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Problems

16) [1991 BC6] A certain rumor spreads through a community at the rate of $\frac{dy}{dt} = 2y(1-y)$, where y is the proportion of the population that has heard the rumor at time t.

A)

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = 0 \Rightarrow 2\frac{dy}{dt} - 4y\frac{dy}{dt} = 0 \Rightarrow (2-4y)(2y-2y^2) = 0$$

$$\Rightarrow 4y(2y^2 - 3y + 1) = 0 \Rightarrow 4y(2y-1)(y-1) \Rightarrow y = 0, 1, \frac{1}{2}$$

When $y = 0$ or 1, $\frac{dy}{dt} = 0$. When $y = \frac{1}{2}$, $\frac{dy}{dt} = \frac{1}{2}$

The population is growing fastest when $y = \frac{1}{2}$, or when 1/2 the population has heard it.

B)

$$\frac{dy}{dt} = 2y(1-y) \Rightarrow \frac{dy}{y(1-y)} = 2dt \Rightarrow \int \frac{dy}{y(1-y)} = \int 2dt$$

$$\int \frac{1}{y}dy - \int \frac{1}{1-y}dy = \int 2dt \Rightarrow \ln|y| - \ln|1-y| = 2t + C$$

$$\ln\left(\frac{.1}{.9}\right) = 0 + C \Rightarrow C = \ln\left(\frac{1}{9}\right) = -\ln 9 \quad \ln\frac{y}{1-y} = 2t - \ln 9$$

$$\frac{y}{1-y} = e^{2t-\ln 9} = \frac{e^{2t}}{9} \Rightarrow 9y = e^{2t} - ye^{2t} \Rightarrow y = \frac{e^{2t}}{9+e^{2t}}$$

C) $\frac{1}{2} = \frac{e^{2t}}{9+e^{2t}} \Rightarrow 9 + e^{2t} = 2e^{2t} \Rightarrow e^{2t} = 9 \Rightarrow t = \frac{\ln 9}{2} = \ln 3$

17) [1997 AB6/BC6] Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2v - 32$ with initial condition $v(0) = -50$.

A) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

$$\frac{dv}{v+16} = -2dt \Rightarrow \int \frac{dv}{v+16} = -2\int dt \Rightarrow \ln|v+16| = -2t + C$$

$$\ln 34 = C \quad \ln|v+16| = -2t + \ln 34 \Rightarrow |v+16| = e^{-2t+\ln 34}$$

$$|v+16| = 34e^{-2t} \quad v = -34e^{-2t} - 16$$

B) $\lim_{t\to\infty}\left(-34e^{-2t} - 16\right) = -16$ ft/sec

C) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

speed $= |v(t)| = |-34e^{-2t} - 16| = 34e^{-2t} + 16$

$34e^{-2t} + 16 = 20 \Rightarrow e^{-2t} = \dfrac{4}{34} \Rightarrow t = \dfrac{\ln\left(\frac{2}{17}\right)}{-2}$

18) [2001 AB6] The function *f* is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x,y) on the graph is given by $\dfrac{dy}{dx} = y^2(6-2x)$

A) Find $\dfrac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6-2x) - 2y^2 = 2y\left(y^2(6-2x)\right)(6-2x) - 2y^2 = 2y^3(6-2x)^2 - 2y^2$$

At $\left(3, \frac{1}{4}\right)$ $\dfrac{d^2y}{dx^2} = 2\left(\frac{1}{4}\right)^3(6-6)^2 - 2\left(\frac{1}{4}\right)^2 = -\dfrac{1}{8}$

B) Find $y = f(x)$ by solving the differential equation $\dfrac{dy}{dx} = y^2(6-2x)$ with the initial condition $f(3) = \dfrac{1}{4}$.

$$\frac{dy}{y^2} = (6-2x)dx \Rightarrow \int \frac{dy}{y^2} = \int (6-2x)dx \Rightarrow -\frac{1}{y} = 6x - x^2 + C$$

$-4 = 18 - 9 + C \Rightarrow C = -13 \quad -\dfrac{1}{y} = 6x - x^2 - 13 \quad y = \dfrac{1}{x^2 - 6x + 13}$