**Calculus and Inverse Functions** Name: Answers

*AP Calculus*

We derived the theorem that gives us the derivative of the inverse of a function:

Given that f(x) is differentiable and one-to-one with inverse g(x) and given that b belongs to the domain of g(x) and $f'(g(b)) \neq 0$, then g'(b) exists and:

$$g'(b) = \frac{1}{f'(g(b))}$$

$\frac{d}{dx}[f(g(x)) = x]$

$f'(g(x)) \cdot g'(x) = 1$

$g'(x) = \frac{1}{f'(g(x))}$

This can be a handy theorem but you may use any method you choose to do the problems below:

1) Given $f(x) = x + e^x$ and given g(x) is the inverse of f(x), find $g'(1)$.

$x + e^x = 1$

$x = 0$ $(0,1)$

$f(0) = 1$

so $g(1) = 0$

Method 1 → take inverse + derive

$x = y + e^y$

$1 = \frac{dy}{dx} + e^y \frac{dy}{dx}$

$\frac{1}{1+e^y} = \frac{dy}{dx}$

since $g(1) = 0$ $\left.\frac{dy}{dx}\right|_{(1,0)} = \frac{1}{1+e^0} = \boxed{\frac{1}{2}}$

Method 2 – Use theorem

$g'(x) = \frac{1}{1 + e^{g(x)}}$

$g'(1) = \frac{1}{1+e^0} = \boxed{\frac{1}{2}}$

2) Given $f(x) = 4x^3 - 2x$ and given g(x) is the inverse of f(x), find $g'(-2)$

$4x^3 - 2x = -2$

solve graphically

$f(-1) = -2$

so $g(-2) = -1$

Method 1

$x = 4y^3 - 2y$

$1 = 12y^2 \frac{dy}{dx} - 2\frac{dy}{dx}$

$\frac{1}{12y^2 - 2} = \frac{dy}{dx}$

$\left.\frac{dy}{dx}\right|_{(-2,-1)} = \frac{1}{12(-1)^2 - 2} = \boxed{\frac{1}{10}}$

Method 2

$g'(x) = \frac{1}{12(g(x))^2 - 2}$

$= \frac{1}{12(-1)^2 - 2}$

$= \boxed{\frac{1}{10}}$

Here are the derivatives of some inverse trig functions....(Remember: $\sin^{-1}(x)$ and arcsin(x) are two different ways to indicate the inverse of sine.)

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Find the derivative:

chain rule

3) $f(x) = \sin^{-1}(3x)$

$$f'(x) = \frac{3}{\sqrt{1-(3x)^2}} = \frac{3}{\sqrt{1-9x^2}}$$

4) $g(x) = x\tan^{-1}(\frac{x}{2})$

$$g'(x) = 1\cdot\tan^{-1}\left(\frac{x}{2}\right) + x\cdot\frac{1}{2}\left(\frac{1}{1+(\frac{1}{2}x)^2}\right)$$

$$= \tan^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\left(\frac{1}{1+\frac{1}{4}x^2}\right)$$

5) $y = e^{\arccos(x)}$

$$y' = e^{\arccos x} \cdot -\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{-e^{\arccos x}}{\sqrt{1-x^2}}$$

6) $f(x) = (\arcsin x)^4$

$$f'(x) = 4(\arcsin x)^3\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{4}{\sqrt{1-x^2}}(\arcsin x)^3$$

Evaluate (no calculators!):

7) $\displaystyle\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \Big]_0^{.5}$

$$= \arcsin(.5) - \arcsin(0)$$

$$= \frac{\pi}{6} - 0 = \boxed{\frac{\pi}{6}}$$

8) $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2} = \arctan x \Big]_{-1}^{1}$

$$= \arctan(1) - \arctan(-1)$$

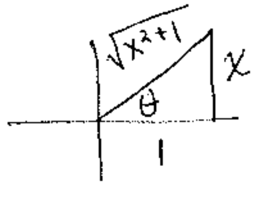
$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \boxed{\frac{\pi}{2}}$$

9) $\displaystyle\int_{-0.5}^{0.5} \frac{dx}{\sqrt{1-(2x)^2}} = \frac{1}{2}\arcsin(2x)$

$$= \frac{1}{2}\Big[\arcsin(2x)\Big]_{-.5}^{.5} = \frac{1}{2}\left(\arcsin(1) - \arcsin(-1)\right) = \frac{1}{2}\left(\frac{\pi}{2} - -\frac{\pi}{2}\right)$$

$$= \boxed{\frac{\pi}{2}}$$

<u>Challenge</u>: Prove that $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$




$$\tan\theta = x$$

$$\sec^2\theta\,\frac{d\theta}{dx} = 1$$

$$\frac{d\theta}{dx} = \frac{1}{\sec^2\theta}$$

$$\frac{d\theta}{dx} = \frac{1}{(\sqrt{x^2+1})^2} = \frac{1}{x^2+1}$$