# Section 7.2
# Sample Proportions

## Learning Objectives

After this section, you should be able to…

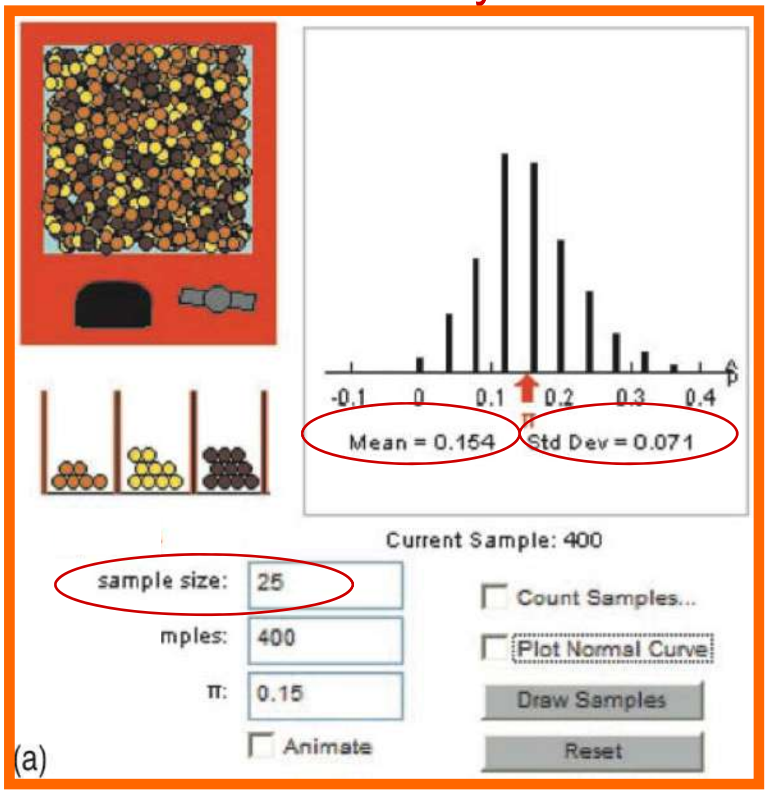
- ✓ FIND the mean and standard deviation of the sampling distribution of a sample proportion
- ✓ DETERMINE whether or not it is appropriate to use the Normal approximation to calculate probabilities involving the sample proportion
- ✓ CALCULATE probabilities involving the sample proportion
- ✓ EVALUATE a claim about a population proportion using the sampling distribution of the sample proportion

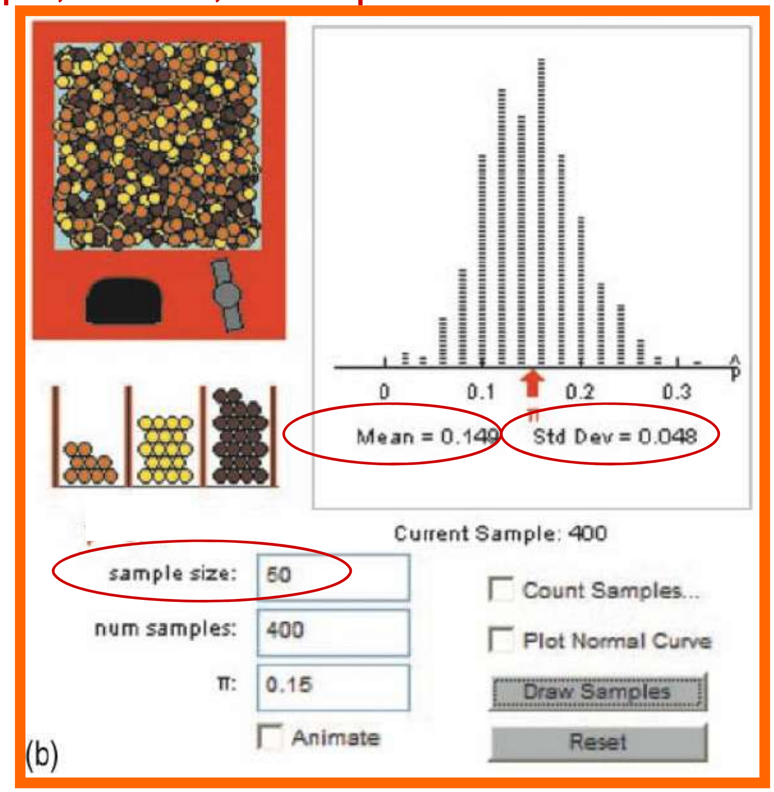
# The Sampling Distribution for the Statistic $\hat{p}$

Consider the approximate sampling distributions generated by a simulation in which SRSs of *Reese's Pieces* are drawn from a population whose proportion of orange candies is 0.15.

What happens to $\hat{p}$ as the sample size increases from 25 to 50?

What do you notice about the shape, center, and spread?




How good is the statistic $\hat{p}$ as an estimate of the parameter $p$?

The sampling distribution of $\hat{p}$ answers this question.

# ■The Sampling Distribution for the Statistic $\hat{p}$

**You should have noticed the sampling distribution has the following characteristics for shape, center, and spread:**

**Shape** : In some cases, the sampling distribution of $\hat{p}$ can be approximated by a Normal curve. This seems to depend on both the sample size $n$ and the population proportion $p$.

**Center** : The mean of the distribution is $\mu_{\hat{p}} = p$. This makes sense because the sample proportion $\hat{p}$ is an unbiased estimator of $p$.

**Spread** : For a specific value of $p$ , the standard deviation $\sigma_{\hat{p}}$ gets smaller as $n$ gets larger. The value of $\sigma_{\hat{p}}$ depends on both $n$ and $p$.

## The Connection between THE STATISTIC $\hat{p}$ and a random variable X

There is an important connection between the sample proportion $\hat{p}$ and the number of "successes" for the random variable $X$ in the sample.

$$\hat{p} = \frac{\text{count of successes in sample}}{\text{size of sample}} = \frac{X}{n}$$

**REMEMBER: for a binomial random variable *X*,** the mean and standard deviation **are:**

$$\mu_X = np \qquad \sigma_X = \sqrt{np(1-p)}$$

Since $\hat{p} = X/n$ — THEN → $\hat{p} = (1/n)\cdot X$

we are just multiplying the random variable $X$ by a constant $(1/n)$ to get the random variable $\hat{p}$.

**Now we can use algebra to calculate $\mu_{\hat{p}}$ and $\sigma_{\hat{p}}$**

## The Connection between THE STATISTIC $\hat{p}$ and a random variable X

| | | |
|---|---|---|
| **Binomial random variable *X* are:** | $\mu_X = np$ | $\sigma_X = \sqrt{np(1-p)}$ |

Since $\hat{p} = X/n$ then $\hat{p} = (1/n) \cdot X$

Therefore…

$$\mu_{\hat{p}} = \frac{1}{n}(np) = p$$

$\hat{p}$ is an unbiased estimator for $p$

$$\sigma_{\hat{p}} = \frac{1}{n}\sqrt{np(1-p)} = \sqrt{\frac{np(1-p)}{n^2}} = \sqrt{\frac{p(1-p)}{n}}$$

As sample size increases, the spread decreases.

## Using the Normal Approximation for $\hat{p}$

Inference about a population proportion $p$ is based on the sampling distribution of $\hat{p}$. when the sample size is large enough.

You must check the following 2 conditions have been met

$np \geq 10$ and
$n(1-p) \geq 10$

then the sampling distribution of $\hat{p}$ is approximately Normal.

$$\text{We can summarize the facts about the sampling distribution of } \hat{p} \text{ as follows:}$$

## Sampling Distribution of a Sample Proportion

Choose an SRS of size $n$ from a population of size $N$ with proportion $p$ of successes. Let $\hat{p}$ be the sample proportion of successes. Then:

The **mean** of the sampling distribution of $\hat{p}$ is $\mu_{\hat{p}} = p$

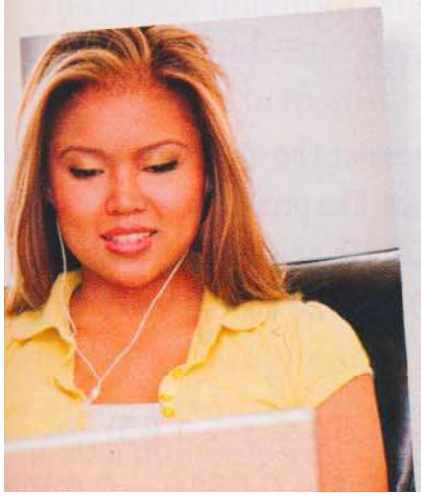
The **standard deviation** of the sampling distribution of $\hat{p}$ is

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

as long as the *10% condition* is satisfied: $n \leq (1/10)N$.

As *n* increases, the sampling distribution becomes **approximately Normal**. Before you perform Normal calculations, check that the *Normal condition* is satisfied: $np \geq 10$ and $n(1-p) \geq 10$.

## Example 1:



### CHECK YOUR UNDERSTANDING

About 75% of young adult Internet users (ages 18 to 29) watch online video. Suppose that a sample survey contacts an SRS of 1000 young adult Internet users and calculates the proportion $\hat{p}$ in this sample who watch online video.

1. What is the mean of the sampling distribution of $\hat{p}$? Explain.
2. Find the standard deviation of the sampling distribution of $\hat{p}$. Check that the 10% condition is met.
3. Is the sampling distribution of $\hat{p}$ approximately Normal? Check that the Normal condition is met.
4. If the sample size were 9000 rather than 1000, how would this change the sampling distribution of $\hat{p}$?

See next slide for worked out solution

## Example 1:

### CHECK YOUR UNDERSTANDING

(a) About 75% of young adult Internet users (ages 18 to 29) watch online video. Suppose that a sample survey contacts an SRS of 1000 young adult Internet users and calculates the proportion $\hat{p}$ in this sample who watch online video.

1. What is the mean of the sampling distribution of $\hat{p}$? Explain.
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3. Is the sampling distribution of $\hat{p}$ approximately Normal? Check that the Normal condition is met.
4. If the sample size w... distribution of $\hat{p}$?

(a) Given information: $p = .75$ (the population parameter for a proportion)

(1) The mean of the sampling distribution ($\mu_{\hat{p}}$) is the same as the population proportion → $\mu_{\hat{p}} = .15$

(2) 10% Condition: SRS = 1,000 AND IT IS FAIR TO ASSUME THE POPULATION IS OVER 10,000 young adults

$$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}} = \sqrt{\frac{(.75)(.25)}{1000}} = .0137$$

(3) The sampling distribution is approximately normal because Normal conditions met:

$np = 1000(.75) = 750 \geq 10$ ✓
$nq = 1000(.25) = 250 \geq 10$ ✓

(4) SRS $n = 9{,}000$

$$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}} = \sqrt{\frac{(.75)(.25)}{9000}} = .0046$$ (NOTICE IT DECREASES)

- **Example 2:**

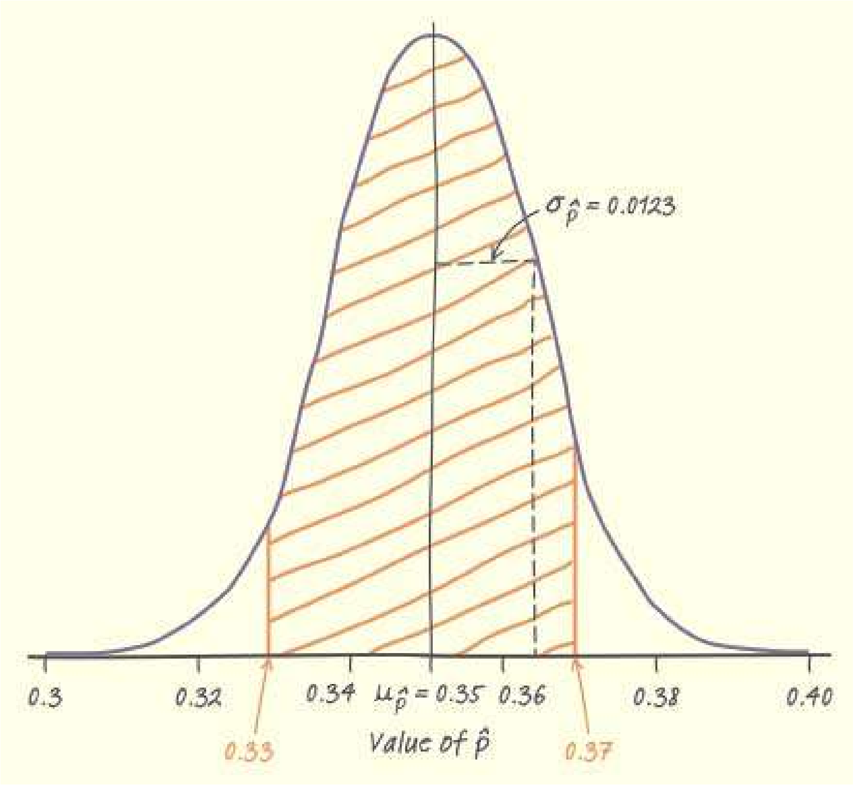
**A polling organization asks an SRS of 1500 first-year college students how far away their home is. Suppose that 35% of all first-year students actually attend college within 50 miles of home. What is the probability that the random sample of 1500 students will give a result within 2 percentage points of this true value?**

**So what are they asking?**
**Draw a picture!**

## ■ Example 2:

A polling organization asks an SRS of 1500 first-year college students how far away their home is. Suppose that 35% of all first-year students actually attend college within 50 miles of home. What is the probability that the random sample of 1500 students will give a result within 2 percentage points of this true value?

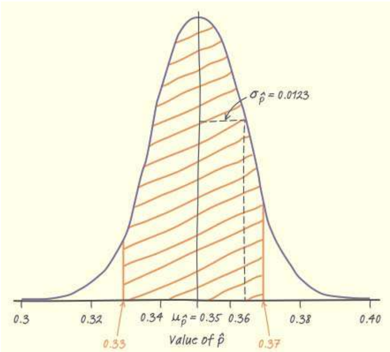
**STATE:** **We want to find the probability that the sample proportion falls between 0.33 and 0.37 (within 2 percentage points, or 0.02, of 0.35).**




**PLAN**: We have an SRS of size $n$ = 1500 drawn from a population in which the proportion $p$ = 0.35 attend college within 50 miles of home.

**Keep Going!**

## Example 2 (Cont):




**Since we know p** ($p$ = 0.35) **and n** ($n$ = 1500) **then we can find the mean and standard deviation**:

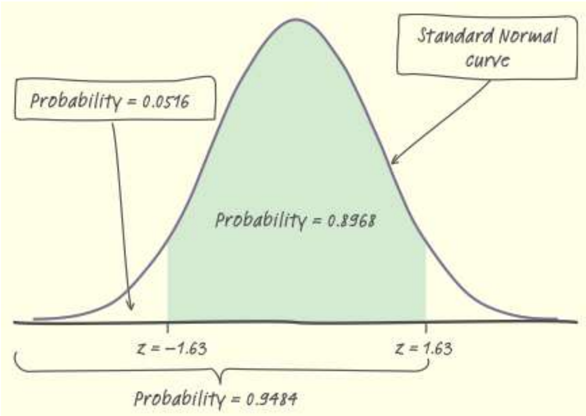
$$\mu_{\hat{p}} = 0.35$$

$$\sigma_{\hat{p}} = \sqrt{\frac{(0.35)(0.65)}{1500}} = 0.0123$$

**Can we use the normal model?**

- Since $np$ = 1500(0.35) = 525 and $n(1 - p)$ = 1500(0.65)=975
- And both are both greater than 10, we can use the normal model.

**•Next standardize to find the desired probability.**




$$z = \frac{0.33 - 0.35}{0.0123} = -1.63 \qquad z = \frac{0.37 - 0.35}{0.0123} = 1.63$$

$$P(0.33 \le \hat{p} \le 0.37) = P(-1.63 \le Z \le 1.63) = 0.9484 - 0.0516 = 0.8968$$

**CONCLUDE**: **About 90% of all SRSs of size 1500 will give a result within 2 percentage points of the truth about the population.**

**Example 3:** The Superintendent of a large school wants to know the proportion of high school students in her district are planning to attend a four-year college or university. Suppose that 80% of all high school students in her district are planning to attend a four-year college or university. What is the probability that an SRS of size 125 will give a result within 7 percentage points of the true value?

See next slide for worked out solution

**Example 3:** The Superintendent of a large school wants to know the proportion of high school students in her district are planning to attend a four-year college or university. Suppose that 80% of all high school students in her district are planning to attend a four-year college or university. What is the probability that an SRS of size 125 will give a result within 7 percentage points of the true value?

$\hat{p} = .8$ = proportion of HS students planning to attend 4-yr college

SRSs $n = 125$

Find Probability $\hat{p} = .8 \pm 7\% \longleftrightarrow P(.73 \le \hat{p} \le .87)$

Check conditions

(1) 10% Condition – IS THE SCHOOL DISTRICT LARGE ENOUGH?

$n = 125 * 10 = 1{,}250$ (we assume the school has 1,250 HS students which seems reasonable for a large school)

(2) Normal condition – $np = 125(.8) = 100 \ge 10$ ✓

met ✓ $nq = 125(.2) = 25 \ge 10$ ✓

we can use the Normal approximation

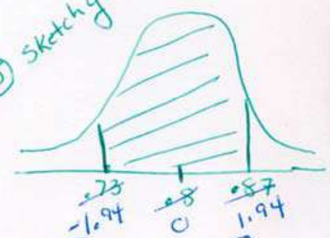
(3) Find mean and std dev:

$\mu_{\hat{p}} = .8$ $\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}} = \sqrt{\frac{(.8)(.2)}{125}}$

$\sigma_{\hat{p}} = .036$

(4) State model $N(.8, .036)$

(5) sketch graph




(6) Find Probability by using Z scores

$P(.73 \le \hat{p} \le .87) = P(-1.94 \le \hat{p} \le 1.94)$

$z = \frac{.73 - .8}{.036}$ $z = \frac{.87 - .8}{.036}$

$z = -1.94$ $z = 1.94$

What would you guess the probability?

(7) Since the Z scores are ± 1.94 (about 2 std deviations)

Remember the 68-95-99.7 rule!

The probability should be around 95%

FIND $P(-1.94 \leq \hat{p} \leq 1.94) = .9476$

$N(0,1) \rightarrow$ normalcdf(−1.94, 1.94, 0, 1)

Conclude: About 95% of all SRSs of size 125 will give a sample proportion within 7 points of the true population proportion of high school students who are planning to attend a 4 year college or university.

# Sample Proportions

## Summary

In this section, we learned that…

- ✓ When we want information about the population proportion $p$ of successes, we often take an SRS and use the sample proportion $\hat{p}$ to estimate the unknown parameter $p$. The **sampling distribution** of $\hat{p}$ describes how the statistic varies in all possible samples from the population.
- ✓ The **mean** of the sampling distribution of $\hat{p}$ is equal to the population proportion $p$. That is, $\hat{p}$ is an unbiased estimator of $p$.
- ✓ The **standard deviation** of the sampling distribution of $\hat{p}$ is $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$ for an SRS of size $n$. This formula can be used if the population is at least 10 times as large as the sample (the 10% condition). The standard deviation of $\hat{p}$ gets smaller as the sample size $n$ gets larger.
- ✓ When the sample size $n$ is larger, the sampling distribution of $\hat{p}$ is close to a Normal distribution with mean $p$ and standard deviation $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$.
- ✓ In practice, use this Normal approximation when both $np \geq 10$ and $n(1 - p) \geq 10$ (the Normal condition).