Calculus page 511ff solutions

1) The expansion is of the form of $\frac{1}{1-x}$ centered at $x = -5$. The radius of convergence is 1, so this will be an identity for $-6 < x < -4$.

2) The expansion is of the form of $\frac{1}{1-x}$ centered at $x = 0$. The radius of convergence is 1, so this will be an identity for $-1 < x < 1$.

3) $\frac{x^{3n}}{2n!+1} \leq \frac{(x^3)^n}{n!}$, which is the Taylor series term for e^{x^3}, which converges for all x, so by direct comparison the original series must converge.

4) $\frac{x^{2n}}{n!+2} \leq \frac{(x^2)^n}{n!}$, which is the Taylor series term for e^{x^2}, which converges for all x, so by direct comparison the original series must converge.

5) $\frac{(\cos x)^n}{n!+1} \leq \frac{1}{n!}$, which is the Taylor series term for e^1, which converges to e, so by direct comparison the original series must converge.

6) $\frac{2(\sin x)^n}{n!+3} \leq \frac{2}{n!}$, which is the Taylor series term for e^2, which converges to e^2, so by direct comparison the original series must converge.

7) Geometric Series: converges if $|x| < 1$. Radius of convergence is 1.

8) Geometric Series: converges if $|x + 5| < 1$. Radius of convergence is 1.

9) Geometric Series: converges if $|4x + 1| < 1$. Radius of convergence is ¼.

10) $\frac{(3x-2)^n}{n} < (3x-2)^n$, which is geometric, and converges if $|3x - 2| < 1$. Radius of convergence is 1/3.

11) Geometric Series: converges if $\left|\frac{x-2}{10}\right| < 1$. Radius of convergence is 10.

12) Ratio Test: $\lim_{n\to\infty}\left(\frac{(n+1)\left|x^{n+1}\right|}{n+3}\right)\left(\frac{n+2}{n\left|x^n\right|}\right) = \lim_{n\to\infty}\left(\frac{(n+1)(n+2)}{n(n+3)}\right)(|x|) = |x|$, which converges if $|x| < 1$. Radius of convergence is 1.

13) $\frac{x^n}{n\sqrt{n}\,3^n} \leq \left(\frac{x}{3}\right)^n$, which is geometric, and converges if $\left|\frac{x}{3}\right| < 1$. Radius of convergence is 3.

14) $\dfrac{x^{2n+1}}{n!} = x\left(\dfrac{x^{2n}}{n!}\right) = x\left(\dfrac{\left(x^2\right)^n}{n!}\right)$, which is xe^x, and converges for all values of x. Radius of convergence is ∞.

15) Ratio Test: $\lim_{n\to\infty}\left(\dfrac{(n+1)\left|(x+3)^{n+1}\right|}{5^{n+1}}\right)\left(\dfrac{5^n}{n\left|(x+3)^n\right|}\right) = \lim_{n\to\infty}\left(\dfrac{n+1}{n}\right)\left(\dfrac{|x+3|}{5}\right) = \dfrac{|x+3|}{5}$

converges if $\dfrac{|x+3|}{5} < 1$. Radius of convergence is 5.

16) Ratio Test: $\lim_{n\to\infty}\left(\dfrac{(n+1)\left|(x)^{n+1}\right|}{4^{n+1}\left((n+1)^2+1\right)}\right)\left(\dfrac{4^n\left(n^2+1\right)}{n\left|(x)^n\right|}\right) = \lim_{n\to\infty}\left(\dfrac{(n+1)\left(n^2+1\right)}{n\left((n+1)^2+1\right)}\right)\left(\dfrac{|x|}{4}\right) = \dfrac{|x|}{4}$, which converges if $\left|\dfrac{x}{4}\right| < 1$.

Radius of convergence is 4.

17) Fails the nth term test. Converges only if $x = 4$. Radius of convergence is 0.

18) Ratio Test: $\lim_{n\to\infty}\left(\dfrac{\sqrt{n+1}\left|(x)^{n+1}\right|}{3^{n+1}}\right)\left(\dfrac{3^n}{\sqrt{n}\left|(x)^n\right|}\right) = \lim_{n\to\infty}\left(\sqrt{1+\dfrac{1}{n}}\right)\left(\dfrac{|x|}{3}\right) = \dfrac{|x|}{3}$, which converges if $\left|\dfrac{x}{3}\right| < 1$. Radius of convergence is 3.

19) Ratio Test: $\lim_{n\to\infty}\left(\dfrac{(n+2)\left|(2-2x)^{n+1}\right|}{(n+1)\left|(2-2x)^n\right|}\right) = \lim_{n\to\infty}\left(\dfrac{n+2}{n+1}\right)\left(|2-2x|\right) = |2-2x|$, which converges if $|2-2x| < 1$. Radius of convergence is ½.

20) Ratio Test: $\lim_{n\to\infty}\left(\dfrac{\left|(4x-5)^{2n+3}\right|}{(n+1)^{3/2}}\right)\left(\dfrac{n^{3/2}}{\left|(4x-5)^{2n+1}\right|}\right) = \lim_{n\to\infty}\left(\dfrac{n}{n+1}\right)^{3/2}\left(|4x-5|\right)^2 = \left(|4x-5|\right)^2$, which converges if $|4x-5| < 1$. Radius of convergence is ¼.

21) Ratio Test: $\lim_{n\to\infty}\left(\dfrac{\left|(x+\pi)^{n+1}\right|}{\sqrt{n+1}}\right)\left(\dfrac{\sqrt{n}}{\left|(x+\pi)^n\right|}\right) = \lim_{n\to\infty}\left(\sqrt{\dfrac{n}{n+1}}\right)\left(|x+\pi|\right) = |x+\pi|$, which converges if $|x+\pi| < 1$. Radius of convergence is 1.

22) Ratio Test: $\lim_{n\to\infty}\left(\frac{\left|\left(x-\sqrt{2}\right)^{2n+3}\right|}{2^{n+1}}\right)\left(\frac{2^n}{\left|\left(x-\sqrt{2}\right)^{2n+1}\right|}\right)=\lim_{n\to\infty}\left(\frac{\left|\left(x-\sqrt{2}\right)^2\right|}{2}\right)=\frac{\left|\left(x-\sqrt{2}\right)^2\right|}{2}$, which converges if $\frac{\left|\left(x-\sqrt{2}\right)^2\right|}{2}<1$. Radius of convergence is $\sqrt{2}$.

23) $\frac{(x-1)^{2n}}{4^n}=\frac{(x-1)^{2n}}{\left(2^2\right)^n}=\frac{(x-1)^{2n}}{2^{2n}}=\left(\frac{x-1}{2}\right)^{2n}$. This is geometric, so the interval of convergence is $-1<\frac{x-1}{2}<1\Rightarrow -2<x-1<2\Rightarrow -1<x<3$. The sum is $\frac{1}{1-\left(\frac{x-1}{2}\right)^2}=\frac{4}{4-(x-1)^2}$

24) $\frac{(x+1)^{2n}}{9^n}=\frac{(x+1)^{2n}}{\left(3^2\right)^n}=\frac{(x+1)^{2n}}{3^{2n}}=\left(\frac{x+1}{3}\right)^{2n}$. This is geometric, so the interval of convergence is $-1<\frac{x+1}{3}<1\Rightarrow -3<x+1<3\Rightarrow -4<x<2$. The sum is $\frac{1}{1-\left(\frac{x+1}{3}\right)^2}=\frac{9}{9-(x+1)^2}$

25) This is geometric, so the interval of convergence is

$$-1<\frac{\sqrt{x}}{2}-1<1\Rightarrow -1<\frac{\sqrt{x}-2}{2}<1\Rightarrow -2<\sqrt{x}-2<2\Rightarrow 0<\sqrt{x}<4\Rightarrow 0<x<16.$$

The sum is $\frac{1}{1-\frac{\sqrt{x}-2}{2}}=\frac{2}{4-\sqrt{x}}$

26) This is geometric, so the interval of convergence is $-1<\ln x<1\Rightarrow e^{-1}<x<e\Rightarrow \frac{1}{e}<x<e.$

The sum is $\frac{1}{1-\ln x}$

27) This is geometric, so the interval of convergence is

$$-1<\frac{x^2-1}{3}<1\Rightarrow -3<x^2-1<3\Rightarrow -2<x^2<4\Rightarrow -2<x<2.$$

The sum is $\frac{1}{1-\frac{x^2-1}{3}}=\frac{3}{4-x^2}$

28) This is geometric, so the interval of convergence is $-1<\frac{\sin x}{2}<1\Rightarrow -2<\sin x<2$. This is true for all real x.

The sum is $\frac{1}{1-\frac{\sin x}{2}}=\frac{2}{2-\sin x}$

29) $\lim_{n\to\infty}\frac{n}{n+1}=1$, so the series diverges by the nth term test.

30) $\lim_{n\to\infty}\frac{2^n}{n+1}=(\text{L'H})\lim_{n\to\infty}\frac{2^n\ln 2}{1}=\infty$, so the series diverges by the nth term test.

31) $\lim_{n\to\infty}\frac{n^2-1}{2^n}=(\text{L'H})\lim_{n\to\infty}\frac{2n}{2^n\ln 2}=(\text{L'H})\lim_{n\to\infty}\frac{2}{2^n(\ln 2)^2}=0$, so the series does **not** fail the nth term test.

Ratio test: $\lim_{n\to\infty}\left(\frac{(n+1)^2-1}{2^{n+1}}\right)\left(\frac{2^n}{n^2-1}\right)=\lim_{n\to\infty}\left(\frac{(n+1)^2-1}{2(n^2-1)}\right)=\frac{1}{2}<1$, so the series converges.

32) The series is geometric, with $r=\frac{1}{8}<1$, so the series converges.

33) $\lim_{n\to\infty}\frac{2^n}{3^n+1}=(\text{L'H})\lim_{n\to\infty}\frac{2^n\ln 2}{3^n\ln 3}=(\text{RW})\lim_{n\to\infty}\left(\frac{2}{3}\right)^n\frac{\ln 2}{\ln 3}=0$, so the series does **not** fail the nth term test.

Ratio test: $\lim_{n\to\infty}\left(\frac{2^{n+1}}{3^{n+1}+1}\right)\left(\frac{3^n+1}{2^n}\right)=\lim_{n\to\infty}\left(\frac{2(3^n)+2}{3^{n+1}+1}\right)=(\text{L'H})\lim_{n\to\infty}\left(\frac{2(3^n)\ln 3}{3^{n+1}(\ln 3)}\right)$

$(\text{RW})\lim_{n\to\infty}\left(\frac{2(3^n)}{3(3^n)}\right)=\frac{2}{3}<1$

so the series converges.

34) Direct comparison test: $n\sin\left(\frac{1}{n}\right)\le -n$, which diverges (nth term test)

35) $\lim_{n\to\infty}\frac{n^2}{e^n}=(\text{L'H})\lim_{n\to\infty}\frac{2n}{e^n}=(\text{L'H})\lim_{n\to\infty}\frac{2}{e^n}=0$, so the series does **not** fail the nth term test.

Ratio test: $\lim_{n\to\infty}\left(\frac{(n+1)^2}{e^{n+1}}\right)\left(\frac{e^n}{n^2}\right)=\lim_{n\to\infty}\left(\frac{(n+1)^2}{e(n^2)}\right)=\frac{1}{e}<1$, so the series converges.

36) $\lim_{n\to\infty}\frac{n^{10}}{10^n} = \left(\text{L'H many times}\right)\lim_{n\to\infty}\frac{10!}{10^n\left(\ln 10\right)^{10}} = 0$, so the series does **not** fail the *n*th term test.

Ratio test: $\lim_{n\to\infty}\left(\frac{\left(n+1\right)^{10}}{10^{n+1}}\right)\left(\frac{10^n}{n^{10}}\right) = \lim_{n\to\infty}\left(\frac{\left(n+1\right)^{10}}{10\left(n^{10}\right)}\right) = \frac{1}{10} < 1$, so the series converges.

37) $\lim_{n\to\infty}\frac{(n+3)!}{3!n!3^n} = \left(\text{RW}\right)\lim_{n\to\infty}\frac{\left(n+3\right)\left(n+2\right)\left(n+1\right)}{3!\left(3^n\right)} = \left(\text{L'H 3 times}\right)\lim_{n\to\infty}\frac{6}{3!\left(3^n\right)\left(\ln 3\right)^3} = 0$,

so the series does **not** fail the *n*th term test.

Ratio test: $\lim_{n\to\infty}\left(\frac{\left(n+4\right)!}{3!\left(n+1\right)!3^{n+1}}\right)\left(\frac{3!n!3^n}{\left(n+3\right)!}\right) = \lim_{n\to\infty}\left(\frac{\left(n+4\right)}{\left(n+1\right)\left(3\right)}\right) = \frac{1}{3} < 1$, so the series converges.

38) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \left(\text{RW}\right)\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \left(\text{logarithms}\right)\lim_{n\to\infty} n\ln\left(\frac{n+1}{n}\right) = \left(\text{RW}\right)\lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{n}\right)}{\frac{1}{n}}$

$= \left(\text{L'H}\right)\lim_{n\to\infty}\frac{\frac{n}{n+1}\left(\frac{n-\left(n+1\right)}{n^2}\right)}{-\frac{1}{n^2}} = \left(\text{RW}\right)\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$

so the series diverges, as it fails the *n*th term test.

39) The series converges, as it is geometric with $r = -\frac{2}{3} < 1$.

40) I tried the *n*th term test, but couldn't make anything of it, so I tried the

Ratio Test: $\lim_{n\to\infty}\left(\frac{(n+1)!}{e^{n+1}}\right)\left(\frac{e^n}{n!}\right) = \left(\text{RW}\right)\lim_{n\to\infty}\frac{\left(n+1\right)}{e} = \infty$, so the series diverges.

41) $\lim_{n\to\infty}\frac{3^n}{n^3 2^n} = \left(\text{L'H}\right)\lim_{n\to\infty}\frac{3^n\ln 3}{n^3\left(2^n\ln 2\right)+3n^2 2^n}$ nth term test seems to be going nowhere....

Ratio Test: $\lim_{n\to\infty}\left(\frac{3^{n+1}}{(n+1)^3 2^{n+1}}\right)\left(\frac{n^3 2^n}{3^n}\right) = \lim_{n\to\infty}\frac{3n^3}{(n+1)^3\left(2\right)} = \frac{3}{2} > 1$, so the series diverges.

42) $\lim_{n\to\infty}\frac{n\ln n}{2^n} = (\text{L'H})\lim_{n\to\infty}\frac{n\left(\frac{1}{n}\right)+\ln n}{2^n\ln 2} = (\text{L'H})\lim_{n\to\infty}\frac{\frac{1}{n}}{2^n(\ln 2)^2} = 0$, so the series does **not** fail the *n*th term test.

Ratio Test: $\lim_{n\to\infty}\frac{(n+1)\ln(n+1)}{2^{n+1}}\left(\frac{2^n}{n\ln n}\right) = \lim_{n\to\infty}\frac{(n+1)\ln(n+1)}{2n\ln n} = \frac{1}{2} < 1$, so the series converges.

43) $\lim_{n\to\infty}\frac{n!}{(2n+1)!} = ?????$ No clue where to take this. Try the ratio test.

Ratio Test: $\lim_{n\to\infty}\frac{(n+1)!}{(2n+3)!}\left(\frac{(2n+1)!}{n!}\right) = \lim_{n\to\infty}\frac{(n+1)}{(2n+3)(2n+2)} = 0 < 1$, so the series converges.

44) Ratio Test: $\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\left(\frac{n^n}{n!}\right) = \lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)^{n+1}} = (\text{RW})\lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$

$$(\text{logarithms})\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} = (\text{L'H})\lim_{n\to\infty}\frac{\frac{n+1}{n}\left(\frac{n+1-n}{(n+1)^2}\right)}{-\frac{1}{n^2}} = (\text{RW})\lim_{n\to\infty}\frac{-n}{n+1} = -1$$

$$\lim_{n\to\infty} y = e^{-1} = \frac{1}{e} < 1$$

so the series converges.

48) $\frac{4}{(4n-3)(4n+1)} = \frac{A}{4n-3} + \frac{B}{4n+1}$

$4 = A(4n+1) + B(4n-3) \quad n = -\frac{1}{4}: 4 = 0A + -4B \Rightarrow B = -1 \quad n = \frac{3}{4}: 4 = 4A + 0B \Rightarrow A = 1$

$S_1 = 1 - \frac{1}{5}$

$S_2 = \left(1-\frac{1}{5}\right) + \frac{1}{5} - \frac{1}{9} = 1 - \frac{1}{9}$

$S_3 = \left(1-\frac{1}{9}\right) + \frac{1}{9} - \frac{1}{13} = 1 - \frac{1}{13}$

$\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$

49) $S_n = 1$

$6 = A(2n+1) + B(2n-1) \quad n = -\frac{1}{2}: 6 = 0A + -2B \Rightarrow B = -3 \quad n = \frac{1}{2}: 6 = 2A + 0B \Rightarrow A = 3$

$S_1 = 3 - \frac{3}{3} = 3 - 1$

$S_2 = (3-1) + \frac{3}{3} - \frac{3}{5} = 3 - \frac{3}{5}$

$S_3 = \left(3-\frac{3}{5}\right) + \frac{3}{5} - \frac{3}{7} = 3 - \frac{3}{7}$

50) $$\frac{40n}{(2n-1)^2(2n+1)^2}=\frac{A}{(2n-1)^2}+\frac{B}{(2n+1)^2}$$

$$40n=A(2n+1)^2+B(2n-1)^2 \quad n=-\frac{1}{2}:-20=0A+4B\Rightarrow B=-5 \quad n=\frac{1}{2}:20=4A+0B\Rightarrow A=5$$

$$S_1=5-\frac{5}{9}$$

$$S_2=\left(5-\frac{5}{9}\right)+\frac{5}{9}-\frac{1}{5}=5-\frac{1}{5}$$

$$S_3=\left(5-\frac{1}{5}\right)+\frac{1}{5}-\frac{5}{49}=5-\frac{5}{49}$$

$$S_n=5$$

51) $$\frac{2n+1}{n^2(n+1)^2}=\frac{A}{n^2}+\frac{B}{(n+1)^2}$$

$$2n+1=A(n+1)^2+Bn^2 \quad n=-1:-1=0A+B\Rightarrow B=-1 \quad n=0:1=1A+0B\Rightarrow A=1$$

$$S_1=1-\frac{1}{4}$$

$$S_2=\left(1-\frac{1}{4}\right)+\frac{1}{4}-\frac{1}{9}=1-\frac{1}{9}$$

$$S_3=\left(1-\frac{1}{9}\right)+\frac{1}{9}-\frac{1}{16}=1-\frac{1}{16}$$

$$S_n=1$$

52) $$\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}$$

$$S_1=1-\frac{1}{\sqrt{2}}$$

$$S_2=\left(1-\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}=1-\frac{1}{\sqrt{3}}$$

$$S_3=\left(1-\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}-\frac{1}{2}=1-\frac{1}{2}$$

$$S_n=1$$

53) $$\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}$$

$$S_1=\frac{1}{\ln 3}-\frac{1}{\ln 2}$$

$$S_2=\left(\frac{1}{\ln 3}-\frac{1}{\ln 2}\right)+\frac{1}{\ln 4}-\frac{1}{\ln 3}=-\frac{1}{\ln 2}+\frac{1}{\ln 4}$$

$$S_3=\left(-\frac{1}{\ln 2}+\frac{1}{\ln 4}\right)+\frac{1}{\ln 5}-\frac{1}{\ln 4}=-\frac{1}{\ln 2}+\frac{1}{\ln 5}$$

$$S_n=-\frac{1}{\ln 2}$$

54)

$$\tan^{-1}(n) - \tan^{-1}(n+1)$$

$$S_1 = \frac{\pi}{4} - \tan^{-1}(2)$$

$$S_2 = \left(\frac{\pi}{4} - \tan^{-1}(2)\right) + \tan^{-1}(2) - \tan^{-1}(3) = \frac{\pi}{4} - \tan^{-1}(3)$$

$$S_3 = \left(\frac{\pi}{4} - \tan^{-1}(3)\right) + \tan^{-1}(3) - \tan^{-1}(4) = \frac{\pi}{4} - \tan^{-1}(4)$$

$$S_n = \frac{\pi}{4}$$