End AP Calculus Exam Prep Assignment #9 page 2

Problems

10) [1995 AB 5] Water is draining from a right conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h - 12)$ feet per minute.
(The volume V of a cone with radius r and height h is $V = (1/3)\pi r^2 h$)

A) Write an expression for the volume of water in the conical tank as a function of h.

$$r = \frac{1}{3}h \quad V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{1}{27}\pi h^3$$

B) At what rate is the volume of water in the conical tank changing when $h = 3$?

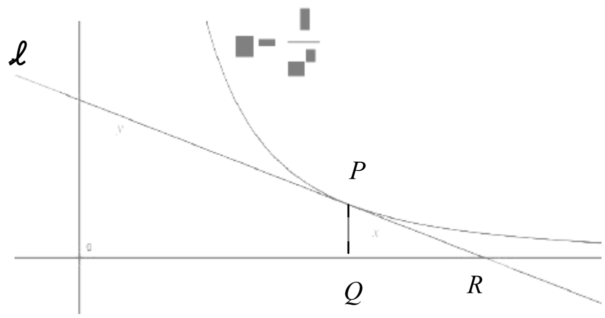
$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt} = \frac{1}{9}\pi(3^2)(3 - 12) = -9\pi \text{ ft}^3/\text{min}$$

C) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure.

Since water is draining from the conical tank at $-9\pi \text{ ft}^3/\text{min}$, the cylindrical tank is filling at a rate of $9\pi \text{ ft}^3/\text{min}$.

$$V = 400\pi y \quad \frac{dV}{dt} = 400\pi\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{9\pi}{400\pi} = \frac{9}{400} \text{ ft/min}$$

11) In the figure at the right, line l is tangent to the graph of $y = \frac{1}{x^2}$ at the point P, with coordinates $\left(w, \frac{1}{w^2}\right)$, where $w > 0$. Point Q has coordinates $(w,0)$. Line l crosses the x-axis at point R, with coordinates $(k,0)$.



A) Find the value of k when $w = 3$.

$$\frac{dy}{dx} = -\frac{2}{x^3} = -\frac{2}{27}$$

$$y - 0 = -\frac{2}{27}(x - k) \Rightarrow \frac{1}{9} = -\frac{2}{27}(3 - k) \Rightarrow -\frac{3}{2} = 3 - k \Rightarrow k = \frac{9}{2}$$

B) For all $w > 0$, find k in terms of w.

$$\frac{1}{w^2} - 0 = -\frac{2}{w^3}(w - k) \Rightarrow -\frac{w}{2} = w - k \Rightarrow k = \frac{3w}{2}$$

C) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of k with respect to time?

$$\frac{dw}{dt} = 7 \quad k = \frac{3w}{2} \Rightarrow \frac{dk}{dt} = \frac{3}{2}\frac{dw}{dt} \quad \frac{dk}{dt} = \frac{3}{2}(7) = 10.5 \text{ units per second.}$$

D) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change in area of ΔPQR with respect to time? Determine whether the area is increasing or decreasing at this instant.

$$A = \frac{1}{2}(k - w)\left(\frac{1}{w^2}\right) = \frac{k - w}{2w^2}$$

$$\frac{dA}{dt} = \frac{2w^2\left(\frac{dk}{dt} - \frac{dw}{dt}\right) - 4w\frac{dw}{dt}(k - w)}{4w^4} \qquad w = 5,\ k = \frac{15}{2},\ \frac{dw}{dt} = 7,\ \frac{dk}{dt} = \frac{21}{2}$$

$$\frac{dA}{dt} = \frac{2(25)\left(\frac{7}{2}\right) - 20(7)\left(\frac{5}{2}\right)}{4(625)} = \frac{175 - 350}{4(625)} = \frac{-175}{4(625)} = -\frac{7}{100}$$

Alternate solution: path taken by 2 students (2008 BC)

$$A = \frac{1}{2}(k - w)\left(\frac{1}{w^2}\right) = \frac{1}{2}\left(\frac{3w}{2} - w\right)\left(\frac{1}{w^2}\right) = \frac{1}{4w}$$

$$\frac{dA}{dt} = -\frac{1}{4w^2}\left(\frac{dw}{dt}\right) \qquad w = 5,\ \frac{dw}{dt} = 7$$

$$\frac{dA}{dt} = -\frac{1}{100}(7) = -\frac{7}{100}$$