# Section 7.4
# Transforming Random Variables

## (DAY 1)

**Learning Objectives**

After this section, you should be able to…

- ✓ DESCRIBE the effect of performing a linear transformation on a random variable **(DAY 1)**
- ✓ COMBINE random variables and CALCULATE the resulting mean and standard deviation **(DAY 1)**
- ✓ CALCULATE and INTERPRET probabilities involving combinations and Normal random variables **(DAY 2)**
- ✓ **HW: 7-4 Homework Handout -- #'s 24-42**



## Linear Transformations

- **In previous Section**, we learned that the mean and standard deviation give us important information about a random variable.
- **In this section**, we'll learn how the mean and standard deviation are affected by transformations on random variables.

**ACTIVITY: The Wolf STAT Company:**

SUMMARY OF ACTIVITY FINDINGS

*1. Adding (or subtracting) a constant, a, to each observation:*

- Adds *a* to measures of center and location.
- Does not change the shape or measures of spread.

*2. Multiplying (or dividing) each observation by a constant, b:*

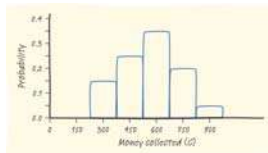
- Multiplies (divides) measures of center and location by *b*.
- Multiplies (divides) measures of spread by $|b|$.
- Does not change the shape of the distribution.

## Linear Transformations – Add/Subtract Constants

Transforming Random Variables

Consider Pete's Jeep Tours again. We defined ***C* as the amount of money Pete collects on a randomly selected day.**

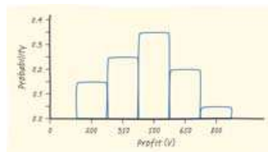
| **Collected $c_i$** | 300 | 450 | 600 | 750 | 900 |
|---|---|---|---|---|---|
| **Probability $p_i$** | 0.15 | 0.25 | 0.35 | 0.20 | 0.05 |



**The mean of *C* is \$562.50 and the standard deviation is \$163.50.**

It costs Pete \$100 per trip to buy permits, gas, and a ferry pass. The random variable *V* describes the profit Pete makes on a randomly selected day.

| **Profit $v_i$** | 200 | 350 | 500 | 650 | 800 |
|---|---|---|---|---|---|
| **Probability $p_i$** | 0.15 | 0.25 | 0.35 | 0.20 | 0.05 |



**The mean of *V* is \$462.50 and the standard deviation is \$163.50.**

***Compare the shape, center, and spread of the two probability distributions.***

## Linear Transformations - Add/Subtract Constants

Transforming Random Variables

How does adding or subtracting a constant affect a random variable?

**Effect on a Random Variable of Adding (or Subtracting) a Constant**

Adding the same number "*a*" (which could be negative) to each value of a random variable:

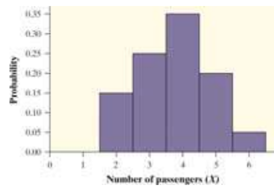
- "*a*" is added to measures of center and location (mean, median, quartiles, percentiles).
- Does NOT change measures of spread (range, *IQR*, standard deviation).
- Does not change the shape of the distribution.

## Linear Transformations – Multiply/Divide Constants

Transforming Random Variables

Pete's Jeep Tours offers a popular half-day trip in a tourist area. There must be at least 2 passengers for the trip to run, and the vehicle will hold up to 6 passengers. **Define $X$ as the number of passengers on a randomly selected day**.

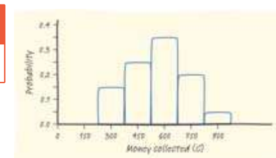
| Passengers $x_i$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Probability $p_i$ | 0.15 | 0.25 | 0.35 | 0.20 | 0.05 |




**The mean of $X$ is 3.75 and the standard deviation is 1.090.**

Pete charges $150 per passenger. **The random variable $C$ describes the amount Pete collects on a randomly selected day.**

| Collected $c_i$ | 300 | 450 | 600 | 750 | 900 |
|---|---|---|---|---|---|
| Probability $p_i$ | 0.15 | 0.25 | 0.35 | 0.20 | 0.05 |



**The mean of $C$ is $562.50 and the standard deviation is $163.50.**

***Compare the shape, center, and spread of the two probability distributions.***

## Linear Transformations – Multiply/Divide Constants

Transforming Random Variables

How does multiplying or dividing by a constant affect a random variable?

**Effect on a Random Variable of Multiplying (Dividing) by a Constant**

Multiplying (or dividing) each value of a random variable by a number *b*:

- **Multiplies** (divides) **measures of center and location** (mean, median, quartiles, percentiles) by *b.*
- **Multiplies** (divides) **measures of spread** (range, *IQR*, standard deviation) **by $|b|$.**
- **Does not change the shape of the distribution.**

**Note: Multiplying a random variable by a constant $b$ multiplies the variance by $b^2$.**

## Linear Transformations - CONCLUSION

Whether we are dealing with data or random variables, the effects of a linear transformation are the same.

**Effect on a Linear Transformation on the Mean and Standard Deviation**

If $Y = a + bX$ is a linear transformation of the random variable $X$, then

- The probability distribution of $Y$ has the same shape as the probability distribution of $X$.
- $\mu_Y = a + b\mu_X$.
- $\sigma_Y = |b|\sigma_X$ (since $b$ could be a negative number).

Transforming Random Variables

# Section 7.4
# Combining Random Variables

## (DAY 2)

**Learning Objectives**

After this section, you should be able to…

- ✓ CALCULATE and INTERPRET probabilities involving combinations **and** Normal random variables **(DAY 2)**
- ✓ **HW: 7-4 Homework Handout -- #'s 49-63**

## Combining Random Variables

***So far, we have looked at settings that involve a single random variable. Many interesting statistics problems require us to examine two or more random variables.***

**Let's investigate the result of adding and subtracting random variables:**

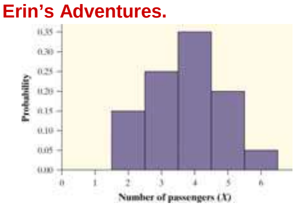
Let $X$= **the number of passengers on a randomly selected trip with Pete's Jeep Tours.**

Let $Y$ = **the number of passengers on a randomly selected trip with Erin's Adventures.**

Define $T = X + Y$. What are the mean and variance of $T$?

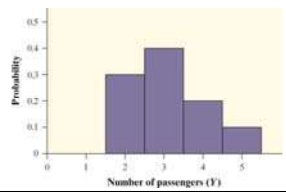
| **Passengers $x_i$** | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| **Probability $p_i$** | 0.15 | 0.25 | 0.35 | 0.20 | 0.05 |

Mean $\mu_X$ = 3.75  Standard Deviation $\sigma_X$ = 1.090




| **Passengers $y_i$** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|
| **Probability $p_i$** | 0.3 | 0.4 | 0.2 | 0.1 |

Mean $\mu_Y$ = 3.10  Standard Deviation $\sigma_Y$ = 0.943




## Combining Random Variables – The Expected Value

**How many total passengers can Pete and Erin expect on a randomly selected day?**

- **Since Pete expects $\mu_X$ = 3.75 and Erin expects $\mu_Y$ = 3.10,**
- **They will average a total of 3.75 + 3.10 = 6.85 passengers per trip.**

- **We can generalize this result as follows:**

**Mean of the Sum of Random Variables**

For any two random variables $X$ and $Y$, if $T = X + Y$, then the expected value of $T$ is

$$E(T) = \mu_T = \mu_X + \mu_Y$$

In general, the mean of the sum of several random variables is the sum of their means.

Combining Random Variables

## Combining Random Variables – **Measure of Variability**

**How much variability is there in the total number of passengers who go on Pete's and Erin's tours on a randomly selected day?**

1. **To determine this, we need to find the probability distribution of T.**
2. **The only way to determine the probability for any value of T is if X and Y are INDEPENDENT random variables.**

> **Definition:**
> If knowing whether any event involving *X* alone has occurred tells us nothing about the occurrence of any event involving *Y* alone, and vice versa, then *X* and *Y* are **independent random variables.**

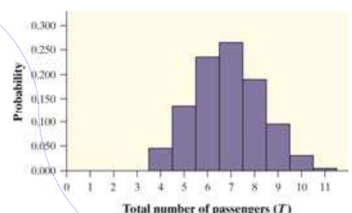
- **Probability models often assume independence when the random variables describe outcomes that appear unrelated to each other.**
- **You should always ask whether the assumption of independence seems reasonable.**
- **In our investigation, it is reasonable to assume *X* and *Y* are independent since the siblings operate their tours in different parts of the country.**

\+ Combining Random Variables

## Combining Random Variables – **Measure of Variability**

**Let *T* = *X* + *Y*. Consider all possible combinations of the values of *X***

| $x_i$ | $p_i$ | $y_i$ | $p_i$ | $t_i = x_i + y_i$ | $p_i$ |
|---|---|---|---|---|---|
| 2 | 0.15 | 2 | 0.3 | 4 | (0.15)(0.3) = 0.045 |
| 2 | 0.15 | 3 | 0.4 | 5 | (0.15)(0.4) = 0.060 |
| 2 | 0.15 | 4 | 0.2 | 6 | (0.15)(0.2) = 0.030 |
| 2 | 0.15 | 5 | 0.1 | 7 | (0.15)(0.1) = 0.015 |
| 3 | 0.25 | 2 | 0.3 | 5 | (0.25)(0.3) = 0.075 |
| 3 | 0.25 | 3 | 0.4 | 6 | (0.25)(0.4) = 0.100 |
| 3 | 0.25 | 4 | 0.2 | 7 | (0.25)(0.2) = 0.050 |
| 3 | 0.25 | 5 | 0.1 | 8 | (0.25)(0.1) = 0.025 |
| 4 | 0.35 | 2 | 0.3 | 6 | (0.35)(0.3) = 0.105 |
| 4 | 0.35 | 3 | 0.4 | 7 | (0.35)(0.4) = 0.140 |
| 4 | 0.35 | 4 | 0.2 | 8 | (0.35)(0.2) = 0.070 |
| 4 | 0.35 | 5 | 0.1 | 9 | (0.35)(0.1) = 0.035 |
| 5 | 0.20 | 2 | 0.3 | 7 | (0.20)(0.3) = 0.060 |
| 5 | 0.20 | 3 | 0.4 | 8 | (0.20)(0.4) = 0.080 |
| 5 | 0.20 | 4 | 0.2 | 9 | (0.20)(0.2) = 0.040 |
| 5 | 0.20 | 5 | 0.1 | 10 | (0.20)(0.1) = 0.020 |
| 6 | 0.05 | 2 | 0.3 | 8 | (0.05)(0.3) = 0.015 |
| 6 | 0.05 | 3 | 0.4 | 9 | (0.05)(0.4) = 0.020 |
| 6 | 0.05 | 4 | 0.2 | 10 | (0.05)(0.2) = 0.010 |
| 6 | 0.05 | 5 | 0.1 | 11 | (0.05)(0.1) = 0.005 |




**Recall: $\mu_T = \mu_X + \mu_Y = 6.85$**

$$\sigma_T^2 = \sum (t_i - \mu_T)^2 p_i$$

$= (4 - 6.85)^2(0.045) + \ldots + (11 - 6.85)^2(0.005) =$ **2.0775**

**Note:** $\sigma_X^2 = 1.1875$ and $\sigma_Y^2 = 0.89$

**What do you notice about the variance of *T*?**

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## Combining Random Variables – **Measure of Variability**

As the preceding example illustrates, **when we add two *independent* random variables, their variances add.** *Standard deviations do not add.*

**Variance of the Sum of Random Variables**

For any two *independent* random variables $X$ and $Y$, if $T = X + Y$, then the variance of $T$ is

$$\sigma_T^2 = \sigma_X^2 + \sigma_Y^2$$

In general, the variance of the sum of several independent random variables is the sum of their variances.

***Remember that you can add variances only if the two random variables are independent,***

***and that you can NEVER add standard deviations!***

**Note: the more random variables you add, means more variability!!!!!!!!!!!**



## Combining Random Variables – General Rules

- The same rules apply when we **subtract** Independent Random Variables:

**Mean of the Difference of Random Variables**

For any two random variables $X$ and $Y$, if $D = X - Y$, then the expected value of $D$ is

$$E(D) = \mu_D = \mu_X - \mu_Y$$

In general, the mean of the difference of several random variables is the difference of their means. *The order of subtraction is important!*

**Variance of the Difference of Random Variables**

For any two *independent* random variables $X$ and $Y$, if $D = X - Y$, then the variance of $D$ is

$$\sigma_D^2 = \sigma_X^2 + \sigma_Y^2$$

In general, the variance of the difference of two independent random variables is the sum of their variances.

## Combining **Normal Random Variables**

If a random variable is Normally distributed, we can use its mean and standard deviation to <u>compute probabilities</u>.

➲ An important fact about Normal random variables is that…

*any sum or difference of independent Normal random variables is also Normally distributed.*

**Example**

Mr. Starnes likes between 8.5 and 9 grams of sugar in his hot tea. Suppose the amount of sugar in a randomly selected packet follows a Normal distribution with mean 2.17 g and standard deviation 0.08 g. If Mr. Starnes selects 4 packets at random, what is the probability his tea will taste right?

**Let $X$ = the amount of sugar in a randomly selected packet.**
**Then, $T = X_1 + X_2 + X_3 + X_4$.**
**We want to find $P(8.5 \le T \le 9)$.**

$$\mu_T = \mu_{X1} + \mu_{X2} + \mu_{X3} + \mu_{X4} = 2.17 + 2.17 + 2.17 + 2.17 = 8.68$$

$$\sigma_T^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2 + \sigma_{X_4}^2 = (0.08)^2 + (0.08)^2 + (0.08)^2 + (0.08)^2 = 0.0256$$

$$\sigma_T = \sqrt{0.0256} = 0.16$$

Combining Random Variables

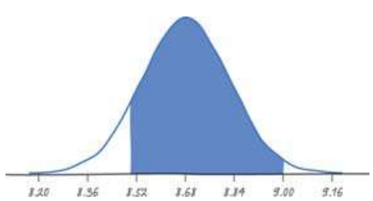
## Combining **Normal Random Variables**

Example (continued)

***X* = the amount of sugar in a randomly selected packet.** $\mu_T = 8.68$
**$T = X_1 + X_2 + X_3 + X_4$.**

$$\sigma_T = \sqrt{0.0256} = 0.16$$

**We want to find $P(8.5 \le T \le 9)$**




**Method 1:** Calculate Z-Scores

$$z = \frac{8.5 - 8.68}{0.16} = -1.13 \quad \text{and} \quad z = \frac{9 - 8.68}{0.16} = 2.00$$

$P(-1.13 \le Z \le 2.00) = 0.9772 – 0.1292 = 0.8480$

<u>**Method 2**</u>**:** State N(8.68,.16). Then use ***normalcdf(8.5, 9, 8.68, .16) = 0.8469***

**There is about an 85% chance Mr. Starnes's tea will taste right.**

## Combining Random Variables - TRY THESE

**Random variables.** Given independent random variables with means and standard deviations as shown, find the mean and standard deviation of:

a) $2Y + 20$
b) $3X$
c) $0.25X + Y$
d) $X - 5Y$
e) $X_1 + X_2 + X_3$

| | Mean | SD |
|---|---|---|
| X | 80 | 12 |
| Y | 12 | 3 |

**Kittens.** In a litter of seven kittens, three are female. You pick two kittens at random.

a) Create a probability model for the number of male kittens you get.
b) What's the expected number of males?
c) What's the standard deviation?

RANDOM VARIABLE EXAMPLE

X and Y are Independent

$\mu_x = 80$ $\mu_y = 12$

$\sigma_x = 12$ $\sigma_y = 3$

(A) $2Y + 20$:

$\mu = 2(12) + 20$ $\boxed{\mu = 44}$

$SD(2Y+20) = 2(SD(Y)) = 2(3)$ $\boxed{\sigma = 6}$

← Constant does not change variability

(B) $3X$:

$E(3X) = 3 \cdot \mu_x = 3 \cdot 80$ $\boxed{\mu = 240}$

$SD(3X) = 3 \cdot \sigma_x = 3 \cdot 12$ $\boxed{\sigma = 36}$

(C) $.25X + Y$:

$E(.25X+Y) = .25(80) + 12 =$ $\boxed{\mu = 32}$

$SD(.25X+Y) = \sqrt{(.25)^2\sigma_x^2 + \sigma_y^2}$

$= \sqrt{.0625(144) + 9} = \sqrt{18}$ $\boxed{\sigma = 4.2426}$

(D) $X - 5Y$:

$E(X-5Y) = \mu_x - 5\mu_y = 80 - 5(12) =$ $\boxed{\mu = 20}$

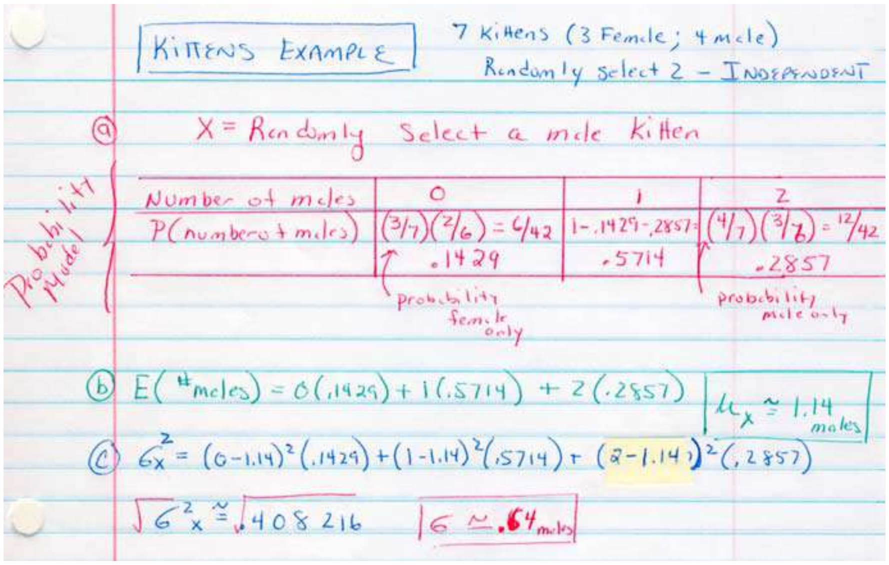
$SD(X-5Y) = \sqrt{(\sigma_x)^2 + (-5)^2(\sigma_y)^2} = \sqrt{12^2 + 25(9)} = \sqrt{369}$

$\boxed{\sigma = 19.209}$

(E) $X_1 + X_2 + X_3$:

$E(X_1+X_2+X_3) = 80 + 80 + 80$ $\boxed{\mu = 240}$

$SD(X_1+X_2+X_3) = \sqrt{12^2+12^2+12^2} = \sqrt{432}$ $\boxed{\sigma = 20.78}$



KITTENS EXAMPLE

7 Kittens (3 Female; 4 male)

Randomly select 2 – INDEPENDENT

(a) X = Randomly Select a male Kitten

Probability Model

| Number of males | 0 | 1 | 2 |
|---|---|---|---|
| P(number of males) | (3/7)(2/6) = 6/42 | 1 - .1429 - .2857 = | (4/7)(3/6) = 12/42 |
| | .1429 | .5714 | .2857 |

probability female only

probability male only

(b) E(# males) = 0(.1429) + 1(.5714) + 2(.2857) $\mu_X \approx 1.14$ males

(c) $\sigma_X^2 = (0-1.14)^2(.1429) + (1-1.14)^2(.5714) + (2-1.14)^2(.2857)$

$\sqrt{\sigma_X^2} \cong \sqrt{.408216}$ $\sigma \simeq .64$ males

# Transforming and Combining Random Variables

## Summary

In this section, we learned that…

- Adding a constant *a* (which could be negative) to a random variable increases (or decreases) the mean of the random variable by *a* but does not affect its standard deviation or the shape of its probability distribution.
- Multiplying a random variable by a constant *b* (which could be negative) multiplies the mean of the random variable by *b* and the standard deviation by |*b*| but does not change the shape of its probability distribution.
- A **linear transformation** of a random variable involves adding a constant *a*, multiplying by a constant *b*, or both. If we write the linear transformation of *X* in the form $Y = a + bX$, the following about are true about *Y*:
  - **Shape:** same as the probability distribution of *X*.
  - **Center:** $\mu_Y = a + b\mu_X$
  - **Spread:** $\sigma_Y = |b|\sigma_X$

# Transforming and Combining Random Variables

## Summary

In this section, we learned that…

- ✓ If *X* and *Y* are any two random variables,

$$\mu_{X \pm Y} = \mu_X \pm \mu_Y$$

- ✓ If *X* and *Y* are **independent random variables**

$$\sigma^2_{X \pm Y} = \sigma^2_X + \sigma^2_Y$$

- ✓ The sum or difference of independent Normal random variables follows a Normal distribution.