AP CALCULUS BC

"Cheat Sheet"

Differentiation Formulas

Important Note: Remember the chain rule whenever you take a derivative! For example, $\frac{d}{dx}e^u = e^u du$. When you look at all these derivatives, remember the chain rule!

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(fg) = fg' + gf'$
3. $\frac{d}{dx}(\frac{f}{g}) = \frac{gf' - fg'}{g^2}$
4. $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\cos x) = -\sin x$
7. $\frac{d}{dx}(\tan x) = \sec^2 x$
8. $\frac{d}{dx}(\cot x) = -\csc^2 x$
9. $\frac{d}{dx}(\sec x) = \sec x \tan x$
10. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = a^x \ln a$
13. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
14. $\frac{d}{dx}(Arc\sin x) = \frac{1}{\sqrt{1-x^2}}$
15. $\frac{d}{dx}(Arc\tan x) = \frac{1}{1+x^2}$
16. $\frac{d}{dx}(Arc\cos x) = \frac{-1}{\sqrt{1-x^2}}$
17. $\frac{dy}{dx} = \frac{dy}{dx} \times \frac{du}{dx}$ *Chain Rule*

Integration Formulas

1. $\int a\, dx = ax + C$
2. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
3. $\int \frac{1}{x} dx = \ln|x| + C$
4. $\int e^x\, dx = e^x + C$
5. $\int a^x dx = \frac{a^x}{\ln a} + C$
6. $\int \ln x\, dx = x \ln x - x + C$ Note: You can figure this out using *uv* substitution
7. $\int \sin x\, dx = -\cos x + C$
8. $\int \cos x\, dx = \sin x + C$

Note: You can figure out #9 and #10 below using substitution techniques

9. $\int \tan x \, dx = \ln|\sec x| + C$ or $-\ln|\cos x| + C$

10. $\int \cot x \, dx = \ln|\sin x| + C$

11. $\int \sec^2 x \, dx = \tan x + C$

12. $\int \sec x \tan x \, dx = \sec x + C$

13. $\int \csc^2 x \, dx = -\cot x + C$

14. $\int \csc x \cot x \, dx = -\csc x + C$

15. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} Arc\tan\left(\frac{x}{a}\right) + C$

16. $\int \frac{dx}{\sqrt{a^2 - x^2}} = Arc\sin\left(\frac{x}{a}\right) + C$

Formulas and Theorems

1. Limits and Continuity:

A function $y = f(x)$ is continuous at $x = a$ if

i). *f(a)* exists

ii). $\lim_{x \to a} f(x)$ exists

iii). $\lim_{x \to a} = f(a)$

Otherwise, *f* is discontinuous at x = a.

The limit $\lim_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal – that is,

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

2. Even and Odd Functions

1. A function $y = f(x)$ is even if $f(-x) = f(x)$ for every x in the function's domain. Every even function is symmetric about the y-axis.

2. A function $y = f(x)$ is odd if $f(-x) = -f(x)$ for every x in the function's domain. Every odd function is symmetric about the origin.

4. Intermediate-Value Theorem

A function $y = f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

Note: If f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a,b).

5. Limits of Rational Functions as $x \to \pm\infty$

i). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^2 - 2x}{x^3 + 3} = 0$

ii). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is infinite if the degrees of $f(x) >$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^3 + 2x}{x^2 - 8} = \infty$

iii). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{2x^2 - 3x + 2}{10x - 5x^2} = -\frac{2}{5}$

6. Horizontal and Vertical Asymptotes

1. A line $y = b$ is a horizontal asymptote of the graph $y = f(x)$ if either

$\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.

2. A line $x = a$ is a vertical asymptote of the graph $y = f(x)$ if either

$\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} = \pm\infty$.

7. Average and Instantaneous Rate of Change

i). *Average Rate of Change*: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is $\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$.

ii). *Instantaneous Rate of Change*: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

8. Definition of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The latter definition of the derivative is the instantaneous rate of change of $f(x)$ with respect to x at x = a.

Geometrically, the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.

9. The Number $\boldsymbol{e}$ as a limit (This is cool to know, but not essential)

i). $$\lim_{n \to +\infty} \left(1+\frac{1}{n}\right)^n = e$$

ii). $$\lim_{n \to 0} \left(1+\frac{n}{1}\right)^{\frac{1}{n}} = e$$

10. Rolle's Theorem (Note: This is simply a special case of the MVT below)
If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a,b) such that $f'(c) = 0$.

11. Mean Value Theorem (MVT)
If f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in (a,b) such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

12. Extreme-Value Theorem (EVT)
If f is continuous on a closed interval $[a,b]$, then $f(x)$ has both a maximum and minimum on $[a,b]$.

13. Intermediate Value Theorem (IVT)
If a continuous function, f, with an interval, [a, b], as its domain, takes values f(a) and f(b) at each end of the interval, then it also takes any value between f(a) and f(b) at some point within the interval.

14. ***Absolute Mins and Maxs***: To find the maximum and minimum values of a function $y = f(x)$, locate
   1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist.
   2. the end points, if any, on the domain of $f(x)$.
   3. Plug those values into $f(x)$ to see which gives you the max and which gives you this min values (the x-value is where that value occurs)

   Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a minimum.

15. **Increasing and Decreasing**: Let f be differentiable for $a < x < b$ and continuous for a $a \le x \le b$,
   1. If $f'(x) > 0$ for every x in (a,b), then f is increasing on $[a,b]$.
   2. If $f'(x) < 0$ for every x in (a,b), then f is decreasing on $[a,b]$.

16. **Concavity:** Suppose that $f''(x)$ exists on the interval (a,b)
   1. If $f''(x) > 0$ in (a,b), then f is concave upward in (a,b).
   2. If $f''(x) < 0$ in (a,b), then f is concave downward in (a,b).

   To locate the ***points of inflection*** of $y = f(x)$, find the points where $f''(x) = 0$ or where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

17. If a function is differentiable at point $x = a$, it is continuous at that point. The converse is false, in other words, continuity does not imply differentiability.

18. Local Linearity and Linear Approximations

The linear approximation to $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$ for x sufficiently close to x_0.

To estimate the slope of a graph at a point – just draw a tangent line to the graph at that point. Another way is (by using a graphing calculator) to "zoom in" around the point in question until the graph "looks" straight. This method almost always works. If we "zoom in" and the graph looks straight at a point, say $(a, f(a))$, then the function is locally linear at that point.

The graph of $y = |x|$ has a sharp corner at $x = 0$. This corner cannot be smoothed out by "zooming in" repeatedly. Consequently, the derivative of $|x|$ does not exist at $x = 0$, hence, is not locally linear at $x = 0$.

19. Dominance and Comparison of Rates of Change

Logarithm functions grow slower than any power function (x^n).

Among power functions, those with higher powers grow faster than those with lower powers.

All power functions grow slower than any exponential function $(a^x, a > 1)$.

Among exponential functions, those with larger bases grow faster than those with smaller bases.

We say, that as $x \to \infty$:

1. $f(x)$ grows faster than $g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(\ )} = \infty$ or if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

   If $f(x)$ grows faster than $g(x)$ as $x \to \infty$, then $g(x)$ grows slower than $f(x)$ as $x \to \infty$.

2. $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$ (L is finite and nonzero).

For example,

1. e^x grows faster than x^3 as $x \to \infty$ since $\lim_{x\to\infty} \frac{e^x}{x^3} = \infty$

2. x^4 grows faster than $\ln x$ as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^4}{\ln x} = \infty$

3. $x^2 + 2x$ grows at the same rate as x^2 as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^2 + 2x}{x^2} = 1$

To find some of these limits as $x \to \infty$, you may use the graphing calculator. Make sure that an appropriate viewing window is used.

20. ***L'Hôpital's Rule***

If $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$.

21. Inverse function

1. If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f, then f and g are inverse functions of each other.
2. A function f has an inverse if and only if no horizontal line intersects its graph more than once.
3. If f is strictly either increasing or decreasing in an interval, then f has an inverse.
4. If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and if the point (a,b) is on $f(x)$, then the point (b,a) is on $g = f^{-1}(x)$; furthermore $g'(b) = \frac{1}{f'(a)}$.

22. Properties of $y = e^x$

1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.
2. The domain is the set of all real numbers, $-\infty < x < \infty$.
3. The range is the set of all positive numbers, $y > 0$.
4. $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$
5. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
6. $y = e^x$ is continuous, increasing, and concave up for all x.
7. $\lim\limits_{x \to +\infty} e^x = +\infty$ and $\lim\limits_{x \to -\infty} e^x = 0$.
8. $e^{\ln x} = x$, for $x > 0$; $\ln(e^x) = x$ for all x.

23. Properties of $y = \ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.
2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.
3. $y = \ln x$ is continuous and increasing everywhere on its domain.
4. $\ln(ab) = \ln a + \ln b$.
5. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.
6. $\ln a^r = r \ln a$.
7. $y = \ln x < 0$ if $0 < x < 1$.

8. $\lim_{x \to +\infty} \ln x = +\infty$ and $\lim_{x \to 0^+} \ln x = -\infty$.

9. $\log_a x = \dfrac{\ln x}{\ln a}$

10. $\dfrac{d}{dx}\left(\ln f(x)\right) = \dfrac{f'(x)}{f(x)}$ and $\dfrac{d}{dx}\left(\ln(x)\right) = \dfrac{1}{x}$

24. Know Left-hand, right-hand, and midpoint Riemann Sums AND how to use trapezoids to approximate signed area under a curve.

25. Definition of Definite Integral as the Limit of a Sum

Suppose that a function $f(x)$ is continuous on the closed interval $[a,b]$. Divide the interval into n equal subintervals, of length $\Delta x = \dfrac{b-a}{n}$. Choose one number in each subinterval, in other words, x_1 in the first, x_2 in the second, …, x_k in the k th,…, and x_n in the n th. Then

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx = F(b) - F(a).$$

26. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

i). $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c.

ii). $\int_a^a f(x)\,dx = 0$

iii). $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

iv). $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c.

v). If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

vi). If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

vii). If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$

viii). If $g(x) \ge f(x)$ on $[a,b]$, then $\int_a^b g(x)\,dx \ge \int_a^b f(x)\,dx$

27. Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F'(x) = f(x), \text{ or } \frac{d}{dx}\int_a^b f(x)\,dx = f(x).$$

28. Second Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{or} \quad \frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = g'(x)f(g(x)) - h'(x)f(h(x))$$

29. Velocity, Speed, and Acceleration

1. The velocity of an object tells how fast it is going and in which direction. Velocity is an instantaneous rate of change. If velocity is positive (graphically above the "x"-axis), then the object is moving away from its point of origin. If velocity is negative (graphically below the "x"-axis), then the object is moving back towards its point of origin. If velocity is 0 (graphically the point(s) where it hits the "x"-axis), then the object is not moving at that time.
2. The speed of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.
   The speed of a particle increases (speeds up) when the velocity and acceleration have the same signs. The speed decreases (slows down) when the velocity and acceleration have opposite signs.
3. The acceleration is the instantaneous rate of change of velocity – it is the derivative of the velocity – that is, $a(t) = v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing (i.e. the velocity graph would be going down at that time), and vice-versa for acceleration increasing. The acceleration gives the rate at which the velocity is changing.

Therefore, if x is the displacement of a moving object and t is time, then:

i) velocity = $v(t) = x'(t) = \frac{dx}{dt}$

ii) acceleration = $a(t) = x''(t) = v'(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

iii) $v(t) = \int a(t)dt$

iv) $x(t) = \int v(t)dt$

Note: The average velocity of a particle over the time interval from t_0 to another time t, is

Average Velocity = $\frac{\text{Change in position}}{\text{Length of time}} = \frac{s(t) - s(t_0)}{t - t_0}$, where $s(t)$ is the position of the particle at time t or $\frac{1}{b-a}\int_a^b v(t)dt$ if given the velocity function.

30. The average value of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

31. Area Between Curves

If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a,b]$, then area between the curves is $\int_a^b [f(x)-g(x)]dx$ or $\int_a^b [top - bottom]\,dx$ or $\int_c^d [right - left]dy$.

32. Integration using "U-substitution"

Step one: Be sure this new idea applies, i.e. check to see whether the expression looks as though it could have arisen from a chain rule being applied.
Step two: Off to the side of the original integration problem, write down what you are going to make u equal to. Often this may be the part of the expression that has the weird exponent.
Step three: Take the derivative of the "u-equals" equation with respect to x
Step four: Look at what you have as u- equals and its derivative and the original expression you were trying to integrate…those pieces should work in a way that now changes all the variables in the original to either a u or a du. Now it should be something you can integrate! For definite integrals, be sure to change your limits if integration to correspond with your new function in terms of u.

33. Integration By "Parts"

If $u = f(x)$ and $v = g(x)$ and if $f'(x)$ and $g'(x)$ are continuous, then
$\int u\,dx = uv - \int v\,du$.

Note: The goal of the procedure is to choose u and dv so that $\int v\,du$ is easier to solve than the original problem.

Suggestion:

When "choosing" u, remember **L.I.A.T.E**, where **L** is the logarithmic function, **I** is an inverse trigonometric function, **A** is an algebraic function, **T** is a trigonometric function, and **E** is the exponential function. Just choose u as the first expression in **L.I.A.T.E** (and dv will be the remaining part of the integrand). For example, when integrating $\int x \ln x\,dx$, choose $u = \ln x$ since **L** comes first in **L.I.A.T.E**, and $dv = x\,dx$. When integrating $\int xe^x\,dx$, choose $u = x$, since x is an algebraic function, and **A** comes before **E** in **L.I.A.T.E**, and $dv = e^x dx$. One more example, when integrating $\int x\,Arc\tan(x)\,dx$, let $u = Arc\tan(x)$, since **I** comes before **A** in **L.I.A.T.E**, and $dv = x\,dx$.

34. Integration using partial fraction decomposition

When integrating a rational function, first see if U-sub works or, if not, determine whether it's in the form of an inverse trig integral. If neither of these work, you may re-write the function as a sum of fractions. Important: Be sure to first do polynomial long division if possible. See the example below:

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

$$\frac{3x+11}{x^2-x-6} = \frac{4}{x-3} - \frac{1}{x+2}$$

then the integral is actually quite simple.

$$\int \frac{3x+11}{x^2-x-6}\,dx = \int \frac{4}{x-3} - \frac{1}{x+2}\,dx$$
$$= 4\ln|x-3| - \ln|x+2| + c$$

35. Improper Integral

$\int_a^b f(x)\,dx$ is an improper integral if

1. f becomes infinite at one or more points of the interval of integration, or
2. one or both of the limits of integration is infinite, or
3. both (1) and (2) hold.

36. Volume of Solids of Revolution (rectangles drawn perpendicular to the axis of revolution)

- Revolving around a horizontal line (y=# or x-axis) where $a \le x \le b$:

Axis of Revolution below the region being revolved:

$$V = \pi \int_a^b (\textit{top function} - a.r.)^2 - (\textit{bottom function} - a.r.)^2\,dx$$

Axis of Revolution above the region being revolved:

$$V = \pi \int_a^b (\textit{bottom function} - a.r.)^2 - (\textit{top function} - a.r.)^2\,dx$$

- Revolving around a vertical line (x=# or y-axis) where $c \le y \le d$:

Axis of Revolution is left of the region being revolved:

$$V = \pi \int_c^d (\textit{right function} - a.r.)^2 - (\textit{left function} - a.r.)^2\,dy$$

Axis of Revolution is right of the region being revolved:

$$V = \pi \int_c^d (\textit{left function} - a.r.)^2 - (\textit{right function} - a.r.)^2\,dy$$

37. Volume of Solids with Known Cross Sections

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume = $\int_a^b A(x)\,dx$.

Cross-sections {if only one function is used then just use that function, if it is between two functions use *top-bottom*} mostly all the same only varying by a constant, with the only exception being the rectangular cross-sections:

Note: You don't need to memorize these; just think about how you'd find the area of each cross-section and go from there…but here are some common ones.

- Square cross-sections:
  $$V = \int_a^b (top\ function - bottom\ function)^2\, dx$$
- Isosceles Right Triangle cross-sections (leg in the *xy* plane):
  $$V = \frac{1}{2}\int_a^b (top\ function - bottom\ function)^2\, dx$$
- Semi-circular cross-sections:
  $$V = \frac{\pi}{8}\int_a^b (top\ function - bottom\ function)^2\, dx$$
- Rectangular cross-sections (height function or value must be given or articulated somehow – notice no "square" on the {top – bottom} part):
  $$V = \int_a^b (top\ function - bottom\ function)(height\ function / value)dx$$

2. For cross sections of area $A(y)$, taken perpendicular to the y-axis, volume = $\int_a^b A(y)\, dy$.

38. Solving Differential Equations: Graphically and Numerically

Slope Fields

At every point (x, y) a differential equation of the form $\frac{dy}{dx} = f(x, y)$ gives the slope of the member of the family of solutions that contains that point. A slope field is a graphical representation of this family of curves. At each point in the plane, a short segment is drawn whose slope is equal to the value of the derivative at that point. These segments are tangent to the solution's graph at the point.

The slope field allows you to sketch the graph of the solution curve even though you do not have its equation. This is done by starting at any point (usually the point given by the initial condition), and moving from one point to the next in the direction indicated by the segments of the slope field.

Know how to solve separable differential equations.

Know how to use the given differential equation and its derivative (i.e. the 2nd derivative) to describe the slope and concavity of a function or determine whether a point is a max or min.

Euler's Method

Euler's Method is a way of approximating points on the solution of a differential equation $\frac{dy}{dx} = f(x, y)$. The calculation uses the tangent line approximation to move from one point to the next. That is, starting with the given point (x_1, y_1) – the initial condition, the point $(x_1 + \Delta x,\ y_1 + f'(x_1, y_1)\Delta x)$ approximates a nearby point on the solution graph. This aproximation may then be used as the starting point to calculate a third point and so on. The accuracy of the method decreases with large values of Δx. The error increases as each successive point is used to find the next. Calculator programs are available for doing this calculation.

Logistics

1. Rate is jointly proportional to its size and the difference between a fixed positive number (L) and its size.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \text{ OR } \frac{dy}{dt} = ky\left(M - y\right) \text{ which yields}$$

$$y = \frac{L}{1 + Ce^{-kt}} \quad \text{through separation of variables}$$

2. $\lim_{t \to \infty} y = L$; L = carrying capacity (Maximum); horizontal asymptote

3. y-coordinate of inflection point is $\frac{L}{2}$, i.e. when it is growing the fastest (or max rate).

39. Definition of Arc Length

If the function given by $y = f(x)$ represents a smooth curve on the interval $[a, b]$, then the arc length of f between a and b is given by $s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

40. Parametric Form of the Derivative

If a smooth curve **C** is given by the parametric equations $x = f(x)$ and $y = g(t)$, then the slope of the curve **C** at (x, y) is $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$, $\frac{dx}{dt} \neq 0$.

Note: The second derivative, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \div \frac{dx}{dt}$.

41. Arc Length in Parametric Form

If a smooth curve **C** is given by $x = f(t)$ and $y = g(t)$ and these functions have continuous first derivatives with respect to t for $a \le t \le b$, and if the point $P(x, y)$ traces the curve exactly once as t moves from $t = a$ to $t = b$, then the length of the curve is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

$$\text{speed} = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

42. Polar Coordinates

1. Cartesian vs. Polar Coordinates. The polar coordinates (r, θ) are related to the Cartesian coordinates (x, y) as follows:

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \text{ and } x^2 + y^2 = r^2$$

2. To find the points of intersection of two polar curves, find (r, θ) satisfying the first equation for which some points $(r, \theta + 2n\pi)$ or $(-r, \theta + \pi + 2n\pi)$ satisfy the second equation. Check separately to see if the origin lies on both curves, i.e. if r can be 0. Sketch the curves.
3. Area in Polar Coordinates: If f is continuous and nonnegative on the interval $[\alpha, \beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

4. Derivative of Polar function: Given $r = f(\theta)$, to find the derivative, use parametric equations.

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

$$\text{Then} \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

5. Arc Length in Polar Form: $s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

43. <u>Sequences and Series</u>

1. If a sequence $\{a_n\}$ has a limit L, that is, $\lim_{n\to\infty} a_n = L$, then the sequence is said to <u>converge</u> to L. If there is no limit, the series <u>diverges</u>. If the sequence $\{a_n\}$ converges, then its limit is unique. Keep in mind that

$\lim_{n\to\infty} \frac{\ln n}{n} = 0$; $\lim_{n\to\infty} x^{\left(\frac{1}{n}\right)} = 1$; $\lim_{n\to\infty} \sqrt[n]{n} = 1$; $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. These limits are useful and arise frequently.

2. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges; the geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$ and $a \neq 0$.

3. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

4. <u>Limit Comparison Test</u>: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of nonnegative terms, with $a_n \neq 0$ for all sufficiently large n, and suppose that $\lim_{n\to\infty} \frac{b_n}{a_n} = c > 0$. Then the two series either both converge or both diverge.

5. <u>Alternating Series</u>: Let $\sum_{n=1}^{\infty} a_n$ be a series such that

   i) the series is alternating

   ii) $|a_{n+1}| \leq |a_n|$ for all n, and

   iii) $\lim_{n\to\infty} a_n = 0$

   Then the series *converges*.

   <u>Alternating Series Remainder</u>: The remainder R_N is less than (or equal to) the first neglected term

$$|R_N| \leq a_{N+1}$$

6. <u>The *n*-th Term Test for Divergence</u>: If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. Note that the converse is *false*, that is, if $\lim_{n\to\infty} a_n = 0$, the series may or may not converge.

7. A series $\sum a_n$ is <u>absolutely convergent</u> if the series $\sum |a_n|$ converges. If $\sum a_n$ converges, but $\sum |a_n|$ does not converge, then the series is <u>conditionally convergent</u>. Keep in mind that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

8. <u>Comparison Test</u>: If $0 \le a_n \le b_n$ for all sufficiently large n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

9. <u>Integral Test</u>: If $f(x)$ is a positive, continuous, and decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ will converge if the improper integral $\int_1^{\infty} f(x)\, dx$ converges. If the improper integral $\int_1^{\infty} f(x)\, dx$ diverges, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

10. <u>Ratio Test</u>: Let $\sum a_n$ be a series with nonzero terms.

    i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then the series converges absolutely.

    ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then the series is divergent.

    iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the test is inconclusive (and another test must be used).

11. <u>Power Series</u>: A power series is a series of the form
$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$ or
$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$ in which the center a and the coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants. The set of all numbers x for which the power series converges is called the <u>interval of convergence</u>.

12. Taylor Series: Let f be a function with derivatives of all orders throughout some intervale containing a as an interior point. Then the Taylor series generated by f at a is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

The remaining terms after the term containing the nth derivative can be expressed as a remainder to Taylor's Theorem:

$$f(x) = f(a) + \sum_{1}^{n} f^{(n)}(a)(x-a)^n + R_n(x) \text{ where } R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Lagrange's form of the remainder: $|f(x) - P_n(x)| = |R_n x| = \frac{f^{(n+1)}(c)\,|(x-a)|^{n+1}}{(n+1)!}$

, where $a < c < x$.

The series will converge for all values of x for which the remainder approaches zero as $x \to \infty$.

Alternating Series Remainder: To find the maximum error between a partial sum and the actual sum of a convergent alternating series, simply look at the first term that was left out. In other words, if you're finding the partial sum of the first 15 terms, then the maximum error is equal to the 16th term.

13. Frequently Used Series and their Interval of Convergence

Note: DO memorize these!! It will save time!

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n, \ |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \ |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \ |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)x^{2n}}{(2n)!}, \ |x| < \infty$$

Trigonometric Formulas

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$
4. $\sin(-\theta) = -\sin\theta$
5. $\cos(-\theta) = \cos\theta$
6. $\tan(-\theta) = -\tan\theta$
7. $\sin(A+B) = \sin A\cos B + \sin B\cos A$
8. $\sin(A-B) = \sin A\cos B - \sin B\cos A$
9. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
10. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
11. $\sin 2\theta = 2\sin\theta\cos\theta$
12. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
13. $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$
14. $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$
15. $\sec\theta = \frac{1}{\cos\theta}$
16. $\csc\theta = \frac{1}{\sin\theta}$
17. $\cos^2\theta = \frac{1}{2}\left(1 + \cos 2\theta\right)$
18. $\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right)$

Note: Trig identities are useful for your future but not necessary on the AP