CHAPTER REVIEW EXERCISES

In Exercises 1–4, refer to the function $f(x)$ whose graph is shown in Figure 1.

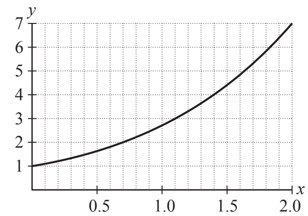


FIGURE 1

1. Compute the average rate of change of $f(x)$ over $[0, 2]$. What is the graphical interpretation of this average rate?

SOLUTION The average rate of change of $f(x)$ over $[0, 2]$ is

$$\frac{f(2) - f(0)}{2 - 0} = \frac{7 - 1}{2 - 0} = 3.$$

Graphically, this average rate of change represents the slope of the secant line through the points $(2, 7)$ and $(0, 1)$ on the graph of $f(x)$.

2. For which value of h is $\dfrac{f(0.7 + h) - f(0.7)}{h}$ equal to the slope of the secant line between the points where $x = 0.7$ and $x = 1.1$?

SOLUTION Because $1.1 = 0.7 + 0.4$, the difference quotient

$$\frac{f(0.7 + h) - f(0.7)}{h}$$

is equal to the slope of the secant line between the points where $x = 0.7$ and $x = 1.1$ for $h = 0.4$.

3. Estimate $\dfrac{f(0.7 + h) - f(0.7)}{h}$ for $h = 0.3$. Is this number larger or smaller than $f'(0.7)$?

SOLUTION For $h = 0.3$,

$$\frac{f(0.7 + h) - f(0.7)}{h} = \frac{f(1) - f(0.7)}{0.3} \approx \frac{2.8 - 2}{0.3} = \frac{8}{3}.$$

Because the curve is concave up, the slope of the secant line is larger than the slope of the tangent line, so the value of the difference quotient should be larger than the value of the derivative.

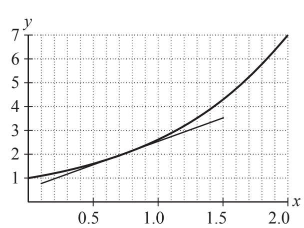
4. Estimate $f'(0.7)$ and $f'(1.1)$.

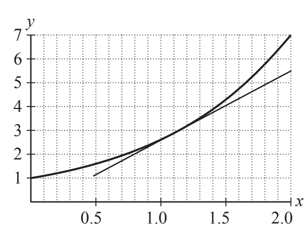
SOLUTION The tangent line sketched in the graph below at the left appears to pass through the points $(0.2, 1)$ and $(1.5, 3.5)$. Thus,

$$f'(0.7) \approx \frac{3.5 - 1}{1.5 - 0.2} = \frac{2.5}{1.3} = 1.923.$$

The tangent line sketched in the graph below at the right appears to pass through the points $(0.8, 2)$ and $(2, 5.5)$. Thus,

$$f'(1.1) \approx \frac{5.5 - 2}{2 - 0.8} = \frac{3.5}{1.2} = 2.917.$$





In Exercises 5–8, compute $f'(a)$ using the limit definition and find an equation of the tangent line to the graph of $f(x)$ at $x = a$.

5. $f(x) = x^2 - x, \quad a = 1$

SOLUTION Let $f(x) = x^2 - x$ and $a = 1$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - (1+h) - (1^2 - 1)}{h}$$
$$= \lim_{h\to 0} \frac{1 + 2h + h^2 - 1 - h}{h} = \lim_{h\to 0} (1+h) = 1$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = 1(x-1) + 0 = x - 1.$$

6. $f(x) = 5 - 3x, \quad a = 2$

SOLUTION Let $f(x) = 5 - 3x$ and $a = 2$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{5 - 3(2+h) - (5-6)}{h} = \lim_{h\to 0} -3 = -3$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = -3(x-2) - 1 = -3x + 5.$$

7. $f(x) = x^{-1}, \quad a = 4$

SOLUTION Let $f(x) = x^{-1}$ and $a = 4$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h} = \lim_{h\to 0} \frac{4 - (4+h)}{4h(4+h)}$$
$$= \lim_{h\to 0} \frac{-1}{4(4+h)} = -\frac{1}{4(4+0)} = -\frac{1}{16}$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = -\frac{1}{16}(x-4) + \frac{1}{4} = -\frac{1}{16}x + \frac{1}{2}.$$

8. $f(x) = x^3, \quad a = -2$

SOLUTION Let $f(x) = x^3$ and $a = -2$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(-2+h)^3 - (-2)^3}{h} = \lim_{h\to 0} \frac{-8 + 12h - 6h^2 + h^3 + 8}{h}$$
$$= \lim_{h\to 0} (12 - 6h + h^2) = 12 - 6(0) + 0^2 = 12$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = 12(x+2) - 8 = 12x + 16.$$

In Exercises 9–12, compute dy/dx using the limit definition.

9. $y = 4 - x^2$

SOLUTION Let $y = 4 - x^2$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{4 - (x+h)^2 - (4 - x^2)}{h} = \lim_{h\to 0} \frac{4 - x^2 - 2xh - h^2 - 4 + x^2}{h} = \lim_{h\to 0} (-2x - h) = -2x - 0 = -2x.$$

10. $y = \sqrt{2x+1}$

SOLUTION Let $y = \sqrt{2x+1}$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} = \lim_{h\to 0} \frac{\sqrt{2x+2h+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2x+2h+1} + \sqrt{2x+1}}{\sqrt{2x+2h+1} + \sqrt{2x+1}}$$
$$= \lim_{h\to 0} \frac{(2x+2h+1) - (2x+1)}{h(\sqrt{2x+2h+1} + \sqrt{2x+1})} = \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}.$$

11. $y = \dfrac{1}{2-x}$

SOLUTION Let $y = \dfrac{1}{2-x}$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{1}{2-(x+h)} - \frac{1}{2-x}}{h} = \lim_{h\to 0} \frac{(2-x)-(2-x-h)}{h(2-x-h)(2-x)} = \lim_{h\to 0} \frac{1}{(2-x-h)(2-x)} = \frac{1}{(2-x)^2}.$$

12. $y = \dfrac{1}{(x-1)^2}$

SOLUTION Let $y = \dfrac{1}{(x-1)^2}$. Then

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\frac{1}{(x+h-1)^2} - \frac{1}{(x-1)^2}}{h} = \lim_{h\to 0} \frac{(x-1)^2-(x+h-1)^2}{h(x+h-1)^2(x-1)^2}$$

$$= \lim_{h\to 0} \frac{x^2-2x+1-(x^2+2xh+h^2-2x-2h+1)}{h(x+h-1)^2(x-1)^2} = \lim_{h\to 0} \frac{-2x-h+2}{(x+h-1)^2(x-1)^2}$$

$$= \frac{-2x+2}{(x-1)^4} = -\frac{2}{(x-1)^3}.$$

In Exercises 13–16, express the limit as a derivative.

13. $\displaystyle\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$

SOLUTION Let $f(x) = \sqrt{x}$. Then

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = f'(1).$$

14. $\displaystyle\lim_{x\to -1} \frac{x^3+1}{x+1}$

SOLUTION Let $f(x) = x^3$. Then

$$\lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)} = f'(-1).$$

15. $\displaystyle\lim_{t\to \pi} \frac{\sin t \cos t}{t-\pi}$

SOLUTION Let $f(t) = \sin t \cos t$ and note that $f(\pi) = \sin \pi \cos \pi = 0$. Then

$$\lim_{t\to \pi} \frac{\sin t \cos t}{t-\pi} = \lim_{t\to \pi} \frac{f(t)-f(\pi)}{t-\pi} = f'(\pi).$$

16. $\displaystyle\lim_{\theta\to \pi} \frac{\cos\theta - \sin\theta + 1}{\theta-\pi}$

SOLUTION Let $f(\theta) = \cos\theta - \sin\theta$ and note that $f(\pi) = -1$. Then

$$\lim_{\theta\to \pi} \frac{\cos\theta - \sin\theta + 1}{\theta-\pi} = \lim_{\theta\to \pi} \frac{f(\theta)-f(\pi)}{\theta-\pi} = f'(\pi).$$

17. Find $f(4)$ and $f'(4)$ if the tangent line to the graph of $f(x)$ at $x = 4$ has equation $y = 3x - 14$.

SOLUTION The equation of the tangent line to the graph of $f(x)$ at $x = 4$ is $y = f'(4)(x-4) + f(4) = f'(4)x + (f(4) - 4f'(4))$. Matching this to $y = 3x - 14$, we see that $f'(4) = 3$ and $f(4) - 4(3) = -14$, so $f(4) = -2$.

18. Each graph in Figure 2 shows the graph of a function $f(x)$ and its derivative $f'(x)$. Determine which is the function and which is the derivative.

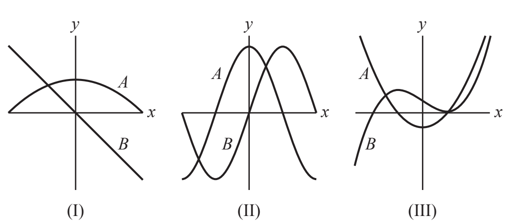


FIGURE 2 Graph of $f(x)$.

SOLUTION

- In (I), the graph labeled A is increasing when the graph labeled B is positive and is decreasing when the graph labeled B is negative. Therefore, the graph labeled A is the function $f(x)$ and the graph labeled B is the derivative $f'(x)$.
- In (II), the graph labeled B is increasing when the graph labeled A is positive and is decreasing when the graph labeled A is negative. Therefore, the graph labeled B is the function $f(x)$ and the graph labeled A is the derivative $f'(x)$.
- In (III), the graph labeled B has horizontal tangent lines at the locations the graph labeled A is zero. Therefore, the graph labeled B is the function $f(x)$ and the graph labeled A is the derivative $f'(x)$.

19. Is (A), (B), or (C) the graph of the derivative of the function $f(x)$ shown in Figure 3?

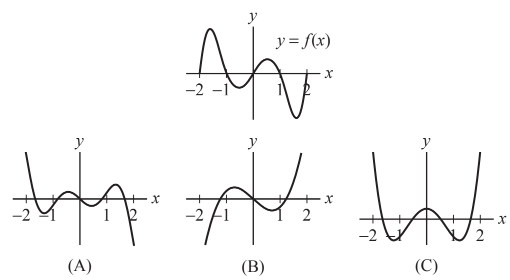


FIGURE 3

SOLUTION The graph of $f(x)$ has four horizontal tangent lines on $[-2, 2]$, so the graph of its derivative must have four x-intercepts on $[-2, 2]$. This eliminates (B). Moreover, $f(x)$ is increasing at both ends of the interval, so its derivative must be positive at both ends. This eliminates (A) and identifies (C) as the graph of $f'(x)$.

20. Let $N(t)$ be the percentage of a state population infected with a flu virus on week t of an epidemic. What percentage is likely to be infected in week 4 if $N(3) = 8$ and $N'(3) = 1.2$?

SOLUTION Because $N(4) - N(3) \approx N'(3)$, we estimate that

$$N(4) \approx N(3) + N'(3) = 8 + 1.2 = 9.2.$$

Thus, 9.2% of the population is likely infected in week 4.

21. A girl's height $h(t)$ (in centimeters) is measured at time t (in years) for $0 \le t \le 14$:

52, 75.1, 87.5, 96.7, 104.5, 111.8, 118.7, 125.2,
131.5, 137.5, 143.3, 149.2, 155.3, 160.8, 164.7

(a) What is the average growth rate over the 14-year period?
(b) Is the average growth rate larger over the first half or the second half of this period?
(c) Estimate $h'(t)$ (in centimeters per year) for $t = 3, 8$.

SOLUTION
(a) The average growth rate over the 14-year period is

$$\frac{164.7 - 52}{14} = 8.05 \text{ cm/year}.$$

(b) Over the first half of the 14-year period, the average growth rate is

$$\frac{125.2 - 52}{7} \approx 10.46 \text{ cm/year},$$

which is larger than the average growth rate over the second half of the 14-year period:

$$\frac{164.7 - 125.2}{7} \approx 5.64 \text{ cm/year}.$$

(c) For $t = 3$,

$$h'(3) \approx \frac{h(4) - h(3)}{4 - 3} = \frac{104.5 - 96.7}{1} = 7.8 \text{ cm/year};$$

for $t = 8$,

$$h'(8) \approx \frac{h(9) - h(8)}{9 - 8} = \frac{137.5 - 131.5}{1} = 6.0 \text{ cm/year}.$$

22. A planet's period P (number of days to complete one revolution around the sun) is approximately $0.199A^{3/2}$, where A is the average distance (in millions of kilometers) from the planet to the sun.
(a) Calculate P and dP/dA for Earth using the value $A = 150$.
(b) Estimate the increase in P if A is increased to 152.

SOLUTION
(a) Let $P = 0.199A^{3/2}$. Then $\frac{dP}{dA} = 0.2985A^{1/2}$. For $A = 150$,

$$P = 0.199(150)^{3/2} \approx 365.6 \text{ days; and}$$

$$\frac{dP}{dA} = 0.2985(150)^{1/2} \approx 3.656 \text{ days/millions of kilometers.}$$

(b) If A is increased to 152, then

$$P(152) - P(150) \approx 2 \times \left.\frac{dP}{dA}\right|_{A=150} = 7.312 \text{ days.}$$

In Exercises 23 and 24, use the following table of values for the number $A(t)$ of automobiles (in millions) manufactured in the United States in year t.

t	1970	1971	1972	1973	1974	1975	1976
$A(t)$	6.55	8.58	8.83	9.67	7.32	6.72	8.50

23. What is the interpretation of $A'(t)$? Estimate $A'(1971)$. Does $A'(1974)$ appear to be positive or negative?

SOLUTION Because $A(t)$ measures the number of automobiles manufactured in the United States in year t, $A'(t)$ measures the rate of change in automobile production in the United States. For $t = 1971$,

$$A'(1971) \approx \frac{A(1972) - A(1971)}{1972 - 1971} = \frac{8.83 - 8.58}{1} = 0.25 \text{ million automobiles/year.}$$

Because $A(t)$ decreases from 1973 to 1974 and from 1974 to 1975, it appears that $A'(1974)$ would be negative.

24. Given the data, which of (A)–(C) in Figure 4 could be the graph of the derivative $A'(t)$? Explain.

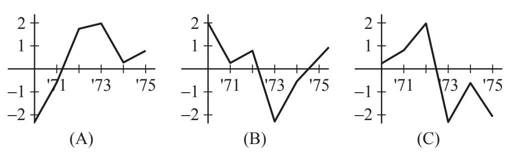


FIGURE 4

SOLUTION The values of $A(t)$ increase, then decrease and finally increase. Thus $A'(t)$ should transition from positive to negative and back to positive. This describes the graph in (B).

25. Which of the following is equal to $\frac{d}{dx}2^x$?

(a) 2^x **(b)** $(\ln 2)2^x$ **(c)** $x2^{x-1}$ **(d)** $\frac{1}{\ln 2}2^x$

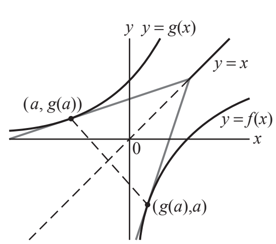
SOLUTION The derivative of $f(x) = 2^x$ is

$$\frac{d}{dx}2^x = 2^x \ln 2.$$

Hence, the correct answer is **(b)**.

26. Describe the graphical interpretation of the relation $g'(x) = 1/f'(g(x))$, where $f(x)$ and $g(x)$ are inverses of each other.

SOLUTION Suppose $f(x)$ and $g(x)$ are inverse functions. Consider a point on the graph of $y = f(x)$ – say (a, b) – and the point on the graph of $y = g(x)$ symmetric with respect to the line $y = x$ – that is, (b, a). The relation $g'(x) = 1/f'(g(x))$ indicates that the lines tangent to the two graphs at these symmetric points have slopes that are reciprocals of one another.



27. Show that if $f(x)$ is a function satisfying $f'(x) = f(x)^2$, then its inverse $g(x)$ satisfies $g'(x) = x^{-2}$.

SOLUTION

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f(g(x))^2} = \frac{1}{x^2} = x^{-2}.$$

28. Find $g'(8)$, where $g(x)$ is the inverse of a differentiable function $f(x)$ such that $f(-1) = 8$ and $f'(-1) = 12$.

SOLUTION The Theorem on the derivative of an inverse function states

$$g'(x) = \frac{1}{f'(g(x))}.$$

Setting $x = 8$, we obtain

$$g'(8) = \frac{1}{f'(g(8))}.$$

Because $f(-1) = 8$, it follows that $g(8) = -1$. Thus,

$$g'(8) = \frac{1}{f'(-1)} = \frac{1}{12}.$$

In Exercises 29–80, compute the derivative.

29. $y = 3x^5 - 7x^2 + 4$

SOLUTION Let $y = 3x^5 - 7x^2 + 4$. Then

$$\frac{dy}{dx} = 15x^4 - 14x.$$

30. $y = 4x^{-3/2}$

SOLUTION Let $y = 4x^{-3/2}$. Then

$$\frac{dy}{dx} = -6x^{-5/2}.$$

31. $y = t^{-7.3}$

SOLUTION Let $y = t^{-7.3}$. Then

$$\frac{dy}{dt} = -7.3t^{-8.3}.$$

32. $y = 4x^2 - x^{-2}$

SOLUTION Let $y = 4x^2 - x^{-2}$. Then

$$\frac{dy}{dx} = 8x + 2x^{-3}.$$

33. $y = \dfrac{x+1}{x^2+1}$

SOLUTION Let $y = \dfrac{x+1}{x^2+1}$. Then

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (x+1)(2x)}{(x^2+1)^2} = \frac{1 - 2x - x^2}{(x^2+1)^2}.$$

34. $y = \dfrac{3t-2}{4t-9}$

SOLUTION Let $y = \dfrac{3t-2}{4t-9}$. Then

$$\frac{dy}{dt} = \frac{(4t-9)(3) - (3t-2)(4)}{(4t-9)^2} = -\frac{19}{(4t-9)^2}.$$

35. $y = (x^4 - 9x)^6$

SOLUTION Let $y = (x^4 - 9x)^6$. Then

$$\frac{dy}{dx} = 6(x^4 - 9x)^5 \frac{d}{dx}(x^4 - 9x) = 6(4x^3 - 9)(x^4 - 9x)^5.$$

36. $y = (3t^2 + 20t^{-3})^6$

SOLUTION Let $y = (3t^2 + 20t^{-3})^6$. Then

$$\frac{dy}{dt} = 6(3t^2 + 20t^{-3})^5 \frac{d}{dt}(3t^2 + 20t^{-3}) = 6(6t - 60t^{-4})(3t^2 + 20t^{-3})^5.$$

37. $y = (2 + 9x^2)^{3/2}$

SOLUTION Let $y = (2 + 9x^2)^{3/2}$. Then

$$\frac{dy}{dx} = \frac{3}{2}(2 + 9x^2)^{1/2} \frac{d}{dx}(2 + 9x^2) = 27x(2 + 9x^2)^{1/2}.$$

38. $y = (x + 1)^3(x + 4)^4$

SOLUTION Let $y = (x + 1)^3(x + 4)^4$. Then

$$\begin{aligned}\frac{dy}{dx} &= 4(x + 1)^3(x + 4)^3 + 3(x + 1)^2(x + 4)^4 = (x + 1)^2(x + 4)^3(4x + 4 + 3x + 12)\\ &= (7x + 16)(x + 1)^2(x + 4)^3.\end{aligned}$$

39. $y = \dfrac{z}{\sqrt{1 - z}}$

SOLUTION Let $y = \dfrac{z}{\sqrt{1 - z}}$. Then

$$\frac{dy}{dz} = \frac{\sqrt{1 - z} - (-\frac{z}{2})\frac{1}{\sqrt{1-z}}}{1 - z} = \frac{1 - z + \frac{z}{2}}{(1 - z)^{3/2}} = \frac{2 - z}{2(1 - z)^{3/2}}.$$

40. $y = \left(1 + \dfrac{1}{x}\right)^3$

SOLUTION Let $y = \left(1 + \dfrac{1}{x}\right)^3$. Then

$$\frac{dy}{dx} = 3\left(1 + \frac{1}{x}\right)^2 \frac{d}{dx}\left(1 + \frac{1}{x}\right) = -\frac{3}{x^2}\left(1 + \frac{1}{x}\right)^2.$$

41. $y = \dfrac{x^4 + \sqrt{x}}{x^2}$

SOLUTION Let

$$y = \frac{x^4 + \sqrt{x}}{x^2} = x^2 + x^{-3/2}.$$

Then

$$\frac{dy}{dx} = 2x - \frac{3}{2}x^{-5/2}.$$

42. $y = \dfrac{1}{(1 - x)\sqrt{2 - x}}$

SOLUTION Let $y = \dfrac{1}{(1 - x)\sqrt{2 - x}} = \left((1 - x)\sqrt{2 - x}\right)^{-1}$. Then

$$\begin{aligned}\frac{dy}{dx} &= -\left((1 - x)\sqrt{2 - x}\right)^{-2} \frac{d}{dx}\left((1 - x)\sqrt{2 - x}\right) = -\left((1 - x)\sqrt{2 - x}\right)^{-2}\left(-\frac{1 - x}{2\sqrt{2 - x}} - \sqrt{2 - x}\right)\\ &= \frac{5 - 3x}{2(1 - x)^2(2 - x)^{3/2}}.\end{aligned}$$

43. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

SOLUTION Let $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \frac{d}{dx}\left(x + \sqrt{x + \sqrt{x}}\right)\\ &= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left(1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2} \frac{d}{dx}\left(x + \sqrt{x}\right)\right)\end{aligned}$$

$$= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left(1 + \frac{1}{2}\left(x + \sqrt{x}\right)^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)\right).$$

44. $h(z) = \left(z + (z+1)^{1/2}\right)^{-3/2}$

SOLUTION

$$\frac{d}{dz}\left(z + (z+1)^{1/2}\right)^{-3/2} = -\frac{3}{2}\left(z + (z+1)^{1/2}\right)^{-5/2}\frac{d}{dz}\left(z + (z+1)^{1/2}\right)$$
$$= -\frac{3}{2}\left(z + (z+1)^{1/2}\right)^{-5/2}\left(1 + \frac{1}{2}(z+1)^{-1/2}\right).$$

45. $y = \tan(t^{-3})$

SOLUTION Let $y = \tan(t^{-3})$. Then

$$\frac{dy}{dt} = \sec^2(t^{-3})\frac{d}{dt}t^{-3} = -3t^{-4}\sec^2(t^{-3}).$$

46. $y = 4\cos(2 - 3x)$

SOLUTION Let $y = 4\cos(2 - 3x)$. Then

$$\frac{dy}{dx} = -4\sin(2 - 3x)\frac{d}{dx}(2 - 3x) = 12\sin(2 - 3x).$$

47. $y = \sin(2x)\cos^2 x$

SOLUTION Let $y = \sin(2x)\cos^2 x = 2\sin x\cos^3 x$. Then

$$\frac{dy}{dx} = -6\sin^2 x\cos^2 x + 2\cos^4 x.$$

48. $y = \sin\left(\frac{4}{\theta}\right)$

SOLUTION Let $y = \sin\left(\frac{4}{\theta}\right)$. Then

$$\frac{dy}{d\theta} = \cos\left(\frac{4}{\theta}\right)\frac{d}{d\theta}\left(\frac{4}{\theta}\right) = -\frac{4}{\theta^2}\cos\left(\frac{4}{\theta}\right).$$

49. $y = \frac{t}{1 + \sec t}$

SOLUTION Let $y = \frac{t}{1 + \sec t}$. Then

$$\frac{dy}{dt} = \frac{1 + \sec t - t\sec t\tan t}{(1 + \sec t)^2}.$$

50. $y = z\csc(9z + 1)$

SOLUTION Let $y = z\csc(9z + 1)$. Then

$$\frac{dy}{dz} = -9z\csc(9z + 1)\cot(9z + 1) + \csc(9z + 1).$$

51. $y = \frac{8}{1 + \cot\theta}$

SOLUTION Let $y = \frac{8}{1 + \cot\theta} = 8(1 + \cot\theta)^{-1}$. Then

$$\frac{dy}{d\theta} = -8(1 + \cot\theta)^{-2}\frac{d}{d\theta}(1 + \cot\theta) = \frac{8\csc^2\theta}{(1 + \cot\theta)^2}.$$

52. $y = \tan(\cos x)$

SOLUTION Let $y = \tan(\cos x)$. Then

$$\frac{dy}{dx} = \sec^2(\cos x)\frac{d}{dx}\cos x = -\sin x\sec^2(\cos x).$$

53. $y = \tan(\sqrt{1 + \csc\theta})$

SOLUTION

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(\sqrt{1+\csc\theta})\frac{d}{dx}\sqrt{1+\csc\theta}\\ &= \sec^2(\sqrt{1+\csc\theta})\cdot\frac{1}{2}(1+\csc\theta)^{-1/2}\frac{d}{dx}(1+\csc\theta)\\ &= -\frac{\sec^2(\sqrt{1+\csc\theta})\csc\theta\cot\theta}{2(\sqrt{1+\csc\theta})}.\end{aligned}$$

54. $y = \cos(\cos(\cos(\theta)))$

SOLUTION Let $y = \cos(\cos(\cos(\theta)))$. Then

$$\begin{aligned}\frac{dy}{d\theta} &= -\sin(\cos(\cos(\theta)))\frac{d}{d\theta}\cos(\cos(\theta)) = \sin(\cos(\cos(\theta)))\sin(\cos(\theta))\frac{d}{d\theta}\cos(\theta)\\ &= -\sin(\cos(\cos(\theta)))\sin(\cos(\theta))\sin(\theta).\end{aligned}$$

55. $f(x) = 9e^{-4x}$

SOLUTION $\dfrac{d}{dx}9e^{-4x} = -36e^{-4x}$.

56. $f(x) = \dfrac{e^{-x}}{x}$

SOLUTION $\dfrac{d}{dx}\left(\dfrac{e^{-x}}{x}\right) = \dfrac{-xe^{-x} - e^{-x}}{x^2} = -\dfrac{e^{-x}(x+1)}{x^2}$.

57. $g(t) = e^{4t-t^2}$

SOLUTION $\dfrac{d}{dt}e^{4t-t^2} = (4-2t)e^{4t-t^2}$.

58. $g(t) = t^2e^{1/t}$

SOLUTION $\dfrac{d}{dt}t^2e^{1/t} = 2te^{1/t} + t^2\left(-\dfrac{1}{t^2}\right)e^{1/t} = (2t-1)e^{1/t}$.

59. $f(x) = \ln(4x^2+1)$

SOLUTION $\dfrac{d}{dx}\ln(4x^2+1) = \dfrac{8x}{4x^2+1}$.

60. $f(x) = \ln(e^x - 4x)$

SOLUTION $\dfrac{d}{dx}\ln(e^x - 4x) = \dfrac{e^x - 4}{e^x - 4x}$.

61. $G(s) = (\ln(s))^2$

SOLUTION $\dfrac{d}{ds}(\ln s)^2 = \dfrac{2\ln s}{s}$.

62. $G(s) = \ln(s^2)$

SOLUTION $\dfrac{d}{ds}\ln(s^2) = 2\dfrac{d}{ds}\ln s = \dfrac{2}{s}$.

63. $f(\theta) = \ln(\sin\theta)$

SOLUTION $\dfrac{d}{d\theta}\ln(\sin\theta) = \dfrac{\cos\theta}{\sin\theta} = \cot\theta$.

64. $f(\theta) = \sin(\ln\theta)$

SOLUTION $\dfrac{d}{d\theta}\sin(\ln\theta) = \dfrac{\cos(\ln\theta)}{\theta}$.

65. $h(z) = \sec(z + \ln z)$

SOLUTION $\dfrac{d}{dz}\sec(z+\ln z) = \sec(z+\ln z)\tan(z+\ln z)\left(1+\dfrac{1}{z}\right)$.

66. $f(x) = e^{\sin^2 x}$

SOLUTION $\dfrac{d}{dx}e^{\sin^2 x} = 2\sin x\cos x e^{\sin^2 x} = \sin 2x e^{\sin^2 x}$.

67. $f(x) = 7^{-2x}$

SOLUTION $\frac{d}{dx}7^{-2x} = \big(-2\ln 7\big)\big(7^{-2x}\big).$

68. $h(y) = \frac{1+e^y}{1-e^y}$

SOLUTION $\frac{d}{dy}\left(\frac{1+e^y}{1-e^y}\right) = \frac{(1-e^y)e^y - (1+e^y)(-e^y)}{(1-e^y)^2} = \frac{e^y(1-e^y+1+e^y)}{(1-e^y)^2} = \frac{2e^y}{(1-e^y)^2}.$

69. $g(x) = \tan^{-1}(\ln x)$

SOLUTION $\frac{d}{dx}\tan^{-1}(\ln x) = \frac{1}{1+(\ln x)^2}\cdot\frac{1}{x}.$

70. $G(s) = \cos^{-1}(s^{-1})$

SOLUTION $\frac{d}{ds}\cos^{-1}(s^{-1}) = \frac{-1}{\sqrt{1-\left(\frac{1}{s}\right)^2}}\left(-\frac{1}{s^2}\right) = \frac{1}{\sqrt{s^4-s^2}}.$

71. $f(x) = \ln(\csc^{-1} x)$

SOLUTION $\frac{d}{dx}\ln(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}\,\csc^{-1} x}.$

72. $f(x) = e^{\sec^{-1} x}$

SOLUTION $\frac{d}{dx}e^{\sec^{-1} x} = \frac{1}{|x|\sqrt{x^2-1}}e^{\sec^{-1} x}.$

73. $R(s) = s^{\ln s}$

SOLUTION Rewrite

$$R(s) = \left(e^{\ln s}\right)^{\ln s} = e^{(\ln s)^2}.$$

Then

$$\frac{dR}{ds} = e^{(\ln s)^2}\cdot 2\ln s\cdot\frac{1}{s} = \frac{2\ln s}{s}s^{\ln s}.$$

Alternately, $R(s) = s^{\ln s}$ implies that $\ln R = \ln\left(s^{\ln s}\right) = (\ln s)^2$. Thus,

$$\frac{1}{R}\frac{dR}{ds} = 2\ln s\cdot\frac{1}{s} \quad\text{or}\quad \frac{dR}{ds} = \frac{2\ln s}{s}s^{\ln s}.$$

74. $f(x) = (\cos^2 x)^{\cos x}$

SOLUTION Rewrite

$$f(x) = \left(e^{\ln\cos^2 x}\right)^{\cos x} = e^{2\cos x\ln\cos x}.$$

Then

$$\begin{aligned}\frac{df}{dx} &= e^{2\cos x\ln\cos x}\left(2\cos x\cdot\frac{-\sin x}{\cos x} - 2\sin x\ln\cos x\right)\\ &= -2\sin x(\cos^2 x)^{\cos x}(1+\ln\cos x).\end{aligned}$$

Alternately, $f(x) = (\cos^2 x)^{\cos x}$ implies that $\ln f = \cos x\ln\cos^2 x = 2\cos x\ln\cos x$. Thus,

$$\begin{aligned}\frac{1}{f}\frac{df}{dx} &= 2\cos x\cdot\frac{-\sin x}{\cos x} - 2\sin x\ln\cos x\\ &= -2\sin x(1+\ln\cos x),\end{aligned}$$

and

$$\frac{df}{dx} = -2\sin x(\cos^2 x)^{\cos x}(1+\ln\cos x).$$

75. $G(t) = (\sin^2 t)^t$

SOLUTION Rewrite

$$G(t) = \left(e^{\ln \sin^2 t}\right)^t = e^{2t \ln \sin t}.$$

Then

$$\frac{dG}{dt} = e^{2t \ln \sin t}\left(2t \cdot \frac{\cos t}{\sin t} + 2 \ln \sin t\right) = 2(\sin^2 t)^t (t \cot t + \ln \sin t).$$

Alternately, $G(t) = (\sin^2 t)^t$ implies that $\ln G = t \ln \sin^2 t = 2t \ln \sin t$. Thus,

$$\frac{1}{G}\frac{dG}{dt} = 2t \cdot \frac{\cos t}{\sin t} + 2 \ln \sin t,$$

and

$$\frac{dG}{dt} = 2(\sin^2 t)^t (t \cot t + \ln \sin t).$$

76. $h(t) = t^{(t^t)}$

SOLUTION Let $h(t) = t^{(t^t)}$. Then $\ln h = t^t \ln t$ and

$$\begin{aligned}\ln(\ln h) &= \ln(t^t \ln t) = \ln t^t + \ln(\ln t)\\ &= t \ln t + \ln(\ln t).\end{aligned}$$

Thus,

$$\frac{1}{h \ln h}\frac{dh}{dt} = t \cdot \frac{1}{t} + \ln t + \frac{1}{t \ln t} = 1 + \ln t + \frac{1}{t \ln t},$$

and

$$\frac{dh}{dt} = t^{(t^t)} t^t \ln t \left(1 + \ln t + \frac{1}{t \ln t}\right).$$

77. $g(t) = \sinh(t^2)$

SOLUTION $\frac{d}{dt}\sinh(t^2) = 2t \cosh(t^2)$.

78. $h(y) = y \tanh(4y)$

SOLUTION $\frac{d}{dy} y \tanh(4y) = \tanh(4y) + 4y \operatorname{sech}^2(4y)$.

79. $g(x) = \tanh^{-1}(e^x)$

SOLUTION $\frac{d}{dx}\tanh^{-1}(e^x) = \frac{1}{1-(e^x)^2} e^x = \frac{e^x}{1-e^{2x}}$.

80. $g(t) = \sqrt{t^2-1}\,\sinh^{-1} t$

SOLUTION $\frac{d}{dt}\sqrt{t^2-1}\,\sinh^{-1} t = \frac{t}{\sqrt{t^2-1}}\sinh^{-1} t + \sqrt{t^2-1} \cdot \frac{1}{\sqrt{t^2+1}} = \frac{t \sinh^{-1} t}{\sqrt{t^2-1}} + \sqrt{\frac{t^2-1}{t^2+1}}$.

81. For which values of α is $f(x) = |x|^\alpha$ differentiable at $x = 0$?

SOLUTION Let $f(x) = |x|^\alpha$. If $\alpha < 0$, then $f(x)$ is not continuous at $x = 0$ and therefore cannot be differentiable at $x = 0$. If $\alpha = 0$, then the function reduces to $f(x) = 1$, which is differentiable at $x = 0$. Now, suppose $\alpha > 0$ and consider the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x|^\alpha}{x}.$$

If $0 < \alpha < 1$, then

$$\lim_{x\to 0-} \frac{|x|^\alpha}{x} = -\infty \qquad \text{while} \qquad \lim_{x\to 0+} \frac{|x|^\alpha}{x} = \infty$$

and $f'(0)$ does not exist. If $\alpha = 1$, then

$$\lim_{x\to 0-} \frac{|x|}{x} = -1 \qquad \text{while} \qquad \lim_{x\to 0+} \frac{|x|}{x} = 1$$

and $f'(0)$ again does not exist. Finally, if $\alpha > 1$, then

$$\lim_{x\to 0} \frac{|x|^\alpha}{x} = 0,$$

so $f'(0)$ does exist.

In summary, $f(x) = |x|^\alpha$ is differentiable at $x = 0$ when $\alpha = 0$ and when $\alpha > 1$.

82. Find $f'(2)$ if $f(g(x)) = e^{x^2}$, $g(1) = 2$, and $g'(1) = 4$.

SOLUTION We differentiate both sides of the equation $f(g(x)) = e^{x^2}$ to obtain,

$$f'(g(x))\, g'(x) = 2xe^{x^2}.$$

Setting $x = 1$ yields

$$f'(g(1))\, g'(1) = 2e.$$

Since $g(1) = 2$ and $g'(1) = 4$, we find

$$f'(2) \cdot 4 = 2e,$$

or

$$f'(2) = \frac{e}{2}.$$

In Exercises 83 and 84, let $f(x) = xe^{-x}$.

83. Show that $f(x)$ has an inverse on $[1, \infty)$. Let $g(x)$ be this inverse. Find the domain and range of $g(x)$ and compute $g'(2e^{-2})$.

SOLUTION Let $f(x) = xe^{-x}$. Then $f'(x) = e^{-x}(1 - x)$. On $[1, \infty)$, $f'(x) < 0$, so $f(x)$ is decreasing and therefore one-to-one. It follows that $f(x)$ has an inverse on $[1, \infty)$. Let $g(x)$ denote this inverse. Because $f(1) = e^{-1}$ and $f(x) \to 0$ as $x \to \infty$, the domain of $g(x)$ is $(0, e^{-1}]$, and the range is $[1, \infty)$.

To determine $g'(2e^{-2})$, we use the formula $g'(x) = 1/f'(g(x))$. Because $f(2) = 2e^{-2}$, it follows that $g(2e^{-2}) = 2$. Then,

$$g'(2e^{-2}) = \frac{1}{f'(g(2e^{-2}))} = \frac{1}{f'(2)} = \frac{1}{-e^{-2}} = -e^2.$$

84. Show that $f(x) = c$ has two solutions if $0 < c < e^{-1}$.

SOLUTION First note that $f(x) < 0$ for $x < 0$, so we only need to examine $(0, \infty)$ for solutions to $f(x) = c$ when $c > 0$. Next, because $f'(x) = e^{-x}(1 - x)$, f is decreasing on $(1, \infty)$ and increasing on $(0, 1)$. Therefore, f is one-to-one on each of these intervals, which guarantees that the equation $f(x) = c$ can have at most one solution on each of these intervals for any value of c.

Now, let $0 < c < e^{-1}$ and consider the interval $[1, \infty)$. Because

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} = 0,$$

it follows that there exists a $d \in (1, \infty)$ such that $f(d) < c$. With $f(1) = e^{-1} > c$, it follows from the Intermediate Value Theorem that the equation $f(x) = c$ has a solution on $[1, \infty)$. Next, consider the interval $[0, 1]$. With $f(0) = 0 < c$ and $f(1) = e^{-1} > c$, it follows from the Intermediate Value Theorem that the equation $f(x) = c$ has a solution on $[0, 1]$.

In summary, the equation $f(x) = c$ has exactly two solutions for any c between 0 and e^{-1}.

In Exercises 85–90, use the following table of values to calculate the derivative of the given function at $x = 2$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	5	4	−3	9
4	3	2	−2	3

85. $S(x) = 3f(x) - 2g(x)$

SOLUTION Let $S(x) = 3f(x) - 2g(x)$. Then $S'(x) = 3f'(x) - 2g'(x)$ and

$$S'(2) = 3f'(2) - 2g'(2) = 3(-3) - 2(9) = -27.$$

86. $H(x) = f(x)g(x)$

SOLUTION Let $H(x) = f(x)g(x)$. Then $H'(x) = f(x)g'(x) + f'(x)g(x)$ and

$$H'(2) = f(2)g'(2) + f'(2)g(2) = 5(9) + (-3)(4) = 33.$$

87. $R(x) = \dfrac{f(x)}{g(x)}$

SOLUTION Let $R(x) = f(x)/g(x)$. Then

$$R'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

and

$$R'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{g(2)^2} = \frac{4(-3) - 5(9)}{4^2} = -\frac{57}{16}.$$

88. $G(x) = f(g(x))$

SOLUTION Let $G(x) = f(g(x))$. Then $G'(x) = f'(g(x))g'(x)$ and

$$G'(2) = f'(g(2))g'(2) = f'(4)g'(2) = -2(9) = -18.$$

89. $F(x) = f(g(2x))$

SOLUTION Let $F(x) = f(g(2x))$. Then $F'(x) = 2f'(g(2x))g'(2x)$ and

$$F'(2) = 2f'(g(4))g'(4) = 2f'(2)g'(4) = 2(-3)(3) = -18.$$

90. $K(x) = f(x^2)$

SOLUTION Let $K(x) = f(x^2)$. Then $K'(x) = 2xf'(x^2)$ and

$$K'(2) = 2(2)f'(4) = 4(-2) = -8.$$

91. Find the points on the graph of $f(x) = x^3 - 3x^2 + x + 4$ where the tangent line has slope 10.

SOLUTION Let $f(x) = x^3 - 3x^2 + x + 4$. Then $f'(x) = 3x^2 - 6x + 1$. The tangent line to the graph of $f(x)$ will have slope 10 when $f'(x) = 10$. Solving the quadratic equation $3x^2 - 6x + 1 = 10$ yields $x = -1$ and $x = 3$. Thus, the points on the graph of $f(x)$ where the tangent line has slope 10 are $(-1, -1)$ and $(3, 7)$.

92. Find the points on the graph of $x^{2/3} + y^{2/3} = 1$ where the tangent line has slope 1.

SOLUTION Suppose $x^{2/3} + y^{2/3} = 1$. Differentiating with respect to x leads to

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-1/3} = -\left(\frac{y}{x}\right)^{1/3}.$$

Tangents to the curve therefore have slope 1 when $y = -x$. Substituting $y = -x$ into the equation for the curve yields $2x^{2/3} = 1$, so $x = \pm\frac{\sqrt{2}}{4}$. Thus, the points along the curve $x^{2/3} + y^{2/3} = 1$ where the tangent line has slope 1 are:

$$\left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}\right) \quad \text{and} \quad \left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right).$$

93. Find a such that the tangent lines $y = x^3 - 2x^2 + x + 1$ at $x = a$ and $x = a + 1$ are parallel.

SOLUTION Let $f(x) = x^3 - 2x^2 + x + 1$. Then $f'(x) = 3x^2 - 4x + 1$ and the slope of the tangent line at $x = a$ is $f'(a) = 3a^2 - 4a + 1$, while the slope of the tangent line at $x = a + 1$ is

$$f'(a+1) = 3(a+1)^2 - 4(a+1) + 1 = 3(a^2 + 2a + 1) - 4a - 4 + 1 = 3a^2 + 2a.$$

In order for the tangent lines at $x = a$ and $x = a + 1$ to have the same slope, we must have $f'(a) = f'(a+1)$, or

$$3a^2 - 4a + 1 = 3a^2 + 2a.$$

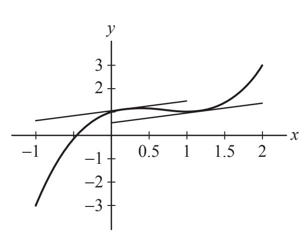
The only solution to this equation is $a = \frac{1}{6}$. The equation of the tangent line at $x = \frac{1}{6}$ is

$$y = f'\left(\frac{1}{6}\right)\left(x - \frac{1}{6}\right) + f\left(\frac{1}{6}\right) = \frac{5}{12}\left(x - \frac{1}{6}\right) + \frac{241}{216} = \frac{5}{12}x + \frac{113}{108},$$

and the equation of the tangent line at $x = \frac{7}{6}$ is

$$y = f'\left(\frac{7}{6}\right)\left(x - \frac{7}{6}\right) + f\left(\frac{7}{6}\right) = \frac{5}{12}\left(x - \frac{7}{6}\right) + \frac{223}{216} = \frac{5}{12}x + \frac{59}{108}.$$

The graphs of $f(x)$ and the two tangent lines appear below.



94. Use the table to compute the average rate of change of Candidate A's percentage of votes over the intervals from day 20 to day 15, day 15 to day 10, and day 10 to day 5. If this trend continues over the last 5 days before the election, will Candidate A win?

Days Before Election	20	15	10	5
Candidate A	44.8%	46.8%	48.3%	49.3%
Candidate B	55.2%	53.2%	51.7%	50.7%

SOLUTION The average rate of change of A's percentage for the period from day 20 to day 15 is

$$\frac{46.8 - 44.8}{5} = 0.4\%/\text{day}.$$

For the period from day 15 to day 10, the average rate of change is

$$\frac{48.3 - 46.8}{5} = 0.3\%/\text{day}.$$

Finally, for the period from day 10 to day 5, the average rate of change is

$$\frac{49.3 - 48.3}{5} = 0.2\%/\text{day}.$$

If this trend continues over the last five days before the election, the average rate of change will drop to 0.1 %/day, so A's percentage will increase another 0.5% to 49.8%. Accordingly, A will *not* win the election.

In Exercises 95–100, calculate y''.

95. $y = 12x^3 - 5x^2 + 3x$

SOLUTION Let $y = 12x^3 - 5x^2 + 3x$. Then

$$y' = 36x^2 - 10x + 3 \quad \text{and} \quad y'' = 72x - 10.$$

96. $y = x^{-2/5}$

SOLUTION Let $y = x^{-2/5}$. Then

$$y' = -\frac{2}{5}x^{-7/5} \quad \text{and} \quad y'' = \frac{14}{25}x^{-12/5}.$$

97. $y = \sqrt{2x+3}$

SOLUTION Let $y = \sqrt{2x+3} = (2x+3)^{1/2}$. Then

$$y' = \frac{1}{2}(2x+3)^{-1/2}\frac{d}{dx}(2x+3) = (2x+3)^{-1/2} \quad \text{and} \quad y'' = -\frac{1}{2}(2x+3)^{-3/2}\frac{d}{dx}(2x+3) = -(2x+3)^{-3/2}.$$

98. $y = \dfrac{4x}{x+1}$

SOLUTION Let $y = \dfrac{4x}{x+1}$. Then

$$y' = \frac{(x+1)(4) - 4x}{(x+1)^2} = \frac{4}{(x+1)^2} \quad \text{and} \quad y'' = -\frac{8}{(x+1)^3}.$$

99. $y = \tan(x^2)$

SOLUTION Let $y = \tan(x^2)$. Then

$$y' = 2x\sec^2(x^2) \quad \text{and}$$

$$y'' = 2x\left(2\sec(x^2)\frac{d}{dx}\sec(x^2)\right) + 2\sec^2(x^2) = 8x^2\sec^2(x^2)\tan(x^2) + 2\sec^2(x^2).$$

100. $y = \sin^2(4x+9)$

SOLUTION Let $y = \sin^2(4x+9)$. Then

$$y' = 8\sin(4x+9)\cos(4x+9) = 4\sin(8x+18) \quad \text{and} \quad y'' = 32\cos(8x+18).$$

In Exercises 101–106, compute $\frac{dy}{dx}$.

101. $x^3 - y^3 = 4$

SOLUTION Consider the equation $x^3 - y^3 = 4$. Differentiating with respect to x yields

$$3x^2 - 3y^2 \frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

102. $4x^2 - 9y^2 = 36$

SOLUTION Consider the equation $4x^2 - 9y^2 = 36$. Differentiating with respect to x yields

$$8x - 18y \frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx} = \frac{4x}{9y}.$$

103. $y = xy^2 + 2x^2$

SOLUTION Consider the equation $y = xy^2 + 2x^2$. Differentiating with respect to x yields

$$\frac{dy}{dx} = 2xy \frac{dy}{dx} + y^2 + 4x.$$

Therefore,

$$\frac{dy}{dx} = \frac{y^2 + 4x}{1 - 2xy}.$$

104. $\frac{y}{x} = x + y$

SOLUTION Solving $\frac{y}{x} = x + y$ for y yields

$$y = \frac{x^2}{1 - x}.$$

By the quotient rule,

$$\frac{dy}{dx} = \frac{(1-x)(2x) - x^2(-1)}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}.$$

105. $y = \sin(x + y)$

SOLUTION Consider the equation $y = \sin(x + y)$. Differentiating with respect to x yields

$$\frac{dy}{dx} = \cos(x + y) \left(1 + \frac{dy}{dx}\right).$$

Therefore,

$$\frac{dy}{dx} = \frac{\cos(x + y)}{1 - \cos(x + y)}.$$

106. $\tan(x + y) = xy$

SOLUTION Consider the equation $\tan(x + y) = xy$. Differentiating with respect to x yields

$$\sec^2(x + y) \left(1 + \frac{dy}{dx}\right) = x \frac{dy}{dx} + y.$$

Therefore,

$$\frac{dy}{dx} = \frac{y - \sec^2(x + y)}{\sec^2(x + y) - x}.$$

107. In Figure 5, label the graphs f, f', and f''.

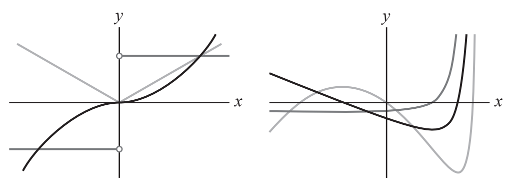


FIGURE 5

SOLUTION First consider the plot on the left. Observe that the green curve is nonnegative whereas the red curve is increasing, suggesting that the green curve is the derivative of the red curve. Moreover, the green curve is linear with negative slope for $x < 0$ and linear with positive slope for $x > 0$ while the blue curve is a negative constant for $x < 0$ and a positive constant for $x > 0$, suggesting the blue curve is the derivative of the green curve. Thus, the red, green and blue curves, respectively, are the graphs of f, f' and f''.

Now consider the plot on the right. Because the red curve is decreasing when the blue curve is negative and increasing when the blue curve is positive and the green curve is decreasing when the red curve is negative and increasing when the red curve is positive, it follows that the green, red and blue curves, respectively, are the graphs of f, f' and f''.

108. Let $f(x) = x^2 \sin(x^{-1})$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(x)$ exists for all x (including $x = 0$) but that $f'(x)$ is not continuous at $x = 0$ (Figure 6).

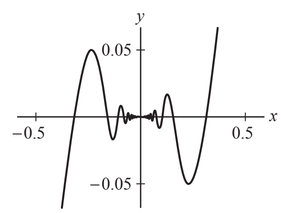


FIGURE 6 Graph of $f(x) = x^2 \sin(x^{-1})$.

SOLUTION Let $f(x) = x^2 \sin(x^{-1})$ for $x \neq 0$ and $f(0) = 0$. For $x \neq 0$, the product rule and the chain rule give

$$f'(x) = 2x \sin(x^{-1}) - x^2 \cos(x^{-1})(x^{-2}) = 2x \sin(x^{-1}) - \cos(x^{-1}),$$

which exists for all $x \neq 0$. At $x = 0$ we use the limit definition of the derivative:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{1}{h}(h^2 \sin(h^{-1})) = \lim_{h\to 0} h \sin(h^{-1}) = 0,$$

by the Squeeze Theorem, since $-h \le h \sin \frac{1}{h} \le h$. Thus, $f'(x)$ exists for all x. However,

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left(2x \sin(x^{-1}) - \cos(x^{-1})\right)$$

does not exist, so $f'(x)$ is not continuous at $x = 0$.

In Exercises 109–114, use logarithmic differentiation to find the derivative.

109. $y = \dfrac{(x+1)^3}{(4x-2)^2}$

SOLUTION Let $y = \dfrac{(x+1)^3}{(4x-2)^2}$. Then

$$\ln y = \ln\left(\frac{(x+1)^3}{(4x-2)^2}\right) = \ln(x+1)^3 - \ln(4x-2)^2 = 3\ln(x+1) - 2\ln(4x-2).$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{3}{x+1} - \frac{2}{4x-2} \cdot 4 = \frac{3}{x+1} - \frac{4}{2x-1},$$

so

$$y' = \frac{(x+1)^3}{(4x-2)^2}\left(\frac{3}{x+1} - \frac{4}{2x-1}\right).$$

110. $y = \dfrac{(x+1)(x+2)^2}{(x+3)(x+4)}$

SOLUTION Let $y = \dfrac{(x+1)(x+2)^2}{(x+3)(x+4)}$. Then

$$\begin{aligned}\ln y &= \ln\left((x+1)(x+2)^2\right) - \ln\left((x+3)(x+4)\right)\\ &= \ln(x+1) + 2\ln(x+2) - \ln(x+3) - \ln(x+4).\end{aligned}$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{1}{x+1} + \frac{2}{x+2} - \frac{1}{x+3} - \frac{1}{x+4},$$

so

$$y' = \frac{(x+1)(x+2)^2}{(x+3)(x+4)}\left(\frac{1}{x+1} + \frac{2}{x+2} - \frac{1}{x+3} - \frac{1}{x+4}\right).$$

111. $y = e^{(x-1)^2}e^{(x-3)^2}$

SOLUTION Let $y = e^{(x-1)^2}e^{(x-3)^2}$. Then

$$\ln y = \ln\left(e^{(x-1)^2}e^{(x-3)^2}\right) = \ln\left(e^{(x-1)^2+(x-3)^2}\right) = (x-1)^2 + (x-3)^2.$$

By logarithmic differentiation,

$$\frac{y'}{y} = 2(x-1) + 2(x-3) = 4x - 8,$$

so

$$y' = 4e^{(x-1)^2}e^{(x-3)^2}(x-2).$$

112. $y = \dfrac{e^x \sin^{-1} x}{\ln x}$

SOLUTION Let $y = \dfrac{e^x \sin^{-1} x}{\ln x}$. Then

$$\begin{aligned}\ln y &= \ln\left(\frac{e^x\sin^{-1}x}{\ln x}\right) = \ln(e^x\sin^{-1}x) - \ln(\ln x)\\ &= \ln\left(e^x\right) + \ln(\sin^{-1}x) - \ln(\ln x) = x + \ln(\sin^{-1}x) - \ln(\ln x).\end{aligned}$$

By logarithmic differentiation,

$$\frac{y'}{y} = 1 + \frac{1}{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}} - \frac{1}{\ln x}\cdot\frac{1}{x},$$

so

$$y' = \frac{e^x\sin^{-1}x}{\ln x}\left(1 + \frac{1}{\sqrt{1-x^2}\sin^{-1}x} - \frac{1}{x\ln x}\right).$$

113. $y = \dfrac{e^{3x}(x-2)^2}{(x+1)^2}$

SOLUTION Let $y = \dfrac{e^{3x}(x-2)^2}{(x+1)^2}$. Then

$$\begin{aligned}\ln y &= \ln\left(\frac{e^{3x}(x-2)^2}{(x+1)^2}\right) = \ln e^{3x} + \ln(x-2)^2 - \ln(x+1)^2\\ &= 3x + 2\ln(x-2) - 2\ln(x+1).\end{aligned}$$

By logarithmic differentiation,

$$\frac{y'}{y} = 3 + \frac{2}{x-2} - \frac{2}{x+1},$$

so

$$y = \frac{e^{3x}(x-2)^2}{(x+1)^2}\left(3 + \frac{2}{x-2} - \frac{2}{x+1}\right).$$

114. $y = x^{\sqrt{x}}(x^{\ln x})$

SOLUTION Let $y = x^{\sqrt{x}}(x^{\ln x})$. Then

$$\ln y = \sqrt{x}\ln x + (\ln x)^2$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x} + 2(\ln x)\cdot\frac{1}{x} = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} + \frac{2\ln x}{x},$$

so

$$y' = x^{\sqrt{x}}(x^{\ln x})\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} + \frac{2\ln x}{x}\right).$$

Exercises 115–117: Let q be the number of units of a product (cell phones, barrels of oil, etc.) that can be sold at the price p. The ***price elasticity of demand*** *E is defined as the percentage rate of change of q with respect to p. In terms of derivatives,*

$$E = \frac{p}{q}\frac{dq}{dp} = \lim_{\Delta p \to 0}\frac{(100\Delta q)/q}{(100\Delta p)/p}$$

115. Show that the total revenue $R = pq$ satisfies $\dfrac{dR}{dp} = q(1+E)$.

SOLUTION Let $R = pq$. Then

$$\frac{dR}{dp} = p\frac{dq}{dp} + q = q\frac{p}{q}\frac{dq}{dp} + q = q(E+1).$$

116. A commercial bakery can sell q chocolate cakes per week at price $\$p$, where $q = 50p(10-p)$ for $5 < p < 10$.

(a) Show that $E(p) = \dfrac{2p-10}{p-10}$.

(b) Show, by computing $E(8)$, that if $p = \$8$, then a 1% increase in price reduces demand by approximately 3%.

SOLUTION

(a) Let $q = 50p(10-p) = 500p - 50p^2$. Then $q'(p) = 500 - 100p$ and

$$E(p) = \left(\frac{p}{q}\right)\frac{dq}{dp} = \frac{p}{50p(10-p)}(500-100p) = \frac{10-2p}{10-p} = \frac{2p-10}{p-10}.$$

(b) From part (a),

$$E(8) = \frac{2(8)-10}{8-10} = -3.$$

Thus, with the price set at \$8, a 1% increase in price results in a 3% decrease in demand.

117. The monthly demand (in thousands) for flights between Chicago and St. Louis at the price p is $q = 40 - 0.2p$. Calculate the price elasticity of demand when $p = \$150$ and estimate the percentage increase in number of additional passengers if the ticket price is lowered by 1%.

SOLUTION Let $q = 40 - 0.2p$. Then $q'(p) = -0.2$ and

$$E(p) = \left(\frac{p}{q}\right)\frac{dq}{dp} = \frac{0.2p}{0.2p - 40}.$$

For $p = 150$,

$$E(150) = \frac{0.2(150)}{0.2(150) - 40} = -3,$$

so a 1% decrease in price increases demand by 3%. The demand when $p = 150$ is $q = 40 - 0.2(150) = 10$, or 10000 passengers. Therefore, a 1% increase in demand translates to 300 additional passengers.

118. How fast does the water level rise in the tank in Figure 7 when the water level is $h = 4$ m and water pours in at 20 m^3/min?

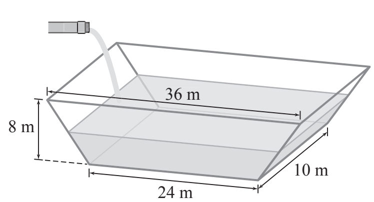


FIGURE 7

SOLUTION When the water level is at height h, the length of the upper surface of the water is $24 + \frac{3}{2}h$ and the volume of water in the trough is

$$V = \frac{1}{2}h\left(24 + 24 + \frac{3}{2}h\right)(10) = 240h + \frac{15}{2}h^2.$$

Therefore,

$$\frac{dV}{dt} = (240 + 15h)\frac{dh}{dt} = 20 \text{ m}^3/\text{min}.$$

When $h = 4$, we have

$$\frac{dh}{dt} = \frac{20}{240 + 15(4)} = \frac{1}{15} \text{ m/min}.$$

119. The minute hand of a clock is 8 cm long, and the hour hand is 5 cm long. How fast is the distance between the tips of the hands changing at 3 o'clock?

SOLUTION Let S be the distance between the tips of the two hands. By the law of cosines

$$S^2 = 8^2 + 5^2 - 2 \cdot 8 \cdot 5\cos(\theta),$$

where θ is the angle between the hands. Thus

$$2S\frac{dS}{dt} = 80\sin(\theta)\frac{d\theta}{dt}.$$

At three o'clock $\theta = \pi/2$, $S = \sqrt{89}$, and

$$\frac{d\theta}{dt} = \left(\frac{\pi}{360} - \frac{\pi}{30}\right) \text{ rad/min} = -\frac{11\pi}{360} \text{ rad/min},$$

so

$$\frac{dS}{dt} = \frac{1}{2\sqrt{89}}(80)(1)\frac{-11\pi}{360} \approx -0.407 \text{ cm/min}.$$

120. Chloe and Bao are in motorboats at the center of a lake. At time $t = 0$, Chloe begins traveling south at a speed of 50 km/h. One minute later, Bao takes off, heading east at a speed of 40 km/h. At what rate is the distance between them increasing at $t = 12$ min?

SOLUTION Take the center of the lake to be origin of our coordinate system. Because Chloe travels at 50 km/h = $\frac{5}{6}$ km/min due south, her position at time $t > 0$ is $(0, \frac{5}{6}t)$; because Bao travels at 40 km/h = $\frac{2}{3}$ km/min due east, her position at time $t > 1$ is $(\frac{2}{3}(t-1), 0)$. Thus, the distance between the two motorboats at time $t > 1$ is

$$s = \sqrt{\frac{4}{9}(t-1)^2 + \frac{25}{36}t^2} = \frac{1}{6}\sqrt{41t^2 - 32t + 16},$$

and

$$\frac{ds}{dt} = \frac{41t - 16}{6\sqrt{41t^2 - 32t + 16}}.$$

At $t = 12$, it follows that

$$\frac{ds}{dt} = \frac{476}{6\sqrt{5536}} \approx 1.066 \text{ km/min}.$$

121. A bead slides down the curve $xy = 10$. Find the bead's horizontal velocity at time $t = 2$ s if its height at time t seconds is $y = 400 - 16t^2$ cm.

SOLUTION Let $xy = 10$. Then $x = 10/y$ and

$$\frac{dx}{dt} = -\frac{10}{y^2}\frac{dy}{dt}.$$

If $y = 400 - 16t^2$, then $\frac{dy}{dt} = -32t$ and

$$\frac{dx}{dt} = -\frac{10}{(400 - 16t^2)^2}(-32t) = \frac{320t}{(400 - 16t^2)^2}.$$

Thus, at $t = 2$,

$$\frac{dx}{dt} = \frac{640}{(336)^2} \approx 0.00567 \text{ cm/s}.$$

122. In Figure 8, x is increasing at 2 cm/s, y is increasing at 3 cm/s, and θ is decreasing such that the area of the triangle has the constant value 4 cm^2.

(a) How fast is θ decreasing when $x = 4$, $y = 4$?

(b) How fast is the distance between P and Q changing when $x = 4$, $y = 4$?

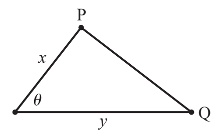


FIGURE 8

SOLUTION

(a) The area of the triangle is

$$A = \frac{1}{2}xy\sin\theta = 4.$$

Differentiating with respect to t, we obtain

$$\frac{dA}{dt} = \frac{1}{2}xy\cos\theta\frac{d\theta}{dt} + \frac{1}{2}y\sin\theta\frac{dx}{dt} + x\sin\theta\frac{dy}{dt} = 0.$$

When $x = y = 4$, we have $\frac{1}{2}(4)(4)\sin\theta = 4$, so $\sin\theta = \frac{1}{2}$. Thus, $\theta = \frac{\pi}{6}$ and

$$\frac{1}{2}(4)(4)\frac{\sqrt{3}}{2}\frac{d\theta}{dt} + \frac{1}{2}(4)\left(\frac{1}{2}\right)(2) + \frac{1}{2}(4)\left(\frac{1}{2}\right)(3) = 0.$$

Solving for $d\theta/dt$, we find

$$\frac{d\theta}{dt} = -\frac{5}{4\sqrt{3}} \approx -0.72 \text{ rad/s}.$$

(b) By the Law of Cosines, the distance D between P and Q satisfies

$$D^2 = x^2 + y^2 - 2xy\cos\theta,$$

so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2xy\sin\theta\frac{d\theta}{dt} - 2x\cos\theta\frac{dy}{dt} - 2y\cos\theta\frac{dx}{dt}.$$

With $x = y = 4$ and $\theta = \frac{\pi}{6}$,

$$D = \sqrt{4^2 + 4^2 - 2(4)(4)\frac{\sqrt{3}}{2}} = 4\sqrt{2-\sqrt{3}}.$$

Therefore,

$$\frac{dD}{dt} = \frac{16 + 24 - \frac{20}{\sqrt{3}} - 12\sqrt{3} - 8\sqrt{3}}{8\sqrt{2-\sqrt{3}}} \approx -1.50 \text{ cm/s}.$$

123. A light moving at 0.8 m/s approaches a man standing 4 m from a wall (Figure 9). The light is 1 m above the ground. How fast is the tip P of the man's shadow moving when the light is 7 m from the wall?

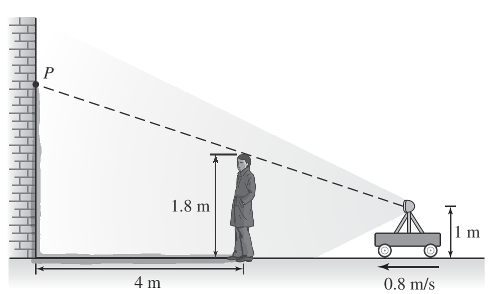


FIGURE 9

SOLUTION Let x denote the distance between the man and the light. Using similar triangles, we find

$$\frac{0.8}{x} = \frac{P-1}{4+x} \qquad \text{or} \qquad P = \frac{3.2}{x} + 1.8.$$

Therefore,

$$\frac{dP}{dt} = -\frac{3.2}{x^2}\frac{dx}{dt}.$$

When the light is 7 meters from the wall, $x = 3$. With $\frac{dx}{dt} = -0.8$, we have

$$\frac{dP}{dt} = -\frac{3.2}{3^2}(-0.8) = 0.284 \text{ m/s}.$$