Journal Club Background

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1 Introduction

Just to make clear what we are trying to do here, let's start with some overview of what Lanczos calls "vectorial mechanics" and work our way to the more generally applicable analytic mechanics. The goal of vectorial mechanics is to describe the motion of point particles due to the influence of external forces and other point particles. We assume a physical body, $\mathcal{B}$ to be a collection of particles such that every particle has an associated mass which is assumed to be a positive, non-zero real number. There exists some map

$$m : \mathcal{B} \to \mathbb{R}^+_*$$

simply meaning that every point in the body has mass. Every point in the body also has an associated position at any given time. That is, for every $b \in \mathcal{B}$, there exists a twice-differentiable function called the trajectory of the particle

$$\boldsymbol{r}_b : [0, t_f] \to \mathbb{R}^3$$

(we make no distinction here between the Euclidean space $\mathbb{E}^3$ and the real vector space $\mathbb{R}^3$ as the distinction just unnecessarily complicates matters). The key observation made by Isaac Newton in the 1680s was that the function $\boldsymbol{r}_b(t)$ obeys the relation

$$\boldsymbol{F}_b = m_b \frac{d^2 \boldsymbol{r}_b}{dt^2} \quad \text{for all } b \in \mathbb{B}$$

where the force acting, which is also a function of time in general, $\boldsymbol{F} : [0, t_f] \to \mathbb{R}^3$, depends on the nature of the system. Usually it depends on the distance between pairs of particles but at this level of generality, we make no claim about the nature of interactions (there are some general requirements on the force, namely some integrability requirements, but for now we'll not worry too much and assume it's reasonably well-behaved). We call the set of differential equations describing the positions of the particles the equations of motion of the system. In general, to solve the system of equations we need to know the initial positions of the particles $\boldsymbol{r}_b(0)$ and initial velocities of the particles $\frac{d\boldsymbol{r}_b}{dt}\big|_{t=0} \equiv \boldsymbol{v}(0)$. As we know, this is Newton's second law (although the original second law replaces the right-hand side with the time derivative of momentum, $\boldsymbol{p} : [0, t_f] \to \mathbb{R}^3$, defined by $m\frac{d\boldsymbol{r}}{dt}$, this definition is identical under the assumption that the body particles don't change mass). The first law follows naturally,

$$\boldsymbol{0} = m \frac{d^2 \boldsymbol{r}}{dt^2} \implies \frac{d\boldsymbol{r}}{dt} = const.$$

namely that if there is no force acting on a particle, it will move in a constant, straight-line path. This is an elementary statement of the conservation of linear momentum of a particle. Later, we will show that this is a result of a more general translational symmetry of nature, which will be applicable even to systems outside of the scope of classical mechanics where the notion of trajectory is not well-defined. The third law which claims that forces only come in pairs which sum to 0 is more of a statement about the nature of forces. For now, we'll take it to be true without justification (later on it will be justified, don't worry).

2 Example: 1D Simple Harmonic Oscillator

From here, in principle, if we knew the fundamental interactions between all particles and were somehow able to solve all the coupled second order ordinary differential equations, we would be done. Arguably the most important example of a system we can solve with vectorial methods is the 1D simple harmonic oscillator, defined as a single point with mass, m, subjected to a force given by

$$F = -kx$$

where x is the position of the point particle and k is a positive real number called the spring constant which describes the stiffness of the oscillator. The solution to the equation of motion,

$$-kx = m\frac{d^2x}{dt^2}$$

is oscillatory.

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}\ t\right) + B\cos\left(\sqrt{\frac{k}{m}}\ t\right)$$

where A and B are determined by the initial conditions of the system. To be clear about definitions, the quantity $\sqrt{\frac{k}{m}}$ is called the angular frequency of the oscillator and is denoted by ω. It is related to the number of oscillations per unit time, simply called the frequency and denoted by ν by $\omega = 2\pi\nu$. The period of oscillation, denoted by T, is defined by the amount of time it takes to make one complete oscillation, which can be found by solving $\omega T = 2\pi$, *i.e.,*

$$T = \frac{2\pi}{\omega}.$$

This simple system is important for modeling anything from a simple mass on a spring to the quantized modes of the electromagnetic field, *i.e.,* photons. It really can't be understated how fundamentally important it is.

3 Work and Energy

In general, we don't know all the interactions between particles and even if we did, it would be too difficult to solve all the equations of motion for practically all systems of interest. To circumvent this problem, we defined the work done by force $\boldsymbol{F}$ which only depends on position, in moving a particle along some parametric curve $\phi : [0,1] \to \mathbb{R}^3$, to be the line integral

$$W_\phi = \int_\phi \boldsymbol{F} \cdot d\boldsymbol{r}.$$

The line element is an object called differential one-form (which hopefully later in our club meetings we will formalize a bit more), given by,

$$d\boldsymbol{r} = \frac{d\boldsymbol{r}}{dt}dt = \boldsymbol{v}dt$$

By Newton's second law we have for any ϕ,

$$\begin{aligned}
\int_\phi \boldsymbol{F} \cdot d\boldsymbol{r} &= m\int_\phi \frac{d^2\boldsymbol{r}}{dt^2} \cdot d\boldsymbol{r} \\
&= m\int_\phi \frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v}\ \ dt \\
&= \frac{m}{2}\int_\phi \frac{d}{dt}(v^2)dt \\
&= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2.
\end{aligned}$$

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The quantity $\frac{1}{2}mv^2$ is called the kinetic energy of the particle and is denoted by T. This is a statement of the so-called Work-Energy theorem: The work done on a particle along some trajectory due to forces which only depend on the position of the particle is equal to the change in the kinetic energy of the particle.

$$W = T_f - T_i$$

If the work done on a particle around a closed loop is equal to zero,

$$\oint \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

the force field is conservative and hence $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$ which implies that there exists a function $V : \mathbb{R}^3 \to \mathbb{R}$ called the potential energy such that (questions about the differentiability and continuity of V are ignored for now),

$$\boldsymbol{F} = -\boldsymbol{\nabla}V.$$

The negative sign is a matter of convention, but we'll see why it's convenient in a moment. Hence, for any ϕ where $\boldsymbol{F}$ is conservative,

$$\begin{aligned} W_\phi &= -\int_\phi \boldsymbol{\nabla} V \cdot d\boldsymbol{r} \\ &= V_i - V_f \end{aligned}$$

and therefore by the Work-Energy theorem,

$$T_f - T_i = V_i - V_f$$

or

$$T_i + V_i = T_f + V_f.$$

The total energy is defined to be the sum of the kinetic energy and potential energy, so what we have shown is that for any conservative force field, the total energy is conserved along any trajectory taken by a particle.

4 Segway into Lagrangian Mechanics

At this point, I suggest we skip the following topics until later:

1. Rotational Motion
2. Fundamental Forces (i.e. Gravitational, Electromagnetic, etc)
3. Simplifications due to Center of Mass

and start talking about a more modern way of thinking about physics, namely an approach based on variational methods. The assumptions made in discussing varational mechanics are as follows:

1. Every mechanical system of s degrees of freedom can in principle be characterized completely by the list of differentiable functions called generalized coordinates, $(q_1, q_2, \ldots, q_s)$. To be clear, each $q_i : [0, t_f] \to \mathbb{R}$ is a real-valued function of time. It's important to emphasize that these numbers don't necessarily represent position, we only require that they uniquely describe a specific state of the entire mechanical system.

2. There exists a definite function of the generalized coordinates, their derivatives (called generalized velocities), and time denoted by $L(q_1, \ldots, q_s\ , \dot{q}_1, \ldots, \dot{q}_s\ , t)$ and called the Lagrangian of the mechanical system.

3. The trajectory taken by a mechanical system is that which extremizes the integral of the Lagrangian with respect to time, *i.e.*,

$$S = \int_0^{t_f} L(q_1, \ldots, q_s\ , \dot{q}_1, \ldots, \dot{q}_s\ , t)\, dt.$$

   The quantity S is called the action, and the principle that the action is extremized is called the principle of least action or Hamilton's principle.

Let's extremize the action. For simplicity let's consider a system with just one degree of freedom. Let $q(t)$ be the function which minimizes the action (it won't matter if we minimize or maximize the action, the equations of motion will turn out to be the same, if you want to, check it), then for any function which is "small" everywhere in the interval $[0, t_f]$, $\delta q(t)$ and zero at the starting and ending time, $\delta q(0) = \delta q(t_f) = 0$, replacing $q(t)$ by

$$q(t) + \delta q(t)$$

will increase S. We call $\delta q(t)$ the variation of the function $q(t)$. The change in S when $q(t)$ is replaced by $q(t) + \delta q(t)$ is called the first variation of the action and is dentoted by δS,

$$\begin{aligned} \delta S &= \int_0^{t_f} L(q + \delta q,\ \dot{q} + \delta\dot{q}, t) dt - \int_0^{t_f} L(q\ , \dot{q}\ , t)\, dt \\ &= \int_0^{t_f} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right) dt \end{aligned}$$

where we have made $\delta q(t)$ arbitrarily small everywhere (this is justified formally by the calculus of variations). We can integrate the second term by parts, and argue that for an extremum, the variation of S must be 0,

$$0 = \delta S = \frac{\partial L}{\partial \dot{q}}\delta q\Big|_0^{t_f} + \int_0^{t_f} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta q\, dt.$$

Since $\delta q(t)$ was arbitrary, this only follows if

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0.$$

If the system has more than one degree of freedom, say s degrees of freedom, the $q_i(t)$ are varied independently and hence we arrive at s equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0.$$

These are called the Euler-Lagrange's equations of the system.

5 The form of the Lagrangian

To guess at what the Lagrangian is for a real system, we can make some assumptions about the nature of space. We'll go back to a discussion in Cartesian coordinates in one dimension for now for simplicity (we'll call the position coordinate x), but the results generalize naturally. The first assumption we make is that for a free particle, the laws of physics should not change under translation in space (a pendulum should act the same here and 5 feet to the left as long as there are no other forces involved). This hints us to guess that the Lagrangian should not depend directly on the physical coordinates of the system. We also assume that the laws of physics shouldn't depend on what direction you're facing, so the Lagrangian probably shouldn't depend directly on the velocities of the system, but they do need to depend on something. The simplest quantity we can construct that follows these requirements is something like

$$L \propto v^2.$$

This argument can be made more formal by requiring that the Lagrangian be invariant under change of reference frame between moving frames (See Landau Mechanics section 4 if you'd like) but the main idea is straightforward. We call the constant of proportionality $\frac{m}{2}$ (the factor of 1/2 is there for later convenience and is a matter of convention). The Lagrangian of a free particle can be identified as what we have defined earlier to be the kinetic energy. Then the Euler Lagrange equation gives us

$$\frac{d}{dt}\left(\frac{\partial}{\partial v}\frac{1}{2}mv^2\right) = \frac{\partial}{\partial x}\frac{1}{2}mv^2$$

or

$$m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = 0$$

which agrees with Newton's second law for a free particle. If the particle is no longer free but has some potential energy $U(x)$, we claim that the choice

$$L = T - U$$

leads us to results that agree with Newton's second law. We have from the Euler-Lagrange equation

$$m\frac{d^2v}{dt^2} = -\frac{dU}{dx} = f$$

which is identical to what we found before. In general the Lagrangian is always the kinetic energy minus the potential energy. The utility of this approach is that scalar quantities are always invariant under a change of coordinates, so as long as we know how to write down the kinetic and potential energy in terms of any generalized coordinates of our choice, $L = T - U$ still holds and we can use the Euler-Lagrange equations to find the equations of motion of out system.