Calculus Newton's Law of Cooling

Sources: #1 – 5: Smith/Minton Calculus 4th ed. #6 – 8: Thomas/Finney Calculus 9th ed.

Use Newton's Law of Cooling $\left(T - T_S = \left(T_O - T_S\right)e^{-kt}\right)$ to solve the following. Round temperature answers to the nearest tenth of a degree, and time (duration) answers to the nearest hundredth of a minute.

1) A cup of fast-food coffee is 180°F when freshly poured. After 2 minutes in a room at 70°F, the coffee has cooled to 165°F. Find the time that it will take for the coffee to cool to 120°F.

$$165 - 70 = (180 - 70)e^{-2k}$$
$$95 = 110e^{-2k}$$
$$e^{-2k} = \frac{95}{110}$$
$$k = \frac{\ln\left(\frac{95}{110}\right)}{-2}$$

$$120 - 70 = (180 - 70)e^{-\left(\frac{\ln\left(\frac{95}{110}\right)}{-2}\right)t}$$
$$50 = 110e^{\left(\frac{\ln\left(\frac{95}{110}\right)}{2}\right)t}$$
$$e^{\left(\frac{\ln\left(\frac{95}{110}\right)}{2}\right)t} = \frac{50}{110}$$
$$t = \frac{\ln\left(\frac{50}{110}\right)}{\left(\frac{\ln\left(\frac{95}{110}\right)}{2}\right)} \approx 10.76 \text{ minutes}$$

2) A bowl of porridge at 200°F (too hot) is placed in a 70°F room. One minute later the porridge has cooled to 180°F. When will the temperature be 120°F (just right)?

$$180 - 70 = (200 - 70)e^{-k}$$
$$110 = 130e^{-k}$$
$$e^{-k} = \frac{11}{13}$$
$$k = -\ln\left(\frac{11}{13}\right)$$

$$120 - 70 = (200 - 70)e^{\ln\left(\frac{11}{13}\right)t}$$
$$50 = 130e^{\ln\left(\frac{11}{13}\right)t}$$
$$e^{\ln\left(\frac{11}{13}\right)t} = \frac{50}{130}$$
$$t = \frac{\ln\left(\frac{5}{13}\right)}{\ln\left(\frac{11}{13}\right)} \approx 5.72 \text{ minutes}$$

3) A smaller bowl of porridge served at 200°F cools to 160°F in one minute. What temperature (too cold) will this porridge be when the bowl of exercise 2 has reached 120°F (just right)?

$$160-70=(200-70)e^{-k}$$
$$90=130e^{-k}$$
$$e^{-k}=\frac{9}{13}$$
$$k=-\ln\left(\frac{9}{13}\right)$$

$$T-70=(200-70)e^{\ln\left(\frac{9}{13}\right)(5.72)}$$
$$T=130e^{\ln\left(\frac{9}{13}\right)(5.72)}+70$$
$$T\approx 85.9°$$

4) A cold drink is poured out at 50°F. After 2 minutes of sitting in a 70°F room, its temperature has risen to 56°F.
A) What will the temperature be after 10 minutes?
B) When will the drink have warmed to 66°F?

A)

$$56-70=(50-70)e^{-2k}$$
$$-14=-20e^{-2k}$$
$$e^{-2k}=\frac{7}{10}$$
$$k=\frac{\ln\left(\frac{7}{10}\right)}{-2}$$

$$T-70=(50-70)e^{\frac{\ln\left(\frac{7}{10}\right)}{2}(10)}$$
$$T=70+(-20)e^{\frac{\ln\left(\frac{7}{10}\right)}{2}(10)}\approx 66.6°\text{F}$$

B)

$$66-70=(50-70)e^{\frac{\ln(0.7)}{2}t}$$
$$-4=-20e^{\frac{\ln(0.7)}{2}t}$$
$$e^{\frac{\ln(0.7)}{2}t}=\frac{4}{20}$$
$$t=\frac{\ln(0.2)}{\frac{\ln(0.7)}{2}}\approx 9.02 \text{ minutes after it is poured.}$$

5) At 10:07 pm, you find a secret agent murdered. Next to him is a martini that got shaken before he could stir it. Room temperature is 70°F. The martini warms from 60°F to 61°F in the 2 minutes from 10:07 pm to 10:09 pm. If the secret agent's martinis are always served at 40°F, what was the time of death (rounded to the nearest minute)?

$$61-70=(60-70)e^{-2k}$$
$$-9=-10e^{-2k}$$
$$e^{-2k}=\frac{9}{10}$$
$$k=\frac{\ln(0.9)}{-2}$$

$$60-70=(40-70)e^{\frac{\ln(0.9)}{2}t}$$
$$-10=-30e^{\frac{\ln(0.9)}{2}t}$$
$$e^{\frac{\ln(0.9)}{2}t}=\frac{1}{3}$$
$$t=\frac{\ln\left(\frac{1}{3}\right)}{\frac{\ln(0.9)}{2}}\approx 20.85 \text{ minutes}$$

The agent was murdered at approx. 9:46 pm.

6) A hard-boiled egg at 98°C is put into a sink of 18°C water. After 5 minutes, the egg's temperature is 38°C. Assuming that the surrounding water has not warmed appreciably, how much longer will it take the egg to reach 20°C?

$$38-18=(98-18)e^{-5k}$$
$$20=80e^{-5k}$$
$$e^{-5k}=\frac{20}{80}$$
$$k=\frac{\ln(0.25)}{-5}$$

$$20-18=(38-18)e^{\frac{\ln(0.25)}{5}t}$$
$$2=20e^{\frac{\ln(0.25)}{5}t}$$
$$e^{\frac{\ln(0.25)}{5}t}=\frac{2}{20}$$
$$t=\frac{\ln(0.1)}{\frac{\ln(0.25)}{5}}\approx 8.30\text{ minutes}$$

7) Suppose that a cup of soup cooled from 90°C to 60°C after 10 minutes in a room whose temperature was 20°C.

A) How much longer would it take the soup to cool to 35°C?

$$60-20=(90-20)e^{-10k}$$
$$40=70e^{-10k}$$
$$e^{-10k}=\frac{40}{70}$$
$$k=\frac{\ln\left(\frac{4}{7}\right)}{-10}$$

$$35-20=(90-20)e^{\frac{\ln\left(\frac{4}{7}\right)}{10}t}$$
$$15=70e^{\frac{\ln\left(\frac{4}{7}\right)}{10}t}$$
$$e^{\frac{\ln\left(\frac{4}{7}\right)}{10}t}=\frac{15}{70}$$
$$t=\frac{\ln\left(\frac{3}{14}\right)}{\frac{\ln\left(\frac{4}{7}\right)}{10}}\approx 27.53\text{ minutes}$$

B) Instead of being left to stand in the room, the cup of 90°C soup is placed in a freezer whose temperature is -15°C, and it took 5 minutes to cool to 60°C. How long will it take the soup to cool from 90°C to 35°C?

$$60-(-15)=(90-(-15))e^{-5k}$$
$$75=105e^{-5k}$$
$$e^{-5k}=\frac{75}{105}$$
$$k=\frac{\ln\left(\frac{5}{7}\right)}{-5}$$

$$35-(-15)=(90-(-15))e^{\frac{\ln\left(\frac{5}{7}\right)}{5}t}$$
$$50=105e^{\frac{\ln\left(\frac{5}{7}\right)}{5}t}$$
$$e^{\frac{\ln\left(\frac{5}{7}\right)}{5}t}=\frac{50}{105}$$
$$t=\frac{\ln\left(\frac{10}{21}\right)}{\frac{\ln\left(\frac{5}{7}\right)}{5}}\approx 11.03\text{ minutes}$$

8) A pan of warm water (46°C) was put into a refrigerator. Ten minutes later, the water's temperature was 39°C. 10 minutes after that it was 33°C. Use Newton's Law of Cooling to estimate the temperature of the refrigerator.

$$39 - T_s = (46 - T_s)e^{-10k} \Rightarrow e^{-10k} = \frac{39 - T_s}{46 - T_s}$$

$$33 - T_s = (39 - T_s)e^{-10k} \Rightarrow e^{-10k} = \frac{33 - T_s}{39 - T_s}$$

$$\frac{39 - T_s}{46 - T_s} = \frac{33 - T_s}{39 - T_s}$$

$$(39 - T_s)^2 = (33 - T_s)(46 - T_s)$$

$$1521 - 78T_s + T_s^2 = 1518 - 79T_s + T_s^2$$

$$T_s = -3°\text{C}$$