Name KEY _______ Date ________________________ Period ______

Worksheet 2.6—The Chain Rule

Short Answer

Show all work, including rewriting the original problem in a more useful way. No calculator unless otherwise stated.

1. Find the derivative of the following functions with respect to the independent variable. (You do not need to simplify your final answers here.)

(a) $y = (2x-7)^3$

$\frac{dy}{dx} = 3(2x-7)^2 \cdot (2)$

$\frac{dy}{dx} = 6(2x-7)^2$

(b) $y = \dfrac{1}{t^2+3t-1}$

$y = (t^2+3t-1)^{-1}$

$\frac{dy}{dt} = (-1)(t^2+3t-1)^{-2} \cdot (2t+3)$

$\frac{dy}{dt} = -\frac{2t+3}{(t^2+3t-1)^2}$

(c) $y = \left(\dfrac{1}{t-3}\right)^2$

$y = ((t-3)^{-1})^2$

$y = (t-3)^{-2}$

$\frac{dy}{dt} = -2(t-3)^{-3} \cdot (1)$

$\frac{dy}{dt} = \frac{-2}{(t-3)^3}$

(d) $y = \csc^3\left(\dfrac{3x}{2}\right)$

$y = \left(\csc\left(\frac{3}{2}x\right)\right)^3$

$\frac{dy}{dx} = 3\left(\csc\left(\frac{3}{2}x\right)\right)^2 \cdot \left(-\csc\left(\frac{3}{2}x\right)\cot\left(\frac{3}{2}x\right)\right) \cdot \left(\frac{3}{2}\right)$

$\frac{dy}{dx} = -\frac{9}{2}\csc^3\left(\frac{3}{2}x\right)\cot\left(\frac{3}{2}x\right)$

(e) $y = 3\sec^2(\pi t - 1)$

$y = 3\left[\sec(\pi t - 1)\right]^2$

$\frac{dy}{dt} = 6\left[\sec(\pi t - 1)\right]^1 \cdot \sec(\pi t - 1)\tan(\pi t - 1) \cdot \pi$

$\frac{dy}{dt} = 6\pi \sec^2(\pi t - 1)\tan(\pi t - 1)$

(f) $y = \sin\sqrt[3]{x} + \sqrt[3]{\sin x}$

$y = \sin(x^{1/3}) + (\sin x)^{1/3}$

$\frac{dy}{dx} = \cos(x^{1/3}) \cdot \left(\frac{1}{3}x^{-2/3}\right) + \frac{1}{3}(\sin x)^{-2/3} \cdot (\cos x)$

$\frac{dy}{dx} = \frac{\cos\sqrt[3]{x}}{3\sqrt[3]{x^2}} + \frac{\cos x}{3\sqrt[3]{\sin^2 x}}$

(g) $y = x^2 \tan\dfrac{1}{x}$

$y = (x^2)(\tan(x^{-1}))$

$\frac{dy}{dx} = (2x)(\tan(x^{-1})) + (x^2)(\sec^2(x^{-1}) \cdot (-1)(x^{-2}))$

$\frac{dy}{dx} = 2x\tan\left(\frac{1}{x}\right) - \sec^2\left(\frac{1}{x}\right)$

(h) $r = \sec(2\theta)\tan(2\theta)$

$\frac{dr}{d\theta} = \sec 2\theta \cdot \tan 2\theta \cdot 2 \cdot \tan 2\theta + \sec 2\theta \cdot \sec^2 2\theta \cdot 2$

$\frac{dr}{d\theta} = 2\sec 2\theta \cdot \tan^2 2\theta + 2\sec^3 2\theta$

(i) $f(x) = \sqrt[3]{\csc^5 7}$

$f'(x) = 0$

($\sqrt[3]{\csc^5 7}$ is an awesome Konstant)

2. Find the equation of the tangent line (in Taylor Form) for each of the following at the indicated point.

(a) $s(t)=\sqrt{t^2+2t+8}$ at $x=2$

$s(t)=(t^2+2t+8)^{1/2}$ pt: $(2, s(2))=(2,4)$

$s'(t)=\frac{1}{2}(t^2+2t+8)^{-1/2}\cdot(2t+2)$

$s'(t)=\dfrac{2(t+1)}{2\sqrt{t^2+2t+8}}$

$s'(t)=\dfrac{t+1}{\sqrt{t^2+2t+8}}$

$s'(2)=\frac{3}{4}$

equation: $y=s(2)+s'(2)(x-2)$

$y=4+\frac{3}{4}(x-2)$

(b) $f(t)=\dfrac{3t+2}{t-1}$ at $(0,-2)$

$f'(t)=\dfrac{(t-1)(3)-(3t+2)(1)}{(t-1)^2}$

$f'(0)=\dfrac{-3-2}{1}=-5$

eq: $y=-2-5(x-0)$

3. Determine the point(s) in the interval $(0,2\pi)$ at which the graph of $f(x)=2\cos x+\sin 2x$ has a horizontal tangent.

$f(x)=2\cos x+\sin 2x$

$f'(x)=-2\sin x+2\cos 2x=0$

$f'(x)=-2\sin x+2\left[\cos^2 x-\sin^2 x\right]=0$

$f'(x)=-2\sin x+2\cos^2 x-2\sin^2 x=0$

$f'(x)=-2\sin x+2(1-\sin^2 x)-2\sin^2 x=0$

$f'(x)=-2\sin x+2-2\sin^2 x-2\sin^2 x=0$

$f'(x)=-4\sin^2 x-2\sin x+2=0$

$f'(x)=-2(2\sin^2 x+\sin x-1)=0$

$f'(x)=-2(2\sin x-1)(\sin x+1)=0$

$2\sin x-1=0$, $\sin x=\frac{1}{2}$, $x=\pi/6,\ 5\pi/6$

$\sin x+1=0$, $\sin x=-1$, $x=\frac{3\pi}{2}$

or

*"simplify early and often."

$f(x)=2\cos x+\sin 2x$

$f(x)=2\cos x+2\sin x\cos x$

$f(x)=2\cos x(1+\sin x)$

$f'(x)=-2\sin x(1+\sin x)+2\cos x(\cos x)$

$f'(x)=-2\sin x-2\sin^2 x+2\cos^2 x$

$f'(x)=-2\sin^2 x-2\sin x+2(1-\sin^2 x)$

$f'(x)=-2\sin^2 x-2\sin x+2-2\sin^2 x$

$f'(x)=-4\sin^2 x-2\sin x+2$

$f'(x)=-2(2\sin^2 x+\sin x-1)=0$

$f'(x)=-2(2\sin x-1)(\sin x+1)=0$

$2\sin x-1=0$, $\sin x=\frac{1}{2}$, $x=\pi/6,\ 5\pi/6$

$\sin x+1=0$, $\sin x=-1$, $x=\frac{3\pi}{2}$

4. Find the second derivative of each of the following functions. Remember to simplify early and often.

(a) $f(x)=2(x^2-1)^3$

$f'(x)=6(x^2-1)^2(2x)$

$f'(x)=12x(x^2-1)^2$

$f''(x)=(12)(x^2-1)^2+(12x)(2(x^2-1)^1\cdot(2x))$

$f''(x)=12(x^2-1)^2+48x^2(x^2-1)$ *factor out least powers.

or

$f''(x)=12(x^2-1)\left[(x^2-1)+4x^2\right]$

$f''(x)=12(x^2-1)(5x^2-1)$

(b) $f(x)=\sin(x^2)$

$f'(x)=\cos(x^2)\cdot(2x)$

$=2x\cos(x^2)$

$f''(x)=2\cos(x^2)+2x(-\sin(x^2)\cdot 2x)$

$f''(x)=2\cos(x^2)-4x^2\sin(x^2)$

5. If $h(x)=\tan(2x)$, evaluate $h''(x)$ at $\left(\frac{\pi}{6},\sqrt{3}\right)$. Simplify early and often.

$h'(x) = \sec^2(2x)\cdot 2$

$h'(x) = 2\sec^2(2x)$

$h'(x) = 2[\sec(2x)]^2$

$h''(x) = 4[\sec(2x)]^1\cdot\sec(2x)\tan(2x)\cdot 2$

$h''(x) = 8\sec^2(2x)\tan(2x)$

$h''\left(\frac{\pi}{6}\right) = 8\left(\sec\frac{\pi}{3}\right)^2\cdot\tan\left(\frac{\pi}{3}\right)$ $\quad \frac{\pi}{3}:\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

$= 8(2)^2\cdot(\sqrt{3})$

$= \boxed{32\sqrt{3}}$

6. If $g(5)=-3$, $g'(5)=6$, $h(5)=3$, and $h'(5)=-2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x)=\dfrac{g(x)}{h(x)}$

$f'(x) = \dfrac{h(x)\cdot g'(x) - g(x)\cdot h'(x)}{h^2(x)}$

$f'(5) = \dfrac{h(5)\cdot g'(5) - g(5)\cdot h'(5)}{(h(5))^2}$

$= \dfrac{(3)(6)-(-3)(-2)}{(3)^2}$

$= \dfrac{18-6}{9}$

$= \dfrac{12}{9}$

$f'(5) = \dfrac{4}{3}$

(b) $f(x)=g(h(x))$

$f'(x) = g'(h(x))\cdot h'(x)$

$f'(5) = g'(h(5))\cdot h'(5)$

$= g'(3)\cdot(-2)$

can't do. need $g'(3)$

(c) $f(x)=g(x)h(x)$

$f'(x) = g'(x)\cdot h(x) + g(x)\cdot h'(x)$

$f'(5) = g'(5)\cdot h(5) + g(5)\cdot h'(5)$

$= (6)(3) + (-3)(-2)$

$= 18+6$

$f'(5) = 24$

(d) $f(x)=[g(x)]^3$

$f'(x) = 3(g(x))^2\cdot g'(x)$

$f'(5) = 3(g(5))^2\cdot g'(5)$

$= 3(-3)^2(6)$

$= 27\cdot 6$

$= 162$

(e) $f(x)=g(x+h(x))$

$f'(x) = g'(x+h(x))\cdot(1+h'(x))$

$f'(5) = g'(5+h(5))\cdot(1+h'(5))$

$= g'(5+3)\cdot(1+(-2))$

$= g'(8)\cdot(-1)$

$= -g'(8)$

We are not given $g'(8)$, so we cannot proceed with this problem, but we can and may and will proceed to another awesome problem.

(f) $f(x)=(g(x)+h(x))^{-2}$

$f'(x) = -2(g(x)+h(x))^{-3}\cdot(g'(x)+h'(x))$

$f'(5) = \dfrac{-2(g'(5)+h'(5))}{(g(5)+h(5))^2}$

$= \dfrac{-2(6-2)}{(-3+3)^2}$

$= \dfrac{-8}{0}$

$=$ DNE (f has a vertical tangent line at $x=5$)

7. Find the derivative of $f(x) = \sin^2 x + \cos^2 x$ two different ways,

(a) By using the chain rule on the given expression.

$f(x) = (\sin x)^2 + (\cos x)^2$

$f'(x) = 2(\sin x)^1 \cdot (\cos x) + 2(\cos x)^1 \cdot (-\sin x)$

$f'(x) = 2\sin x \cdot \cos x - 2\sin x \cdot \cos x$

$f'(x) = 0$

(b) By using an identity first, then differentiating.

$f(x) = \sin^2 x + \cos^2 x$

$f(x) = 1$ (Pappa PID)

$f'(x) = 0$

(c) What's the moral of THIS story? (Hint: It is NOT *"Flattery is a dangerous weapon in the hands of the enemy."*)

"Simplify early and often."

8. Using calculus and trig Identities, prove that if $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$, then $f'(x) = g'(x)$.

None needed

$f(x) = (\tan x)^2$

$f'(x) = 2(\tan x)^1 \cdot \sec^2 x$

$= 2\sec^2 x \tan x$

$g(x) = (\sec x)^2$

$g'(x) = 2(\sec x)^1 \cdot (\sec x \tan x)$

$= 2\sec^2 x \tan x$

9. Using the chain rule,

(a) Prove that the derivative of an odd function is an even function. That is if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

Let $f(x)$ be an odd function such that $f(-x) = -f(x)$

Differentiating both sides:

$\frac{d}{dx}[f(-x)] = \frac{d}{dx}[-f(x)]$

$f'(-x) \cdot (-1) = -f'(x)$ ÷multiply both sides by -1

$f'(-x) = f'(x)$

So $f'(x)$ is an even function.

(b) What type of function do you think the derivative of an even function is? Justify in a manner similar to part (a).

if f is even, $f(-x) = f(x)$

So, $\frac{d}{dx}[f(-x)] = \frac{d}{dx}[f(x)]$

$f'(-x) \cdot (-1) = f'(x)$

$f'(-x) = -f'(x)$

So, $f'(x)$ is an odd function

10. As demonstrated on the last example in the notes,

(a) Using the chain rule, prove that if $|g(x)| = \sqrt{g^2(x)}$ then $\frac{d}{dx}\left[|g(x)|\right] = \frac{g(x)}{|g(x)|} \cdot g'(x),\ g(x) \neq 0$.

Let $f(x) = |g(x)|$

$f(x) = \sqrt{(g(x))^2}$

$f(x) = ((g(x))^2)^{1/2}$ *don't simplify here

$f'(x) = \frac{1}{2}((g(x))^2)^{-1/2} \cdot 2(g(x))^1 \cdot g'(x)$

$f'(x) = \frac{g(x) \cdot g'(x)}{\sqrt{(g(x))^2}}$

$f'(x) = \frac{g(x)}{|g(x)|} \cdot g'(x)$

(b) Use the result from part (a) to find $\frac{d}{dx}\left[|x^2 - 4|\right]$. , Let $g(x) = x^2 - 4$

So $g'(x) = \frac{x^2-4}{|x^2-4|}(2x)$

$g'(x) = \frac{2x(x^2-4)}{|x^2-4|}$

11. What is the largest value possible for the slope of the curve of $y = \sin\left(\frac{x}{2}\right)$? Justify.

$y' = \cos\left(\frac{x}{2}\right) \cdot \frac{1}{2}$

$y' = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

Range of y': $\left[-\frac{1}{2}, \frac{1}{2}\right]$,

So max slope of y is $\frac{1}{2}$.

12. Find the equation of the normal line to the curve $y = 2\tan\left(\frac{\pi x}{4}\right)$ at $x = 1$.

$y' = 2\sec^2\left(\frac{\pi}{4}x\right) \cdot \left(\frac{\pi}{4}\right)$

$y'(1) = 2\left(\sec\frac{\pi}{4}\right)^2\left(\frac{\pi}{4}\right)$

$y'(1) = \frac{\pi}{2}(\sqrt{2})^2$

$y'(1) = \pi$ = slope of tangent line

pt: $y(1) = 2\tan\left(\frac{\pi}{4}\right)$

$y(1) = 2$

m_N = slope of normal line $= -\frac{1}{\pi}$ (opp. recip)

equation

$m_N = -\frac{1}{\pi}$, pt: $(1,2)$

$y = 2 - \frac{1}{\pi}(x-1)$

13. After the chain rule is applied to find the derivative of a function $F(x)$, the function $F'(x) = f(x) = 4(\cos(3x))^3 \cdot (-\sin(3x)) \cdot 3$ is obtained. Give a possible function for $F(x)$. Check your work by taking the derivative of your guess using the chain rule.

F(x) could be

$F(x) = [\cos(3x)]^4$

$= \cos^4(3x)$

Multiple Choice

A 14. If $f(x)=\dfrac{1}{\sqrt{x^2+3}}$, find $f'(x)$.

(A) $f'(x)=-\dfrac{x}{\sqrt{\left(x^2+3\right)^3}}$

(B) $f'(x)=\dfrac{x}{\sqrt{x^2+3}}$

(C) $f'(x)=-\dfrac{x}{\left(x^2+3\right)\sqrt{2x}}$

(D) $f'(x)=-\dfrac{1}{2\sqrt{\left(x^2+3\right)^3}}$

(E) $f'(x)=-\dfrac{x^2+3x}{x^2+3}$

$f(x) = (x^2+3)^{-1/2}$

$f'(x) = -\frac{1}{2}(x^2+3)^{-3/2} \cdot (2x)$

$f'(x) = \dfrac{-x}{\sqrt{(x^2+3)^3}}$

C 15. If $g(x)=(1-x)^3(4x+1)$, then $g'(x)=$

(A) $-12(1-x)^2$

(B) $(1-x)^2(1+8x)$

(C) $(1-x)^2(1-16x)$

(D) $3(1-x)^2(4x+1)$

(E) $(1-x)^2(16x+7)$

$g'(x)=3(1-x)^2(-1)(4x+1)+(1-x)^3(4)$

$g'(x) = (1-x)^2\left[-3(4x+1)+4(1-x)\right]$ *factor out $(1-x)^2$

$g'(x) = (1-x)^2\left[-12x-3+4-4x\right]$

$g'(x) = (1-x)^2(-16x+1)$

$g'(x) = (1-x)^2(1-16x)$

D 16. $\frac{d}{dx}\left[\left(\frac{x^2-3}{5x^2-9}\right)^5\right] = 5\left(\frac{x^2-3}{5x^2-9}\right)^4 \cdot \left(\frac{(5x^2-9)(2x)-(x^2-3)(10x)}{(5x-9)^2}\right)$

$= \frac{5(x^2-3)^4\left[(5x^2-9)(2x)-(10x)(x^2-3)\right]}{(5x^2-9)^4(5x-9)^2}$

$= \frac{5(x^2-3)^4\left[2x((5x^2-9)-5(x^2-3)\right]}{(5x-9)^6}$

$= \frac{5(x^2-3)^4(2x)\left[5x^2-9-5x^2+15\right]}{(5x-9)^6}$

$= \frac{10x(x^2-3)^4(6)}{(5x-9)^6}$

$= \frac{60x(x^2-3)^4}{(5x-9)^6}$

Love that Algebra!

(A) $\frac{10x(x^2-3)^4(10x^2-17)}{(5x^2-9)^6}$

(B) $\frac{-10x(x^2-3)^4(5x^2-16)}{(5x^2-9)^5}$

(C) $\frac{-240x(x^2-3)^4}{(5x^2-9)^6}$

(D) $\frac{60x(x^2-3)^4}{(5x^2-9)^6}$

(E) $\frac{100x(x^2-3)^4}{(5x^2-9)^6}$

D 17. A derivative of a function $f(x)$ is obtained using the chain rule. The result is $f'(x)=3\sec^3 x\tan x$. Which of the following could be $f(x)$?

I. $f(x)=-\pi+\frac{3}{4}\sec^4 x \rightarrow f'(x)=3\sec^3 x\cdot\sec x\tan x = 3\sec^4 x\tan x$ ✗

II. $f(x)=8+\sec^3 x \rightarrow f'(x)=3\sec^2 x\cdot\sec x\cdot\tan x = 3\sec^3 x\tan x$ ✓

III. $f(x)=\sec x+\sec x\tan^2 x$

$f(x) = \sec x+\sec x(\sec^2 x-1)$

$f(x) = \sec x+\sec^3 x-\sec x = \sec^3 x \rightarrow f'(x)=3(\sec^2 x)\cdot\sec x\cdot\tan x = 3\sec^3 x\cdot\tan x$ ✓

(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III