you can't add apples and oranges. In other words, you can't add a force (in Newtons) to a time (in seconds) because they are different units. Likewise, if you have an equation to calculate a position like: $X = \frac{v_{\mathrm{A}}(2\text{ m})}{v_{\mathrm{A}} + v_B}$ your units on the right hand side of the equation should come out to be meters just like the left hand side of the equation. You can do a unit check as follows:

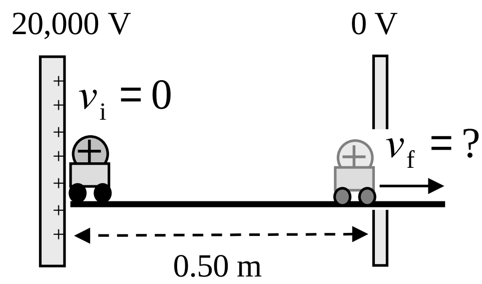
$$\left[\frac{v_{\mathrm{A}}(2\text{ m})}{v_{\mathrm{A}} + v_B}\right] = \frac{\text{m/s}\cdot\text{m}}{\text{m/s}+\text{m/s}} = \frac{\text{m}^2/\text{s}}{\text{m/s}} = \text{m}$$

What we found is that the units of the right hand side of the equation gave us meters at the end. This is consistent with the units of the left hand side of the equation: $[X] = \text{m}$. As with all evaluation techniques, it doesn't prove that the equation is correct, but it does give us more confidence that we calculated it correctly.

Note: The notation of putting square brackets around something means "the units of…" I.e., $[X]$ literally means "the units of X."

4. Independent Solution Method

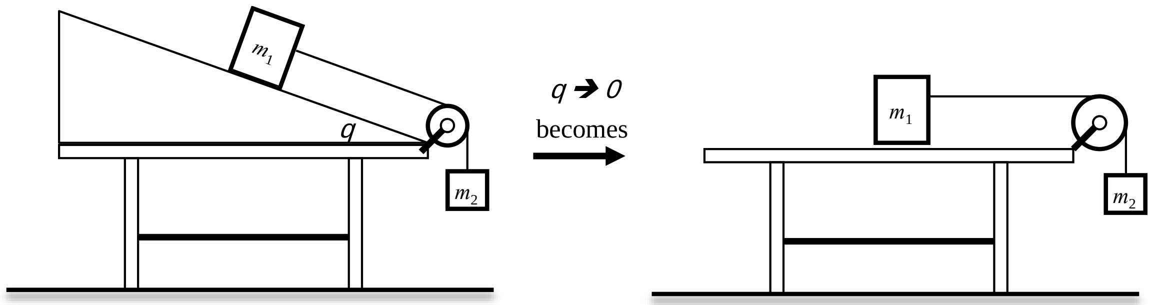
A 2.0 kg cart with a +2.0 x 10^{-4} C charge starts at rest and moves 0.50 m between two metal plates with a 20,000 V potential difference between them. The second plate has a hole in it that allows the cart to pass through. The supporting surface exerts a 6.0 N opposing friction force on the cart. How fast is the cart moving after it has traveled the 0.50 m across the 20,000 V potential difference?



This example can be solved using an energy approach or a forces approach. If I get the same answer both ways, using two completely different approaches, I am more confident that I got the right answer.

5. Reducing a solution to a known case

For example, suppose you are asked to solve an atwood machine type problem on a slope at a certain angle θ, you can let θ → 0, thereby turning your solution into the case of a block on a table with a block hanging over the end. If you already know the solution to the case of the flat table, you can test your slope solution by letting θ → 0 in your solution and see if it gives you the table solution.



6. Cross substitution

Cross substitution is a way to check that the answer you found in one part of a problem is consistent with the answer you found in another part of the problem. To see an example of this, let's return to the example of the two cars colliding:

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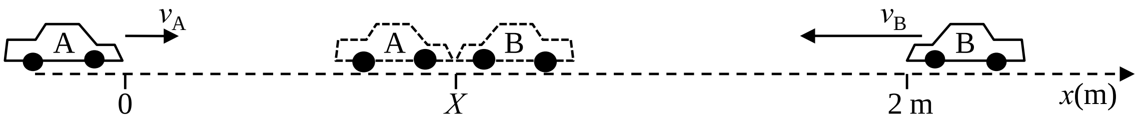

To solve this we start by setting up equations of motion for the position of car A and the position of car B starting from the general equation:

$$x(t) = x_0 + v_{0,x}t + \frac{1}{2}a_x t^2$$

For car A, $x_0 = 0$ m, $v_{0,x} = +3$ m/s, and $a_x = 0$. Therefore:

$$x_A(t) = +(3 \text{ m/s})t$$

For car B $x_0 = 2$ m, $v_{0,x} = -5$ m/s, and $a_x = 0$. Therefore:

$$x_B(t) = 2 \text{ m} - (5 \text{ m/s})t$$

To find the position where the two cars collide we can set these two position equations equal to each other and solve for time:

$$(3 \text{ m/s})t = 2 \text{ m} - (5 \text{ m/s})t \Rightarrow (8 \text{ m/s})t = 2 \text{ m} \Rightarrow t = 0.25 \text{ s}$$

Having found the time it takes for the cars to get from their starting point to the collision point, we can substitute back into the equation for x_A to find the collision point X:

$$X = +(3 \text{ m/s})(0.25 \text{ s}) = 0.75 \text{ m}$$

Now one way to check if we did this right is to do a "cross check" by substituting the time $t = 0.25$ s into the equation for x_B. Our expectation is that when we do this, we will get the same result as we did when we substituted $t = 0.25$ s into the equation for x_A:

$$X = (2 \text{ m}) - (5 \text{ m/s})(0.25 \text{ s}) = (2 \text{ m}) - (1.25 \text{ m}) = 0.75 \text{ m}$$

Indeed, we get the same answer! Our equation for x_B gives an answer that is consistent with our equation for x_A. Once again, this evaluation technique doesn't prove our calculations are correct. It is quite possible that we did the problem internally consistently while still doing the whole thing wrong.