AP CALCULUS (BC) NAME ______________________

Chapter 1 - Limits and Continuity Date ________________________

The **limit** is one of the signature concepts of calculus. It separates elementary function analysis from higher mathematics (such as calculus). The process of "finding" a limit is the engine that drives the study of calculus, and allows for the investigation of: 1) quantities and their **instantaneous rates of change**; and 2) quantities and their **accumulation of change**.

In this chapter we define what a limit is, and we determine the value of the limit of a function. The definition is both simple (in what it says) and yet sophisticated (in how it says it). Most of our limits will be determined by one of the following methods: substitution, graphically, numerically, and algebraically (or some combination), depending on the situation. Other methods will be discussed as needed to handle special circumstances.

Limits will also be used to explore the concept of **continuity of functions**; an important characteristic of many functions used to model natural phenomena.

Section 1.1 - Rates of Change and LIMITS

Part A. Limits

1) Evaluating Limits

The first method attempted in every limit evaluation problem (regardless of the problem directions) must be substitution. You will only need some other method (such as numerical, graphical, or algebraic) if substitution yields an **INDETERMINATE**. An **indeterminate** is an expression of the form $\frac{0}{0}\ or\ \frac{\infty}{\infty}$. Note: A limit evaluation problem that yields an expression of the form $\frac{c}{0}$, $c \neq 0$ is **UNDEFINED** which means the limit DNE (does not exist). An indeterminate limit is not the same as an undefined limit.

Think of an indeterminate limit as a limit <u>whose value cannot be determined by simple substitution</u> and whose value may exist, but which must be determined by some other method; whereas an undefined limit cannot ever exist regardless of method used.

If **SUBSTITUTION** doesn't work (you get an **INDETERMINATE**), then try another method.

When evaluating **GRAPHICALLY**, then you just inspect the graph (no work, just write the limit statement).

VIDEO (KHAN ACADEMY): ESTIMATING LIMIT VALUES FROM GRAPHS
VIDEO (KHAN ACADEMY): CONNECTING LIMITS AND GRAPHICAL BEHAVIOR

CONTINUE →

When evaluating **NUMERICALLY**, then you make a table of values. I prefer a vertical table (as opposed to horizontal which the book uses) with the column on the left for the independent variable and the column on the right for the expression you're trying to determine the limit for.

VIDEO (KHAN ACADEMY): APPROXIMATING LIMITS USING TABLES
VIDEO (KHAN ACADEMY): ESTIMATING LIMITS FROM TABLES
VIDEO (KHAN ACADEMY): ONE-SIDED LIMITS FROM TABLES

If evaluating **ALGEBRAICALLY**, then you must show the algebraic steps and eventual second substitution that yields a value. Try the following problems. Remember, always attempt (and show the steps for) substitution before using the indicated method.

VIDEO (KHAN ACADEMY): LIMITS BY RATIONALIZATION

a. substitution $\lim_{x \to 2} 2x^3 - 7x + 1$ = _____ {See page 67 Example 4 for steps}

b. graphically $\lim_{x\to 0} \frac{1}{x} * \sin x$ = ______ {See page 67 Example 6 for steps}

c. numerically $\lim_{x\to 0} \frac{x+\sin x}{x}$ = ______ {Set up a Table of Values similar to Exercises # 15 – 20 on page 70}

d. algebraically $\lim_{x\to 3} \frac{x^2-9}{x^2-5x+6}$ = ______ {See Exercise # 26 on page 70}

Homework 1.1a: page 70 # 5, 10, 11, 13, 19, 24 – 26, 28, 78, 79

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2) Properties of Limits – Basic facts that allow us to evaluate the limits of many functions.

a. Theorem 1 – Sum/Difference, Product, Constant Multiple, Quotient,
Power Rules, Constant, and Identity Rules (pages 65 – 66)

VIDEO (KHAN ACADEMY): LIMIT PROPERTIES
VIDEO (KHAN ACADEMY): LIMITS OF COMBINED FUNCTIONS

Example 3 (page 66) – Application of Limit Properties

b. Theorem 2 – Polynomial and Rational Functions (page7)

Like many other functions, limits of polynomials and rational functions can be determined by substitution, *where those limit values are defined*!

Example 4 (page 67) – Application of Theorem 2

Example 5 (page 67) – Using the Properties to Evaluate Limits (Try Substitution First)

Example 6 (page 67) – Exploring a Limit That Fails to Exist (DNE)

VIDEO (KHAN ACADEMY): UNDEFINED LIMITS BY DIRECT SUBSTITUTION

Homework 1.1b: page 70 # 22, 23, 30, 32, 33, 55

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c. Theorem 3 – One-sided and Two-sided Limits (and Limits That Fail to Exist)

Right Hand Limit (RHL) and Left Hand Limit (LHL) and Piecewise Functions

VIDEO (KHAN ACADEMY): LIMITS OF PIECEWISE FUNCTIONS
VIDEO (KHAN ACADEMY): LIMITS OF PIECEWISE FUNCTIONS: ABS. VALUE

Example 7 (page 68) – Greatest Integer Function

Example 8 (page 68) – Exploring Limits Graphically

VIDEO (KHAN ACADEMY): ONE-SIDED LIMITS FROM GRAPHS

Homework 1.1c: page 70 # 43, 45, 46, 57, 58, 73 – 76

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d. Theorem 4 – Squeeze (Sandwich) Theorem

Certain limits which cannot be determined directly can be determined **indirectly**; one such method is the Squeeze (Sandwich) Theorem.

Example 9 (page 69)

VIDEO (KHAN ACADEMY): SQUEEZE THEOREM INTRO
VIDEO (KHAN ACADEMY): LIMIT OF [SIN (X)] / X AS X → 0
VIDEO (KHAN ACADEMY): LIMIT OF [1 – COS (X)] / X AS X → 0

Homework 1.1d: page 70 # 68

Part B. Speed of a Particle

1) Average and Instantaneous Speed

The speed of a particle (object) is a specific example of the more general concept of a **rate of change**; speed is the rate at which the *position* of a particle changes with respect to *time*. Therefore any exploration of speed can be extended to the more general notion of a rate of change in any quantity with respect to some other quantity; in many practical applications, the second quantity is very often time, but not always.

To help motivate this concept we will investigate an object in free-fall. According to physics (and of course mathematics), the distance an object travels near the Earth's surface is governed by:

$\mathbf{s = 16t^2}$, where **t** is time measured in seconds, and **s** is distance measured in feet.

a. average speed of a particle during some time interval

i) Determine the average speed during the first three seconds (i.e. on **[0, 3]**);
ii) Determine the average speed on **[1, 4]**;
iii) Determine the average speed on **[2, 3]**;

b. instantaneous speed of a particle (i.e. speed at a specific moment in time)

Determine the instantaneous speed of this particle at $t = 2$ seconds.

i) Determine the average speed on **[2, 3]**; (see above)
ii) Determine the average speed on **[2, 2.1]**;
iii) Determine the average speed on **[2, 2.01]**;
iv) Determine the average speed on **[2, 2.001]**;

How can you use these average speeds to get a "handle" on the instantaneous speed for this particle at t = 2?

c. using the calculator NUMERICALLY determine the instantaneous speed

d. using ALGEBRAIC method, confirm the numerical investigation above {similar to page 64 Example 2}

Homework 1.1d: page 70 # 2, 4, 69

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Section 1.2 - LIMITS Involving Infinity

1) Finite LIMITS as $x \to \pm\infty$ (i.e. limits that exist as $x \to \pm\infty$)

VIDEO (KHAN ACADEMY): INTRODUCTION TO LIMITS AT INFINITY

VIDEO (McTAN UNIVERSITY): LIMITS OF RATIONAL EXPRESSIONS 1
VIDEO (McTAN UNIVERSITY): LIMITS OF RATIONAL EXPRESSIONS 2
VIDEO (McTAN UNIVERSITY): LIMITS OF RATIONAL EXPRESSIONS 3

a. Horizontal Asymptotes

Read opening paragraph on page 74.

VIDEO (McTAN UNIVERSITY): HORIZONTAL ASYMPTOTES 1
VIDEO (McTAN UNIVERSITY): HORIZONTAL ASYMPTOTES 2

VIDEO (KHAN ACADEMY): LIMITS AT INFINITY & HORIZONTAL ASYMPTOTES

Example # 1: Evaluate $\lim_{x\to\infty}\frac{1+2x}{x}$ = ______ **{**graphically, then numerically}

The line y = L is a horizontal asymptote of the graph of the function y = f(x) iff either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Does the function $f(x) = \frac{1 + 2x}{x}$ **have any horizontal asymptotes? If yes, show the steps/work that lead to your conclusion, and write the equation(s).**

Example #2: Determine any horizontal asymptotes for the function:

$g(x) = \frac{x}{\sqrt{x^2+1}}$. Show the steps/work.

VIDEO (McTAN UNIVERSITY): HORIZONTAL ASYMPTOTES w/RADICALS – A
VIDEO (McTAN UNIVERSITY): HORIZONTAL ASYMPTOTES w/RADICALS – B

Do you notice anything unusual about this function's asymptotic behavior?

b. Squeeze (Sandwich) Theorem Revisited

Prove: $\lim_{x\to\infty}\frac{\sin x}{x} = 0$

c. Properties/Theorems of LIMITS as x → ∞ (very similar to LIMITS as x → c)

Example: Use Properties of Limits to Evaluate: $\lim_{x\to\infty}\frac{5x+\sin x}{x}$.

Homework 1.2a: page 80 # 1 – 5, 12, 24

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2) End Behavior Models (EBM)

End Behavior Models are just that – "simple" power functions that model the behavior of given functions at extreme values of x. The models are useful "approximations" for the more complicated given functions as x → ±∞.

a. Right EBM: A function g(x) is a REBM for a function f(x) iff $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

b. Left EBM: A function g(x) is a REBM for a function f(x) iff $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1$.

{Note: If the REBM and LEBM are identical, then it is referred to as the EBM}

Examples: Determine the EBMs for each of the following; and **verify**.
{See page 78 Examples 7 and 8}

i) $f(x) = \frac{2x^5 + x^3 - 4x + 1}{5x^2 - 3x + 7}$

ii) $g(x) = \frac{5x^2 - 3x + 7}{2x^5 + x^3 - 4x + 1}$

iii) $m(x) = \frac{5x^2 - 3x + 7}{3x^2 - 4x + 1}$

iv) $h(x) = x + e^{-x}$

Homework 1.2b: page 80 # 39, 41 (verify #41 using the definition of EBM), 42, 43, 45

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3) Infinite Limits as x → a

Read paragraph at bottom of page 76.

VIDEO (KHAN ACADEMY): INTRODUCTION TO INFINITE LIMITS
VIDEO (KHAN ACADEMY): ANALYZING UNBOUNDED LIMITS: RATIONAL F'NS

a. Vertical Asymptotes

VIDEO (McTAN UNIVERSITY): VERTICAL ASYMPTOTES

Example: Determine: $\lim_{x\to1} \frac{3x-1}{x-1}$

4) Investigating f(x) as x → ±∞ by investigating f(1/x) as x → 0 AND conversely.

a. Substitution – an **indirect method** for evaluating limits.

Example: Determine $\lim_{x\to\infty} \sin\frac{1}{x}$.

Homework 1.2c: page 80 # 13, 14, 15, 27, 29, 54, 55, 62, 63, 64

Quick Quiz for AP Preparation: page 81 # 1 – 4

Review Exercises for Test #1: page 101 # 1, 5, 6, 8, 10, 14, 15 – 20, 27, 33, 34, 35, 42, 54, 57

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Section 1.3 – Continuity (Handout)

1) Read Introduction on page 82

2) Graphical Analysis for Continuity: Example 1 (Pages 82 – 83)
3) Definition – Continuity at a Point

VIDEO (KHAN ACADEMY): CONTINUITY AT POINT

a. interior point – a function, y = f(x), is continuous at an interior point, $c \in D_f$, iff
$\lim_{x \to c} f(x) = f(c)$.

b. endpoint – a function, y = f(x), is continuous at a left (right) endpoint iff

$\lim_{x \to a+} f(x) = f(a)$ or $\lim_{x \to b-} f(x) = f(b)$

If a function, f, is not continuous at x = c, we say that **f is discontinuous at x = c**, and that there is a point of discontinuity at x = c.
{Note: c does not have to be in D_f}

c. Test for Continuity

We say that a function f, is continuous at a point x = c, iff

i) f(c) exists
ii) $\lim_{x \to c} f(x)$ exists

{Now, reconsider the handout, discuss, and have students figure out the next part}

iii) $\lim_{x \to c} f(x) = f(c)$

VIDEO (KHAN ACADEMY): WORKED EXAMPLE CONTINUITY AT A POINT (GRAPHICAL)

4) Algebraic Analysis for Continuity for a Piecewise Defined Functions – page 89 # 48

VIDEO (McTAN UNIVERSITY): CONTINUITY FOR A PIECEWISE FUNCTION 1
VIDEO (McTAN UNIVERSITY): CONTINUITY FOR A PIECEWISE FUNCTION 2
VIDEO (McTAN UNIVERSITY): CONTINUITY FOR A PIECEWISE FUNCTION 3

5) Types of Discontinuity – see page 84

Removable - criteria "c" definitely fails; criteria "b" definitely passes
Jump - criteria "b" definitely fails; other criteria could go either way
Infinite - criteria "b" definitely fails; other criteria could go either way
Oscillating - criteria "b" definitely fails; other criteria could go either way

VIDEO (KHAN ACADEMY): TYPES OF DISCONTINUITIES

Homework 1.3a: page 88 # 2, 3, 7, 11 – 14, 19, 20, 21, 23, 41, 43, 47, 49, 58

6) Continuous Extensions and "Removing Discontinuities" – Exploration #1 (page 85)

VIDEO (McTAN UNIVERSITY): CONTINUOUS EXTENSIONS

7) Continuity over an Interval and Continuous Functions – read page 85

VIDEO (KHAN ACADEMY): CONTINUITY OVER AN INTERVAL

8) Intermediate Value Theorem for Functions – an *Existence Theorem*

If a function ***f*** is continuous on a closed interval **[a,b]**, then ***f(x)*** takes on every y-value between ***f(a)*** and ***f(b)*** at least once.

VIDEO (KHAN ACADEMY): INTERMEDIATE VALUE THEOREM (IVT)
VIDEO (KHAN ACADEMY): WORKED EXAMPLE: USING THE IVT
VIDEO (KHAN ACADEMY): JUSTIFICATION WITH THE IVT: TABLE
VIDEO (KHAN ACADEMY): JUSTIFICATION WITH THE IVT: EQUATION

9) Application – IVT for Functions and the "Existence" of Solutions

Example 5 (Page 87) – Is there any real number which is 3 less than its cube?

Homework 1.3b: page 88 # 15 – 18, 25, 27, 29, 30, 31, 45, 56, 57, 59

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Section 1.4 - Rates of Change and Tangent Lines

1) Average Rates of Change

a. Read Introduction – page 91
b. Example #1: Algebraic – page 91
c. Example #2: Graphical/Numeric – page 91
d. Connection: Average Rate of Change and Slope of a Secant Line

VIDEO (KHAN ACADEMY): SECANT LINES AND AVERAGE RATE OF CHANGE

2) Secant Lines → Tangent Line

3) Instantaneous Rate of Change and Slope of the Tangent Line

a. Read page 92 (bottom)
b. Example #3 – page 93
    i) Slope of a Tangent Line
    ii) Equation of a Tangent Line

Homework 1.4a: page 97 # 2a, 3b, 4a, 6b, 7

4) Slope of a Curve at a Point – Definition

> The **slope of a curve**, y = f(x), at the point P(a, f(a)) is the number
>
> $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided the limit exists.

5) Connections Revisited (Restated)

a. Average Rate of Change = Slope of a Secant Line = Difference Quotient
b. Instantaneous Rate of Change = Slope of the Tangent Line = LIMIT of the DQ

6) Piecewise Functions and Slope of a Tangent Line – see page 98 # 16

7) Another Example of Slope and Tangents – Example #4 – page 94

8) Determining Equations of Normal Lines to Curves

9) Velocity as a Rate of Change – page 95

10) Sensitivity – Page 96 (Example 8)

Homework 1.4b: page 97 # 9, 10, 15, 20, 29, 31, 33, 37, 40abc, 42 – 46, 57
Quick Quiz for AP Preparartion: page 100 # 1 – 4

Review Exercises for Test #2: page 101 # 25, 29, 31, 39, 40, 43, 45, 47, 48, 50, 51, 56