Algebra II
Unit 6: Exponential and Logarithmic Functions

Time Frame: Approximately four weeks

Unit Description

In this unit, students explore exponential and logarithmic functions, their graphs, and applications.

Student Understandings

Students solve exponential and logarithmic equations and graph exponential and logarithmic functions by hand and by using technology. They will compare the speed at which the exponential function increases to that of linear or polynomial functions and determine which type of function best models data. They will comprehend the meaning of a logarithm of a number and know when to use logarithms to solve exponential functions.

Guiding Questions

1. Can students solve exponential equations with variables in the exponents and having a common base?
2. Can students solve exponential equations not having the same base by using logarithms with and without technology?
3. Can students graph and transform exponential functions?
4. Can students graph and transform logarithmic functions?
5. Can student write exponential functions in logarithmic form and vice versa?
6. Can students use the properties of logarithms to solve equations that contain logarithms?
7. Can students find natural logarithms and anti-natural logarithms?
8. Can students use logarithms to solve problems involving exponential growth and decay?
9. Can students look at a table of data and determine what type of function best models that data and create the regression equation?

Unit 6 Grade-Level Expectations (GLEs)

Teacher Note: The individual Algebra II GLEs are sometimes very broad, encompassing a variety of functions. To help determine the portion of the GLE that is being addressed in each unit and in each activity in the unit, the key words have been underlined in the GLE list, and the number of the predominant GLE has been underlined in the activity.

Grade-Level Expectations	
GLE #	**GLE Text and Benchmarks**
Number and Number Relations	
1.	Read, write, and perform basic operations on complex numbers (N-1-H) (N-5-H
2.	Evaluate and perform basic operations on expressions containing rational exponents (N-2-H)
3.	Describe the relationship between exponential and logarithmic equations (N-2-H)
Algebra	
4.	Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)
6.	Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)
7.	Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)
10.	Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)
Data Analysis. Probability, and Discrete Math	
17.	Discuss the differences between samples and populations (D-1-H)
19.	Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)
20.	Interpret and explain, with the use of technology, the regression coefficient and the correlation coefficient for a set of data (D-2-H)
22.	Explain the limitations of predictions based on organized sample sets of data (D-7-H)
Patterns, Relations, and Functions	
24.	Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)
25.	Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)
27.	Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)
28.	Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)
29.	Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)
CCSS for Mathematical Content	
CCSS #	**CCSS Text**
Building Functions	
F.BF.4a	Find inverse functions. a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

Linear, Quadratic, and Exponential Models	
F.LE.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.
ELA CCSS	
CCSS #	**CCSS Text**
Reading Standards for Literacy in Science and Technical Subjects 6-12	
RST.11-12.3	Follow precisely a complex multistep procedure when carrying out experiments, taking measurements, or performing technical tasks; analyze the specific results based on explanations in the text.
RST.11-12.4	Determine the meaning of symbols, key terms, and other domain-specific words and phrases as they are used in a specific scientific or technical context relevant to grades 11–12 texts and topics.
Writing Standards for Literacy in History/Social Studies, Science and Technical Subjects 6-12	
WHST.11-12.7	Conduct short as well as more sustained research projects to answer a question (including a self-generated question) or solve a problem; narrow or broaden the inquiry when appropriate; synthesize multiple sources on the subject, demonstrating understanding of the subject under investigation.

Sample Activities

Ongoing Activity: Little Black Book of Algebra II Properties

Materials List: black marble composition book, Little Black Book of Algebra II Properties BLM

Activity:

- Have students continue to add to the Little Black Books they created in previous units which are modified forms of *vocabulary cards* (view literacy strategy descriptions). When students create *vocabulary cards*, they see connections between words, examples of the word, and the critical attributes associated with the word such as a mathematical formula or theorem. *Vocabulary cards* require students to pay attention to words over time, thus improving their memory of the words. In addition, *vocabulary cards* can become an easily accessible reference for students as they prepare for tests, quizzes, and other activities with the words. These self-made reference books are modified versions of *vocabulary cards* because, instead of creating cards, the students will keep the vocabulary in black marble composition books (thus the name "Little Black Book" or LBB). Like *vocabulary cards*, the LBBs emphasize the important concepts in the unit and reinforce the definitions, formulas, graphs, real-world applications, and symbolic representations.
- At the beginning of the unit, distribute copies of the Little Black Book of Algebra II Properties BLM for Unit 6. This is a list of properties in the order in which they will be learned in the unit. The BLM has been formatted to the size of a composition book so

students can cut the list from the BLM and paste or tape it into their composition books to use as a table of contents.

- The students' description of each property should occupy approximately one-half page in the LBB and include all the information on the list for that property. The students may also add examples for future reference.
- Periodically check the Little Black Books and require that the properties applicable to a general assessment be finished by the day before the test, so pairs of students can use the LBBs to quiz each other on the concepts as a review.

Exponential and Logarithmic Functions

6.1 Laws of Exponents – write rules for adding, subtracting, multiplying and dividing values with exponents, raising an exponent to a power, and using negative and fractional exponents.

6.2 Solving Exponential Equations – write the rules for solving two types of exponential equations: same base and different bases (e.g., solve $2^x = 8^{x-1}$ without calculator; solve $2^x = 3^{x-1}$ with and without calculator).

6.3 Exponential Function with Base *a* – write the definition, give examples of graphs with a > 1 and $0 < a < 1$, and locate three ordered pairs, give the domains, ranges, intercepts, and asymptotes for each.

6.4 Exponential Regression Equation – give a set of data and explain how to use the method of finite differences to determine if it is best modeled with an exponential equation, and explain how to find the regression equation.

6.5 Exponential Function Base *e* – define *e*, graph $y = e^x$ and then locate 3 ordered pairs, and give the domain, range, asymptote, intercepts.

6.6 Compound Interest Formula – define continuous and finite, explain and give an example of each symbol

6.7 Inverse Functions – write the definition, explain one-to-one correspondence, give an example to show the test to determine when two functions are inverses, graph the inverse of a function, find the line of symmetry and the domain and range, explain how to find inverse analytically and how to draw an inverse on the calculator.

6.8 Logarithm – write the definition and explain the symbols used, define common logs, characteristic, and mantissa, and list the properties of logarithms.

6.9 Laws of Logs and Change of Base Formula – list the laws and the change of base formula and give examples of each.

6.10 Solving Logarithmic Equations – explain rules for solving equations, identify the domain for an equation, find $\log_2 8$ and $\log_{25} 125$, and solve each of these equations for *x*: $\log_x 9 = 2$, $\log_4 x = 2$, $\log_4(x-3)+\log_4 x=1$).

6.11 Logarithmic Function Base a – write the definition, graph $y = \log_a x$ with $a < 1$ and $a > 1$ and locate three ordered pairs, identify the domain, range, intercepts, and asymptotes, and find the domain of $y = \log(x^2 + 7x + 10)$.

6.12 Natural Logarithm Function – write the definition and give the approximate value of *e*, graph $y = ln\ x$ and give the domain, range, and asymptote, and locate three ordered pairs, solve $ln\ x = 2$ for *x*.

6.13 Exponential Growth and Decay – define half-life and solve an example problem, give and solve an example of population growth using $A(t) = Pe^{rt}$.

Activity 1: Fractional Exponents (GLEs: 1, 2; CCSSs: RST.11-12.3, RST.11-12.4)

Materials List: paper, pencil, graphing calculator**,** Math Log Bellringer BLM

This activity has not changed because it already incorporates these CCSSs. In this activity, students will review properties of numbers with integral exponents first discussed in Unit 3 and extend them to simplify and evaluate expressions with fractional exponents.

Math Log Bellringer:

Simplify the following.

(1) a^2a^3

(2) $\dfrac{b^7}{b^3}$

(3) $(c^3)^4$

(4) $2x^5 + 3x^5$

(5) $(2x)^3$

(6) $(a + b)^2$

(7) x^0

(8) 2^{-1}

Solutions:

(1) a^5, *(2)* b^4 *(3)* c^{12}, *(4)* $5x^5$,*(5)* $8x^3$, *(6)* $a^2 + 2ab + b^2$, *(7) 1, (8) ½*

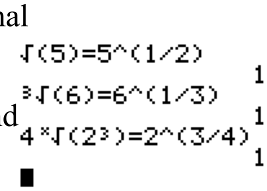
Activity:

- Overview of the Math Log Bellringers:
  - As in previous units, each in-class activity in Unit 6 is started with an activity called a Math Log Bellringer that either reviews past concepts to check for understanding (reflective thinking about what was learned in previous classes or previous courses) or sets the stage for an upcoming concept (predictive thinking for that day's lesson).
  - A math log is a form of a *learning log* (view literacy strategy descriptions) that students keep in order to record ideas, questions, reactions, and new understandings. Documenting ideas in a log about how content's being studied forces students to "put into words" what they know or do not know. This process offers a reflection of understanding that can lead to further study and alternative learning paths. It combines writing and reading with content learning. The Math Log Bellringers will include mathematics done symbolically, graphically, and verbally.
  - Since Bellringers are relatively short, Blackline Masters have not been created for each of them. Write them on the board before students enter class, paste them into an enlarged *Word®* document or *PowerPoint®* slide, and project using a TV or digital projector, or print and display using a document or overhead projector. A sample enlarged Math Log Bellringer *Word®* document has been included in the Blackline Masters. This sample is the Math Log Bellringer for this activity.

- ➢ Have the students write the Math Log Bellringers in their notebooks, preceding the upcoming lesson during beginning-of-class record keeping, and then circulate to give individual attention to students who are weak in that area.

- When students have completed the Bellringer, have them use *discussion* (view literacy strategy descriptions) in the form of Think Pair Square Share. It has been shown that students can improve learning and remembering when they participate in a dialog about class. In Think Pair Square Share, after being given an issue, problem, or question, students are asked to think alone for a short period of time and then pair up with someone to share their thoughts. Then have pairs of students share with other pairs, forming, in effect, small groups of four students. *Discussion* highlights students' understanding of what they know, as well as what they still need to learn, in order to fully comprehend the concept.
  (1) Have each student write two mathematical rules that explain in words the law of exponents he/she used to simplify the expressions in the Bellringer. The rules should be written in sentence form describing the process used.
  (2) Pair students to first check the correctness of their Bellringers and rules. If they have written the same rules, have the pair write an additional rule so they have a total of four rules.
  (3) Divide the students into groups of four to compare their rules. Again if they have written the same rules, have the group write additional rules so they have a total of eight rules. Have the group write the rules they used on large sheets of paper and tape them to the board to compare with other groups.
  (4) Critique the wording as a class, stressing the need for a common base in #1 and #2, a common base and exponent in #4, and a common exponent in #5.
  Rules should look something like this:
  (1) When you multiply 2 variables with the same base, add the exponents.
  (2) When you divide two variables with the same base, subtract the exponents.
  (3) When you raise a variable with an exponent to a power, multiply the exponents.
  (4) When you add two expressions that have the same variable raised to the same exponent, add the coefficients.
  (5) When you raise a product to a power, each of the factors is raised to that power.
  (6) When you raise a sum to a power, FOIL or use the distributive property.
  (7) Any variable or number ≠ 0 raised to the zero power = 1.
  (8) A number or variable raised to a negative exponent is the reciprocal of the number.

- Have the students discover the equivalency of the following expressions in their calculators and write a rule for fractional exponents. This can be done by getting decimal representations, or the students can use the TEST feature on the TI-83 and TI-84 to determine equivalency. Enter $\sqrt{5} = 5 \wedge (1/2)$ (The "=" sign is found under $\boxed{\text{2ND}}$, [TEST], above the $\boxed{\text{MATH}}$ button). If the calculator returns a "1", then the statement is true; if it returns a "0," then the statement is false.



(1) $\sqrt{5}$ *and* $5^{\frac{1}{2}}$

(2) $\sqrt[3]{6}$ *and* $6^{\frac{1}{3}}$

(3) $\sqrt[4]{2^3}$ *and* $2^{\frac{3}{4}}$

Solutions:

All are equivalent. The rule for fractional exponents is if $\sqrt[n]{a}$ *is a real number, then* $a^{\frac{b}{c}} = \sqrt[c]{a^b} = \left(\sqrt[c]{a}\right)^b$

- Have students practice changing radicals to fractional exponents and vice versa using the laws of exponents by simplifying complex radicals. Have students simplify problems such as the following without calculators and use the properties in the Bellringers to simplify similar problems with fractional exponents:

(1) $\left(\frac{1}{100}\right)^{\frac{1}{2}}$

(2) $8^{\frac{1}{3}}$

(3) $625^{\frac{1}{5}}$

(4) $\sqrt{4^3}$

Solutions: (1) $\frac{1}{10}$*, (2) 2, (3) 5, (4) 8*

- Assign additional problems from the math textbook.

- Critical Thinking Writing Assessment: *(See Activity-Specific Assessments at end of unit.)*

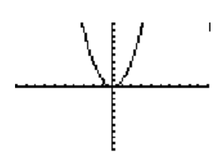
Activity 2: Graphs of Exponential Function (GLEs: 2, 4, 6, 7, 19, 25, 27, 28, 29; CCSS: RST.11-12.3)

Materials List: paper, pencil, graphing calculator, Graphing Exponential Functions Discovery Worksheet BLM

This activity has not changed because it already incorporates this CCSS. In this activity, the students will discover the graph of an exponential function and its domain, range, intercepts, shifts, and effects of differing bases, and will use the graph to explain irrational exponents.

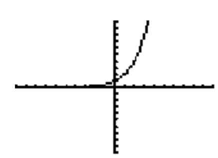
Math Log Bellringer:

(1) Graph $y = x^2$ and $y = 2^x$ on your graphing calculator individually with a window of x: [−10, 10] and y: [−10, 10] and describe the similarities and differences.



(2) Graph them on the same screen with a window of x: [−10, 10] and y: [−10, 100] and describe any additional differences.

Solutions:



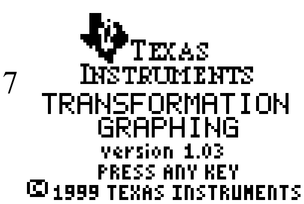
(1) *Both have the same domain, all reals, but the range of $y = x^2$ is $y \geq 0$ and the range of $y = 2^x$ is $y > 0$. There are different y–intercepts, (0, 0) and (0, 1). The end- behavior is the same as x approaches ∞; but as x approaches -∞, the end-behavior of $y = x^2$ approaches ∞ and the end- behavior of $y = 2^x$ approaches 0.*

(2) *$y = 2^x$ grows faster than $y = x^2$.*

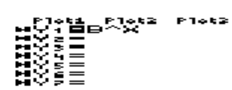
Activity:

- Discuss the Bellringer in terms of how fast the functions increase. Show how fast exponential functions increase by the following demonstration:
  Place 1 penny on the first square of a checker board, double it and place two pennies on the second square, 4 on the next, 8 on the next, and so forth until the piles are extremely high. Have the students determine how many pennies would be on the last square, tracing to that number on their calculators. Measure smaller piles to determine the height of the last pile and compare it to the distance to the sun, which is 93,000,000 miles.

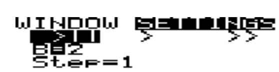
- Graphing Exponential Functions Discovery Worksheet BLM:
  - ➢ On this worksheet, the students will use their graphing calculators to graph the exponential function $f(x) = b^x$ with various changes in the constants to determine how these changes affect the graph.
  - ➢ The students can graph each equation individually or use the Transformation APP on the TI 83 and TI 84 as they did in Activity 7 in Unit 5.



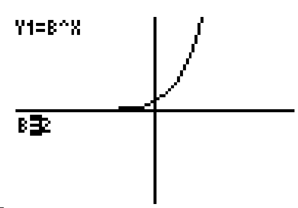
    To use the Transformation APPS:
    - Turn on the application by pressing $\boxed{\text{APPS}}$, Transfrm $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$



    - Enter the equation $y_1 = \text{B}^x$



    - Set the window by pressing $\boxed{\text{WINDOW}}$ and cursor to SETTINGS, set where B will start, in this example B = 2, and adjust the step for B to Step = 1.



    - $\boxed{\text{GRAPH}}$ and use the cursor to change the values of B.
    - When finished, uninstall the transformation APP by pressing $\boxed{\text{APPS}}$, Transfrm, 1:Uninstall

    - For more information see the *TI 83/TI84 Transformation App Guidebook* at http://education.ti.com/downloads/guidebooks/eng/transgraph-eng.
  - ➢ Distribute Graphing Exponential Functions Discovery Worksheet BLM. Graph the first equation together having the students locate the *y*-intercept and trace to high and low *x*

values to determine end-behavior and that there is a horizontal asymptote at $y = 0$. (This is not obvious on the graph.) Have them sketch the graph and dot the horizontal asymptote on the x–axis.

- Arrange the students in pairs to complete the graphs and answer the questions. Circulate to make sure they are graphing correctly.
- When the students finish the worksheet, go over the answers to the questions making sure they have all come to the correct conclusions.

- Examine the graph of $f(x) = 2^x$ in #1 and discuss its continuity by using the trace function on the calculator to determine $f\left(\frac{3}{2}\right), f\left(\sqrt{3}\right), and\ f(2)$. Because it is a continuous function, a number can be raised to any real exponent, rational and irrational, and have a value. Discuss irrational exponents with the students and have them apply the Laws of Exponents to simplify the following expressions:

(1) $5^{\sqrt{3}} \cdot 5^{6\sqrt{3}}$

(2) $\frac{6^{5\sqrt{2}}}{6^{\sqrt{2}}}$

(3) $\frac{8^{\pi}}{2^{\pi}}$

(4) $\frac{2^{\sqrt{5}}4^{3\sqrt{5}}}{16^{-\frac{1}{4}}8^{2\sqrt{5}}}$

Solutions: (1) $5^{7\sqrt{3}}$ *, (2)* $6^{4\sqrt{2}}$ *, (3)* 4^{π} *, (4)* $2^{1+\sqrt{5}}$

- Assign additional graphing problems and irrational exponent problems from the math textbook.

Activity 3: Regression Equation for an Exponential Function (GLEs: 2, 4, 6, 7, 10, 19, 22, 27, 28, 29; CCSSs: RST.11-12.3, RST.11-12.4)

Materials List: paper, pencil, graphing calculator, Exponential Regression Equations BLM

This activity has not changed because it already incorporates these CCSSs. In this activity, the students will enter data into their calculators and change all the parameters for an exponential equation of the form, $y = Ab^{x-C} + D$, to find the best regression equation. They then will use the equation to interpolate and extrapolate.

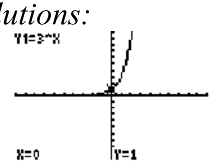
Math Log Bellringer:

Use what you know about shifts and translations to graph the following without a calculator locating asymptotes and y-intercepts.

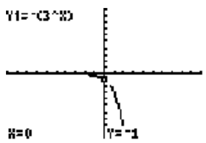
(1) $f(x) = 3^x$

(2) $f(x) = -3^x$
(3) $f(x) = 3^{-x}$
(4) Describe the translations in #2 and #3
(5) $f(x) = 3^{x-4}$
(6) $f(x) = 3^x - 4$
(7) Describe the shifts in #5 and #6
(8) $f(x) = 5(3^x)$

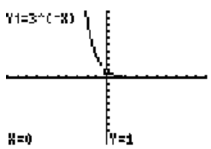
Solutions:



(1) *horizontal asymptote at $y = 0$,*

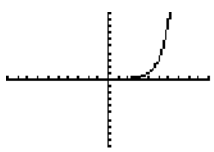


(2) *horizontal asymptote at $y = 0$,*

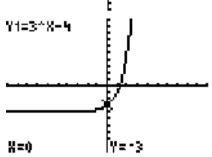


(3) *horizontal asymptote at $y = 0$,*

(4) #2 rotates the parent function through space around the x-axis and #3 rotates it around the y-axis

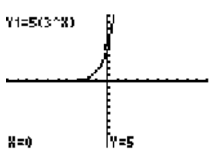


(5) $\left(0, \frac{1}{81}\right)$, *horizontal asymptote at $y = 0$*



(6) *horizontal asymptote at $y = -4$*

(7) #5 shifted the parent function to the right 4 and #6 shifted it down 4



(8) *horizontal asymptote at $y = 0$*

Activity:

- Use the Bellringer to check for understanding of translations.

- Ignite interest in the upcoming activity by using *lesson impressions* (view literacy strategy descriptions). *Lesson impressions* create situational interest in the content to be covered by capitalizing on students' curiosity. By asking students to form a written impression of the topic to be discussed, they become eager to discover how closely their impression text matches the actual content. This strategy is especially helpful to struggling and reluctant learners as it heightens motivation and helps students focus on important content. In this strategy, present a list of ideal words to students and tell students they are to use the words to make a guess as to what will be covered in class that day. Students are then encouraged to write a short descriptive passage, a story, or an essay using the impression words..
  - Write the following impression words on the board: exponential function, data, scatter plot, regression equation, model, best fit equation, interpolate, extrapolate, method of finite differences, point of intersection.
  - Tell students to write a short paragraph that predicts today's lesson activity and incorporates all the impression words.
  - Have students share their paragraphs with their classmates.

- Exponential Regression Equations BLM:
  - In the first section on this Exponential Regression Equations BLM, the students will enter real-world data into their calculators to create a scatter plot, find an exponential regression (prediction) equation, and use the model to interpolate and extrapolate points to answer real-world questions. In the second section, they will be using the method of finite differences to determine which data is exponential and to find its regression equation.
  - Distribute the Exponential Regression Equations BLM and have students work in pairs.
  - If necessary, review with students the steps for making a scatter plot. (To enter data on a TI-84 calculator: $\boxed{\text{STAT}}$, 1:Edit, enter data into L_1 and L_2 . To set up the plot of the data: $\boxed{2}\boxed{^{nd}}$, [STAT PLOT] (above $\boxed{\text{Y=}}$), 1:PLOT1, $\boxed{\text{ENTER}}$, On, Type: Scatter Plot, Xlist: L_1, Ylist: L_2, Mark (any). To graph the scatter plot: $\boxed{\text{ZOOM}}$, 9: ZoomStat).
  - When all the students have found an equation in Section 1, Real-World Exponential Data**,** write all the equations on the board and have the students determine which equation is the best fit.
  - Have students use that best fit equation to answer the interpolation and extrapolation questions in #3.
  - Discuss how they determined the answer to #4. Since the calculator cannot trace to a dependent variable, the best method is to graph $y = 25$ and find the point of intersection. Review this process with the students. On the TI–84, use $\boxed{2}\boxed{^{nd}}$ [CALC] (above $\boxed{\text{TRACE}}$), 5: intersect, enter a lower and upper bound on either side of the point of intersection and $\boxed{\text{ENTER}}$.
  - Review the Method of Finite Differences from Unit 2, Activity 8, and have students apply it to determine which data in Section 2 is exponential then to find a regression equation for each set of data.
  - When all students have completed the BLM, discuss their answers.
  - Have students revisit their *lesson impressions* and choose the one that best predicted the lesson content.

Activity 4: Exponential Data Research (GLEs: 4, 6, 7, <u>10</u>, 19, 22, <u>24</u>, 27, <u>29</u>; CCSS: WHST.11-12.7)

Materials List: paper, pencil, graphing calculator (or computer), Exponential Data Research Project BLM

This activity has not changed because it already incorporates this CCSS. This is an out-of-class activity in which the students will find data that is best modeled by an exponential curve.

Activity:

- Exponential Data Research Project:
  - ➢ Distribute the Exponential Data Research Project BLM and discuss the directions with the students.
  - ➢ State that this is an individual project and each person must have different data, so they should be the first to print out the data and claim the topic. Possible topics include: US Bureau of Statistics, Census, Stocks, Disease, Bacteria Growth, Investments, Land Value, Animal Population, number of stamps produced each year.
  - ➢ Give the students approximately one week to complete the project.
  - ➢ When the students hand in their projects have each student present his/her findings to the class.

Activity 5: Solving Exponential Equations with Common Bases (GLEs: 2, 4, <u>10</u>)

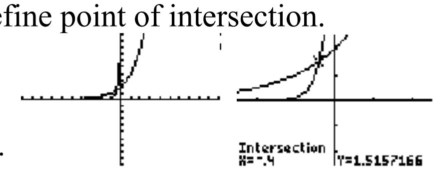
Materials List: paper, pencil, graphing calculator

In this activity, students will use the properties of exponents to solve exponential equations with similar bases.

Math Log Bellringer:

Graph $y = 2^{x+1}$ and $y = 8^{2x+1}$ on your graphing calculator. Zoom in and find the point of intersection. Define point of intersection.

Solution:



A point of intersection is an ordered pair that is a solution for both equations.

Activity:

- Define exponential equation as any equation in which a variable appears in the exponent and have students discuss a method for solving the Bellringer analytically.
  - ➢ Students have a difficult time understanding that a point of intersection is a shared x and y -value; therefore, to solve for a point of intersection analytically, students should solve

the set of equations simultaneously, meaning set $y = 2^{x+1}$ and $y = 8^{2x+1}$ equal to each other, $2^{x+1} = 8^{2x+1}$ and solve for x.

- They should develop the property, necessitating getting the same base and setting the exponents equal to each other.

  Solution:
  $2^{x+1} = 8^{2x+1}$
  $2^{x+1} = (2^3)^{2x+1}$
  $2^{x+1} = 2^{6x+3}$
  $\therefore x + 1 = 6x + 3 \quad x = -\frac{2}{5}$

- Use the property above to solve the following equations.
  (1) $3^{x+2} = 9^{2x}$
  (2) $3^{-x} = 81$
  (3) $\left(\frac{3}{2}\right)^{x+1} = \left(\frac{27}{8}\right)^{x}$
  (4) $8^x = 4$
  (5) $\left(\frac{1}{27}\right)^{x} = 81$

  Solutions: (1) $x = \frac{2}{3}$, *(2)* $x = -4$, *(3)* $x = ½$, *(4)* $x = \frac{2}{3}$, *(5)* $x = -\frac{4}{3}$

- Assign additional problems from the math textbook.

- Critical Thinking Writing Assessment: *(See Activity-Specific Assessments at end of unit.)*

Activity 6: Inverse Functions and Logarithmic Functions (GLEs: 2, 3, 4, 25, 27; CCSSs: F-BF.4a, RST.11-12.4)

Materials List: paper, pencil, graph paper, graphing calculator

This activity has not changed because it already incorporates these CCSSs. In this activity, students will review the concept of inverse functions in order to develop the logarithmic function which is the inverse of an exponential function.

Math Log Bellringer:

(1) Find the domain and range of $f(x) = \frac{2}{x+1}$
(2) Find the inverse $f^{-1}(x)$ of $f(x) = \frac{2}{x+1}$ and state its domain and range.
(3) Discuss what you remember about inverse functions.

Solutions:

(1) D: $x \neq -1$, R: $y \neq 0$

(2) $f^{-1}(x) = \dfrac{2-x}{x}$ *D: $x \neq 0$, R: $y \neq -1$*

(3) The students should generate these statements:

- *Definition: $f^{-1}(x)$ is an inverse function of f(x) if and only if* $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$.
- *You find the inverse of a function by swapping the x and y and solving for y.*
- *The graphs of a function and its inverse are symmetric over the line y = x.*
- *You swap the domains and ranges.*
- *In all ordered pairs, the abscissa and ordinate are swapped.*
- *If an inverse relation is going to be an inverse function, then the original function must have a one-to-one correspondence.*
- *You can tell if an inverse relation is going to be an inverse function from the graph if the original function passes both the vertical and horizontal line test.*

Activity:

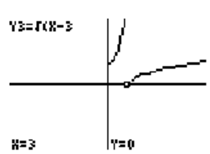
- Review the concepts of an inverse function from Unit 1, Activity 12, and have the students practice finding an inverse function on the following problem:
  (1) Analytically find the inverse of $f(x) = x^2 + 3$ on the restricted domain $x \geq 0$
  (2) Prove they are inverses using the definition $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$
  (3) What is the domain and range of $f(x)$ and $f^{-1}(x)$?
  (4) Graph both by hand on the same graph labeling x- and y-intercepts.
  (5) Graph the line $y = x$ on the same graph and locate one pair of points that are symmetric across the line $y = x$.
  (6) Why is the domain of $f(x)$ restricted?

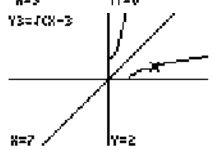
  Solution:

  (1) $f^{-1}(x) = \sqrt{x-3}$

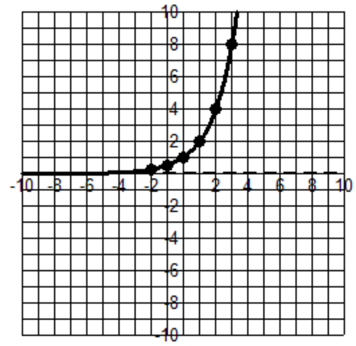
  (2) $\left(\sqrt{x-3}\right)^2 + 3 = \sqrt{x^2+3-3} = x \quad \left(\sqrt{x^2} = x \text{ if } x \geq 0\right)$

  (3) f(x): domain $x \geq 0$, range $y \geq 3$, $f^{-1}(x)$: domain $x \geq 3$, range $y \geq 0$

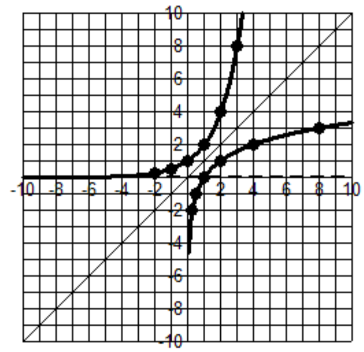
  (4)  *y− intercept of f(x) is (0, 3), x−intercept of $f^{-1}(x)$ is (3, 0)*

  (5)  *Ordered pairs may vary. f(2) = 7, $f^{-1}(7) = 2$*

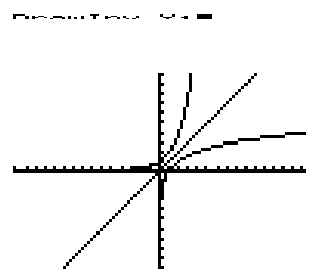
  (6) f(x) would not have a one−to−one correspondence and the inverse would not be a function.



- Give the students graph paper and have them discover the inverse of the exponential function in the following manner:
  - ➢ Graph *f(x)* = 2^x by hand dotting the horizontal asymptote and label the ordered pairs at $x = -2, -1, 0, 1, 2, 3$.
  - ➢ Is this function a one-to-one correspondence? (*Solution*: *yes, therefore an inverse function must exist*)

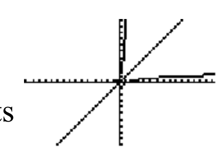


  - ➢ Graph $y = x$ on the same graph and draw the inverse function by plotting ordered pairs of the inverse and dotting the vertical asymptote. Compare the graph of the inverse with the graph of the original function discussing domains, ranges, increasing and decreasing, intercepts, and asymptotes. *Solution:*
    - o *The domain of the original function is all reals and the range is y > 0, but the domain of the inverse is x > 0 and the range is all reals.*
    - o *The inverse is increasing just like the original function.*
    - o *The y-intercept of the original function is 1, but the x-intercept of the inverse is 1.*
    - o *The asymptote of the original function is y=0. The asymptote of the inverse is x=0.*



  - ➢ On the calculator graph $y_1 = 2^x$ and $y_2 = x$. Use the calculator function, $\boxed{\text{ZOOM}}$, 5:ZSquare. Draw the graph of the inverse on graphing calculator ($\boxed{2^{nd}}$, [DRAW], (above $\boxed{\text{PRGM}}$), 8: DrawInv, $\boxed{\text{VARS}}$, Y–VARS, 1:Function, 1:Y_1).

- Have students try to find the inverse of $y = 2^x$ analytically by swapping x and y and attempting to isolate y.
  - ➢ Use this discussion to define logarithm and its relationship to exponents: $\log_b a = c$ if and only if $b^c = a$
  - ➢ Use the definition to rewrite $\log_2 8 = 3$ as an exponential equation. *(Solution: $2^3 = 8$)*
  - ➢ Find $\log_5 25$ by thinking exponentially: "5 raised to what power = 25?" *(Solution: $5^2 = 25$ therefore $\log_5 25 = 2$)*
  - ➢ Define common logarithm as logarithm with base 10 in which the base is understood: $f(x) = \log x$. On the calculator, have the students $\boxed{\text{ZOOM}}$ Square and graph $y_1 = 10^x$, $y_2 = \log x$, $y_3 = x$ to see that y_1 and y_2 are symmetric across the line $y = x$.



  - ➢ Have the students find log 100 without a calculator (*Solution: log 100 = 2 because 10^2 = 100)* and use the definition of logarithm to evaluate the following logarithmic expressions. Have students write "because" and the exponential equivalent after each problem:

(1) $\log_5 125$
(2) $\log 0.001$
(3) $\log_{\frac{1}{4}} 16$
(5) $\log_3 81$
(6) $\log_{\sqrt{3}} 3^{12}$

Solutions:
(1) $\log_5 125 = 3$ *because* $5^3 = 125$
(2) $\log .001 = -3$ *because* $10^{-3} = .001$
(3) $\log_{\frac{1}{4}} 16 = -2$ *because* $\left(\frac{1}{4}\right)^{-2} = 16$
(4) $\log_3 81 = 4$ *because* $3^4 = 81$
(5) $\log_{\sqrt{3}} 3^{12} = 24$ *because* $\left(\sqrt{3}\right)^{24} = \left(3^{\frac{1}{2}}\right)^{24} = 3^{12}$

- Applying the definition of inverses $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$ to logs implies $b^{\log_b x} = \log_b b^x = x$. Use the definition of inverse to simplify the following expressions:

  (1) $3^{\log_3 8}$
  (2) $5^{\log_5 \sqrt{2}}$
  (3) $\log_3 3^{17}$
  (4) $\log_{15} 15^{\sqrt{13}}$

  Solutions: (1) 8, (2) $\sqrt{2}$*, (3) 17, (4)* $\sqrt{13}$

- Assign additional problems from the math textbook to practice these skills.

- Critical Thinking Writing Assessment: *(See Activity-Specific Assessments at end of unit.)*

Activity 7: Graphing Logarithmic Functions (GLEs: 3, 4, 6, 7, 10, 19, 25, 27, 28)

Materials List: paper, pencil, graphing calculator, Graphing Logarithmic Functions Discovery Worksheet BLM

In this activity, students will learn how to graph logarithmic functions, determine the properties of logarithmic functions, and apply shifts and translations.

Math Log Bellringer:

Evaluate the following: If there is no solution, discuss why.
(1) log 100000 =
(2) $\log_2 32$ =
(3) $\log_{\frac{1}{9}} 243 =$
(4) $\log_2(-4) =$

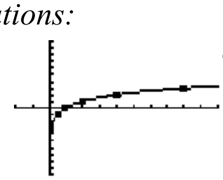
Solutions:
(1) 5 , (2) 5, (3) $-\frac{5}{2}$
(4) no solution, 2 raised to any power will be a positive number.

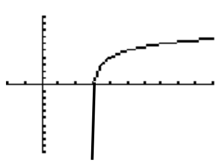
Activity:

- Use the Bellringer to check for understanding of evaluating logarithms in different bases.

- Graphing Logarithmic Functions:
  - In the Graphing Logarithmic Functions Discovery Worksheet, the students will first graph $f(x) = \log x$ by hand by plotting points and discuss its local and global characteristics, then use their knowledge of shifts to graph additional log functions by hand.
  - Distribute the Graphing Logarithmic Functions Discovery Worksheet BLM. Have students work in pairs to complete the first section of the worksheet. This is a noncalculator worksheet, so students can get a better understanding of the logarithm function. Circulate to make sure they are plotting the points correctly. When they have finished the first section, review the answers to the questions.
  - Have students complete the worksheet and review answers to the questions.
  - When they have finished, have students individually graph the following by hand to check for understanding.
    (1) Graph $f(x) = \log_2 x$ plotting and labeling five ordered pairs.
    (2) Graph $f(x) = \log_2(x-3) + 4$

Solutions:

(1)  *Ordered pairs: (½, –1), (1, 0), (2, 1), (4, 2), (8, 3)*

(2) 

Activity 8: Laws of Logarithms and Solving Logarithmic Equations (GLEs: 2, 3, 10; CCSS: WHST.11-12.7)

Materials List: paper, pencil, graphing calculator

This activity has not changed because it already incorporates this CCSS. In this activity, the students will express logarithms in expanded form and as a single log in order to solve logarithmic equations.

Math Log Bellringer:

Solve for x. If there is no solution, discuss why.

(1) $\log_2 x = 3$
(2) $\log_5 25 = x$
(3) $\log_x 16 = 4$
(4) $\log_3(\log_{27}3)=\log_4 x$
(5) $\log_x (-36) = -2$

Solutions:
(1) x = 8
(2) x = 2
(3) x = 2
(4) x = ¼
(5) no solution. Bases must be positive so a positive number raised to any power will be positive.

Activity:

- Use the Bellringer to discuss how to solve different types of logarithmic equations by changing them into exponential equations.

- Give students additional practice problems from the math textbook.

- Have the students discover the Laws of Logarithms using the following modified d*irected learning–thinking activity (DL–TA)* (view literacy strategy descriptions). *DL-TA* is an instructional approach that invites students to make predictions and then to check their predictions during and after the reading. *DL-TA* provides a frame for self-monitoring because of the pauses throughout the reading to ask students questions. This is a modified a *DL–TA* because the students will be calculating not reading.
  - In *DL–TA,* first activate and build background knowledge for the content to be read. This often takes the form of a discussion eliciting information the students may already have, including personal experience, prior to reading. Ask the students to reiterate the first three Laws of Exponents developed in Activity 1 and write the words for the Law on the board.

    Solutions:
    (1) When you multiply two variables with the same base, add exponents.
    (2) When you divide two variables with the same base, subtract the exponents.
    (3) When you raise a variable with an exponent to a power, multiply the exponents.

- Next in *DL–TA*, students are encouraged to make predictions about the text content. Ask the students to list what they think will happen with logarithms and list these on the board.
- Then in *DL–TA,* guide students through a section of text, stopping at predetermined places to ask students to check and revise their predictions. This is a crucial step in *DL-TA* instruction. When a stopping point is reached, ask students to reread the predictions they wrote and change them, if necessary, in light of new evidence that has influenced their thinking. Have the students find the following values in their calculators rounding three places behind the decimal. Once they have finished, have them reread their predictions to see if they want to change one.
  (1) log 4 + log 8 (2) log 32, (3) log ½ + log 100, (4) log 50
  Solutions: (1 & 2) 1.505, (3 & 4) 1.699
- Continue this *DL-TA* cycle with the next set of problems stopping after #8 and #12 to rewrite predictions.
  (5) log 16 – log 2 (6) log 8 (7) log 4 – log 8 (8) log 0.5
  Solutions: (5 & 6) 0.903, (7& 8) –0.301

  (9) 2log 4 (10) log 16 (11) ½ log 9 (12) log 3
  Solutions: (9 & 10) 1.204, (11 & 12) 0 .477
- When the students are finished, their revised predictions should be the Laws of Logarithms. Write the Laws symbolically and verbally. Stress the need for the same base and relate the Laws of Logs back to the Laws of Exponents.
  (1) $\log_b a + \log_b c = \log_b ac$. Adding two logs with the same base is equivalent to taking the log of the product – the inverse operation of the first Law of Exponents.
  (2) $\log_b a - \log_b c = \log_b \frac{a}{c}$. Multiplying two logs with the same base is equivalent to taking the log of the quotient – the inverse operation of the second Law of Exponents.
  (3) $a \log_b c = \log_b c^a$. Multiplying a log by a constant is equivalent to taking the log of the number raised to that exponent – the inverse operation of the third Law of Exponents.
- Check for understanding by asking the students to solve the following problems without a calculator:
  (1) log 4 + log 25
  (2) $\log_3 24 - \log_3 8$
  (3) ½ $\log_2 64$
  Solutions: (1) 10, (2) 1, (3) 3

• Give guided practice problems solving exponential equations by applying the Laws of Logs. Remind students that the domain of logarithms is $x > 0$; therefore, all answers should satisfy this domain.
  (1) $\log x + \log (x - 3) = 1$
  (2) $\log_4 x - \log_4 (x - 1) = ½$
  (3) $\log_5 (x - 2) + \log_5 (x - 1) = \log_5 (4x - 8)$
  Solutions:
  (1) x = 5 is the solution because x = –2 is not in the domain
  (2) x = 2
  (3) x = 5 is the solution because x = 2 is not in the domain of $\log_5 (x - 2)$

- Assign additional problems from the math textbook.

- Critical Thinking Writing Assessment: *(See Activity-Specific Assessments at end of unit.)*

Activity 9: Solving Exponential Equations with Unlike Bases (GLEs: 2, 3, 10; CCSS: F-LE.4)

Materials List: paper, pencil, graphing calculator

This activity has not changed because it already incorporates this CCSS. Students will use logarithms to solve exponential equations of unlike bases and will develop the change of base formula for logarithms.

Math Log Bellringer:

Solve for x: If it cannot be solved by hand, discuss why.
(1) $3^{2x} = 27^{x+1}$ by hand.
(2) $2^{3x} = 6^{4x}$

Solution:
(1) $x = -3$
(2) This problem cannot be solved by hand because 2 and 6 cannot be converted to the same base.

Activity:

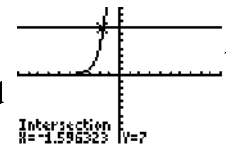
- Use the Bellringer to review solving exponential equations which have the same base.

- Have students find $\log_{10} 6^2$ on the calculator, then change $\log_{10} 6^2 = x$ to the exponential equation $10^x = 6^2$, noting that this is an exponential equation with different bases, 10 and 6. Develop the process for solving exponential equations with different bases using logarithms.
  (1) When x is in the exponent, take the log of both sides using base 10 because that base is on the calculator.
  (2) Apply the 3rd Law of Logarithms to bring the exponent down to the coefficient.
  (3) Isolate x.

  Guided Practice:

$$4^{(x+3)} = 7$$
$$\log 4^{(x+3)} = \log 7$$
$$(x + 3) \log 4 = \log 7$$
$$x + 3 = \frac{\log 7}{\log 4}$$
$$x = \frac{\log 7}{\log 4} - 3$$

- Use the calculator to find the point of intersection of $y = 4^{x+3}$ and $y = 7$. Discuss this alternate process for solving the equation $4^{x+3} = 7$. Compare the decimal answer to the decimal equivalent of the exact answer above, and discuss the difference in an exact answer and decimal approximation.



  (Solution: $x = \dfrac{\log 7}{\log 4} - 3 = -1.596$)

- Application:
  Have students work in pairs to solve the following application problem. When they finish the problem, have several groups describe the steps they used to solve the problem and what properties they used.

  A biologist wants to determine the time t in hours needed for a given culture to grow to 567 bacteria. If the number N of bacteria in the culture is given by the formula $N = 7\left(2^t\right)$, find t. Discuss the steps used to solve this problem and the properties you used. Find both the exact answer and decimal approximation rounded 3 places.

  Solution: Replace N with 567 to get the equation $567 = 7\left(2^t\right)$. *Divide both sides by 7 to get* $81 = \left(2^t\right)$. *Take log base 2 of both sides to solve for t.* $\log_2 81 = 6.340$ *hours.*

- Have students determine $\log_2 8$ by hand and $\dfrac{\log_{10} 8}{\log_{10} 2}$ on the calculator, then formulate a formula for changing the base: $\log_c a = \dfrac{\log_b a}{\log_b c}$. Verify the formula by solving the equation $\log_5 6 = x$ in the following manner:

  $\log_5 6 = x$
  $5^x = 6$
  $\log 5^x = \log 6$
  $x \log 5 = \log 6$

$$x = \frac{\log_{10} 6}{\log_{10} 5}$$

  (Teacher Note: Even though the TI-84 calculator with OS 2.53 can evaluate logs in different bases, the change of base formula is necessary for calculus.)

- Assign additional problems from the math textbook solving exponential equations and changing base of logarithms.

Activity 10: Exponential Growth and Decay (GLEs: 2, 3, 4, 7, 17, 19, 20, 24, 29; CCSSs: F-LE.4, RST.11-12.3, RST.11-12.4)

Materials List: paper, pencil, graphing calculator, *Skittles*® (50 per group), Exponential Growth and Decay Lab BLM**,** 1 cup per group
This activity has not changed because it already incorporates these CCSSs. Students will model exponential growth and apply logarithms to solve the problems.

Math Log Bellringer:

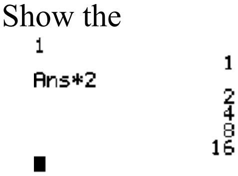
A millionaire philanthropist walks into class and offers to either pay you one cent on the first day, two cents on the second day, and double your salary every day thereafter for thirty days or to pay you one lump sum of exactly one million dollars. Write the exponential equation that models the daily pay and determine which choice you will take.

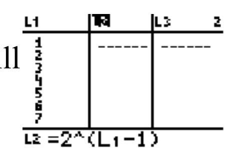
Solution: $y = 2^{x-1}$ if x starts with 1 and ends with 30, $y = 2^x$ if x starts with 0 and ends with 29. If you took the first option, after 30 days you would have \$10,737,418.23.

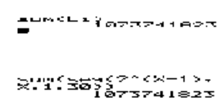
Activity:

- Have students explain the process they used to generate the pay for each of the thirty days to find the answer. Discuss the following calculator skills.

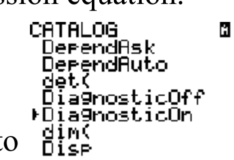
- Most students will have written down the 30 days of pay and added them up. Show the different calculator methods for generating and adding a list of numbers.
  (1) Iteration Method: On the home screen type $\boxed{1}$ $\boxed{\text{ENTER}}$. Then type $\boxed{\text{X}}$ $\boxed{2}$ $\boxed{\text{ENTER}}$. Continue to press $\boxed{\text{ENTER}}$ and count thirty days recording the numbers and adding them up.
  (2) List Method: $\boxed{\text{STAT}}$, EDIT. Put the numbers 1 through 30 in L1. In L2, move the cursor up to highlight L2 and enter 2^(L1 – 1) $\boxed{\text{ENTER}}$ and L2 will fill with the daily salary. On the home screen, type 2nd $\boxed{\text{STAT}}$ (LIST), MATH, 5:sum (L2) and it will add all the numbers in List 2 and give the answer in cents.
  (3) Summing a Sequence: On the home screen, type $\boxed{\text{2ND}}$, [LIST] (above $\boxed{\text{STAT}}$), MATH, 5:sum(, $\boxed{\text{2nd}}$ [LIST] (above $\boxed{\text{STAT}}$), OPS, 5:seq(, 2^(*x*–1), *x*, 1, 30)

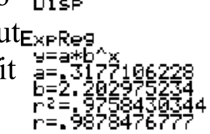






- Exponential Growth and Decay Lab:
  - In this lab, the students will simulate exponential growth and decay using *Skittles*® (or *M & M's*®) to find a regression equation and use that equation to predict the future.
  - Review, if necessary, how to enter data into a calculator and enter a regression equation. (steps in the Activity 3 Exponential Regression Equations BLM)
  - Introduce the correlation coefficient. The correlation coefficient, r^2, is the measure of the fraction of total variation in the values of y. This concept will be covered in depth in future math courses, so it is sufficient to refer to r^2 simply as the percentage of points that are clustered in a small band about the regression equation. Therefore, a higher percentage would be a better fit regression equation. It is interesting to show the students the formula that determines *r*, but the calculator will automatically calculate this value. The





feature must be turned on. 2ND , [CATALOG], (above 0.), DiagnosticOn, ENTER. When the regression equation is created, it will display the correlation coefficient.

$$r=\frac{n\left(\sum xy\right)-\left(\sum x\right)\left(\sum y\right)}{\sqrt{n\left(\sum x^2\right)-\left(\sum x\right)^2}\sqrt{n\left(\sum y^2\right)-\left(\sum y\right)^2}}$$

- Divide the students in groups of four. Give each group a cup with approximately 50 candies in each cup and the Exponential Growth and Decay Lab BLM.
- As the groups finish the Exponential Growth section, circulate and have each group explain the method they used to solve the related questions.
- When the groups have finished both sets of data, combine the statistics and have half of the groups find a regression equation and correlation coefficient for the whole set of growth data. The other groups will find the regression equation and correlation coefficient for the decay data. Discuss the differences in a sample (the 50 candies each group has) and a population (the entire bag of candies), then discuss the accuracy of predictions based on the size of the sample.

Activity 11: Compound Interest and Half-Life Applications (GLEs: 2, 3, 10, 19, 24, 29; CCSSs: F-LE.4, RST.11-12.4)

Materials List: paper, pencil, graphing calculator

This activity has not changed because it already incorporates these CCSSs. Students will develop the compound interest and half-life formulas, then use them to solve application problems.

Math Log Bellringer:

If you have $2000 dollars and you earn 6% interest in one year, how much money will you have at the end of a year? Explain the process you used.

Solution: $2120. Students will have different discussions of how they came up with the answer.

Activity:

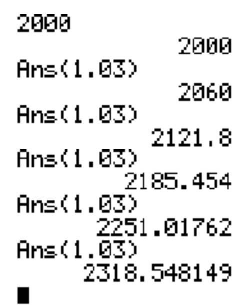
- Use the Bellringer to review the concept of multiplying by 1.06 to get the final amount in a one-step process.

- Discuss the meaning of compounding interest semiannually and quarterly. Draw an empty chart similar to the one below on the board or visual presenter. Guide students through its completion to develop a process to find the value of an account after 2 years.
  - $2000 is invested at 6% APR (annual percentage rate) compounded semiannually (thus 3% each 6 months = 2 times per year). What is the account value after t years?
  - While filling in the chart, record on the board the questions the students ask such as:
    1. Why do you divide .06 by 2?
    2. Why do you have an exponent of $2t$?

3. How did you come up with the pattern?

Time years	Do the Math	Developing the Formula	Account Value
0	$2000	$2000	$2000.00
½	*$2000(1.03)*	*$2000(1+.06/2)*	*$2060.00*
1	*$2060(1.03)*	*$2000(1+.06/2)(1+.06/2)*	*$2121.80*
1½	*$2121.80(1.03)*	*$2000(1+.06/2)(1+.06/2)(1+.06/2)*	*$2185.454*
2	*$2185.454(1.03)*	*$2000(1+.06/2)(1+.06/2)(1+.06/2)(1+.06/2)*	*$2251.01762*
t		*$2000(1+.06/2)^{2t}$*	

- Use the pattern to derive the formula for finding compound interest: $A(t)=P(1+\frac{r}{n})^{nt}$.
  A(t) represents the value of the account in *t* years,
  P – the principal invested,
  r – the APR or annual percentage rate,
  t – the time in years,
  n – the number of times compounded in a year.



- Have students test the formula $A(t)=2000(1+\frac{.06}{2})^{2t}$ by finding $A(10)$, then using the iteration feature of the calculator to find the value after 10 years. *Solution:$3612.22*

- Have the students use a modified form of *questioning the content* (*QtC*) (view literacy strategy descriptions) to work additional problems.
  - The goals of *QtC* are to construct meaning of text, to help students go beyond the words on the page, and to relate outside experiences to the texts being read. Participate in *QtC* as a facilitator, guide, initiator, and responder. Students need to be taught that they can, and should, ask questions of authors as they read.
  - In this modified form of *QtC,* the student is the author. Assign different rows of students to do the calculations for investing $2000 with APR of 6% for ten years if compounded (1) yearly, (2) quarterly, (3) monthly, and (4) daily. Then have the students swap problems with other students and ask the questions developed earlier about compounded interest. Once each student is sure that his/her partner has answered the questions and solved the problem correctly, ask for volunteers to work the problem on the board.

    Solutions:

    (1) yearly: $A(t)=2000(1+\frac{.06}{1})^{1(10)}=\$3581.70$

    (2) quarterly: $A(t)=2000(1+\frac{.06}{4})^{4(10)}=\$3628.04$

    (3) monthly: $A(t)=2000(1+\frac{.06}{12})^{12(10)}=\$3638.79$

(4) daily: $A(t) = 2000(1 + \frac{.06}{365})^{365(10)} = \$3644.06$

- Have students solve the following problem for their situations: How long will it take to double your money in these situations? Again swap problems and once again facilitate the *QtC* process.

  Solutions:

  (1) yearly: $4000 = 2000(1 + \frac{.06}{1})^{1(t)}$ *t =11.896 years*

  (2) quarterly: $\$4000 = 2000(1 + \frac{.06}{4})^{4(t)}$ *t =11.639 years*

  (3) monthly: $4000 = 2000(1 + \frac{.06}{12})^{12(t)}$ *t =11.581 years*

  (4) daily: $4000 = 2000(1 + \frac{.06}{365})^{365(t)}$ *t =11.553 years*

- Define half-life, develop the exponential decay formula, $A = A_0 \frac{1}{2}^{\frac{t}{k}}$ where *k* is the half–life, and use it to solve the following problem:

  A certain substance in the book bag deteriorates from 1000g to 400g in 10 days. Find its half-life.

  Solution:

$$400 = 1000\frac{1}{2}^{\frac{10}{k}}$$

$$0.4 = \frac{1}{2}^{\frac{t}{k}}$$

$$\log 0.4 = \log \frac{1}{2}^{\frac{t}{k}}$$

$$\log 0.4 = \frac{10}{k}\log\frac{1}{2}$$

$$\frac{\log 0.4}{\log 0.5} = \frac{10}{k}$$

$$k = \frac{10\log 0.5}{\log 0.4} = 7.565\ days$$

- Assign additional problems on compound interest and half–life from the math textbook.

Activity 12: Natural Logarithms (GLEs: 2, 3, 4, 6, 10, 24, 27, 29; CCSSs: F-LE.4, RST.11-12.4)

Materials List: paper, pencil, graphing calculator

This activity has not changed because it already incorporates these CCSSs. The students will determine the value of *e* and define natural logarithm.

Math Log Bellringer:

Use your calculator to determine *log* 10 and *ln e*. Draw conclusions.

Solution: log 10 = 1 and ln e = 1. ln must be a logarithm with a base e.

Activity:

- Define *ln* as a natural logarithm base *e*. Have students do the following activity to discover the approximation of *e*. Let students use their calculators to complete the following table. Have them put the equation in y_1 and use the home screen and the notation $y_1(1000)$ to find the values.

n	10	100	1000	10,000	100,000	1,000,000	1,000,000,000
$\left(1+\frac{1}{n}\right)^n$	*2.05937*	*2.07048*	*2.7169*	*2.7181*	*2.7182682*	*2.718280469*	*2.718281827*

- Define *e* as the value that this series approaches as *n* gets larger and larger. It is approximately equal to 2.72 and was named after Leonard Euler in 1750. Stress that *e* is a transcendental number similar to π . Although it looks as if it repeats, the calculator has limitations. The number is really 2.71828182845904590… and is irrational.

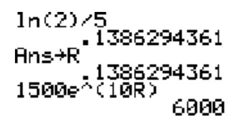
- Graph $y = \ln x$ and $y = e^x$ and discuss inverses and the domain and range of $y = \ln x$. Locate the x-intercept at (1, 0) which establishes the fact that *ln* e = 1.

- Compare $\left(1+\frac{1}{n}\right)^n$ to the compound interest formula, $A(t) = Pe^{rt}$, which is derived by increasing the number of times that compounding occurs until interest has been theoretically compounded an infinite number of times.
  - Revisit the problem from Activity 11 in which the students invested $2000 at 6% APR, but this time compound it continuously for one year and discuss the difference.
    Solution: $3644.24
  - Revisit the problem in Activity 11 of how long it will take to double money. When the students take the log of both sides to solve for *t*, they should use the natural logarithm because ln e = 1.
    Solution:
    $\$4000 = \$2000e^{.06t}$
    $2 = e^{.06t}$
    $\ln 2 = \ln e^{.06t}$

$ln\ 2 = .06t\ ln\ e$
$ln\ 2 = .06t\ (1)$
$\frac{\ln 2}{.06} = t$
$t = 11.552$ *years*

- Discuss use of this formula in population growth. Work with the students on the following two part problem: If the population in Logtown, USA, is 1500 in 2000 and 3000 in 2005, what would the population be in 2015?
  - Most students will answer 6000. Take this opportunity to explain the difference in a proportion, which is a linear equation having a constant slope, and population growth which is an exponential equation that follows the $A(t) = Pe^{rt}$ formula.
  - Part I: Find the rate of growth (*r*)

    $A(t) = Pe^{rt}$
    $3000 = 1500(e^{r(5)})$
    $2 = e^{5r}$
    $\ln 2 = \text{le } e^{5r}$
    $\ln 2 = (5r) \ln e$
    $\ln 2 = 5r$
    $\frac{\ln 2}{5} = r$. Have students store this decimal representation in a letter in the calculator such as R. Discuss how the error can be magnified if a rounded number is used in the middle of a problem.



  - Part II: Use the rate to solve the problem.

    $A(t) = Pe^{rt}$
    $A(15) = 1500(e^{R(15)})$ using the rate stored in R
    $A(10) = 12000$
  - Discuss the difference in what they thought was the answer (6000), which added 1500 every 5 years (linear), and the real answer (12000) which multiplied by 2 every 5 years (exponential).

- Assign additional problems from the math textbook.

- Critical Thinking Writing Assessment: *(See Activity-Specific Assessments at end of unit.)*

Activity 13: Comparing Interest Rates (GLEs: 2, 10, 24, 29; CCSS: F-LE.4)

Materials List: paper, pencil, graphing calculator, Money in the Bank Research Project BLM

This activity has not changed because it already incorporates this CCSS. This is an out-of-class activity. Distribute the Money in the Bank Research Project BLM. Have students choose a financial institution in town or on the Internet. If possible, have each student in a class choose a different bank. Have them contact the bank or go online to find out information about the interest rates available for two different types of accounts and how they are compounded. Have students fill in the following information and solve the following problems. When all projects are in, have students report to the class.

Money in the Bank Research Project

Information Sheet: Name of bank, name of person you spoke to, bank address and phone number or the URL if online, types of accounts, interest rates, and how funds are compounded.

Problem: Create a hypothetical situation in which you invest $500.
(1) Find the equation to model two different accounts for your bank.
(2) Determine how much you will have for each account at the end of high school, at the end of college, and when you retire. (Assume you finish high school in one year, college four years later, and retire 50 years after you finish college.)
(3) Determine how many years it will take you to double your money for each account.
(4) Determine in which account you will put your money and discuss why.
(5) Display all information on a poster board and report to the class.

Sample Assessments

General Assessments

- Use Bellringers as ongoing informal assessments.
- Collect the Little Black Books of Algebra II Properties and grade for completeness at the end of the unit.
- Monitor student progress using small quizzes to check for understanding during the unit on such topics as the following:
  (1) solving exponential equations with same base
  (2) graphing $y = e^x$ and $y = \log x$ with shifts
  (3) evaluating logs such as $\log_2 8$
- Administer two comprehensive assessments:
  (1) exponential equations and graphs, evaluating logs, properties of logs and logarithmic graphs
  (2) solving exponential equations with the same base and different bases, and application problems

Activity-Specific Assessments

- *Teacher Note:* Critical Thinking Writings are used as activity-specific assessments in many of the activities in every unit. Post the following grading rubric on the wall for students to refer to throughout the year.

2 pts.	- answers in paragraph form in complete sentences with proper grammar and punctuation
2 pts.	- correct use of mathematical language
2 pts.	- correct use of mathematical symbols

3 pts./graph - correct graphs (if applicable)
3 pts./solution - correct equations, showing work, correct answer
3 pts./discussion - correct conclusion

- Activity 1: Critical Thinking Writing

  (1) Simplify $\sqrt{(-9)^2}$.
  (2) Simplify $\left(\sqrt{-9}\right)^2$.
  (3) Discuss why the answers to problems 1 and 2 are different.
  (4) Discuss why one of the Laws of Exponents, $a^{\frac{b}{c}} = \sqrt[c]{a^b} = \left(\sqrt[c]{a}\right)^b$, does not apply to problems #1 and #2.

  Solutions:
  (1) 9
  (2) –9
  (3) By order of operations, in problem 1 you have to square the expression first to get 81 and then take the square root to get 9. In problem 2 you have to take the square root first to get 3i, then square it to get –9.
  (4) This Law of Exponents only applies when a $\geq$ 0.

- Activity 4: Evaluate the Exponential Data Research Project (see activity) using the following rubric:

  Grading Rubric for Data Research Project
  10 pts. – table of data with proper documentation (source and date of data)
  10 pts. – scatterplot with model equation from the calculator or spreadsheet (not by hand)
  10 pts. – equations, domain, range,
  10 pts. – real-world problem using extrapolation with correct answer
  10 pts. – discussion of subject and limitations of the prediction
  10 pts. – poster - neatness, completeness, readability
  10 pts. – class presentation

- Activity 5: Critical Thinking Writing

  (1) Solve the two equations: (a) $x^2 = 9$ and (b) $3^x = 9$
  (2) Discuss the family of equations to which they belong.
  (3) Discuss how the equations are alike and how they are different.
  (4) Discuss the two different processes used to solve for x.

  Solutions:
  (1) (a) $x = \pm 3$, (b) $x = 2$
  (2) x^2 belongs to the family of polynomial equations and 3^x is an exponential equation

(3) Both equations have exponent; but in the first the exponent is a number, and in the 2nd the exponent is a variable
(4) (a) Take the square root of both sides. (b) Find the exponent for which you can raise 3 to that power to get 9.

- Activity 6: Critical Thinking Writing

The value of $\log_3 16$ is not a number you can evaluate easily in your head. Discuss how you can determine a good approximation.

Solution:
Answers will vary but should discuss the fact that the answer to a log problem is an exponent and $3^2 = 9$ and $3^3 = 27$ so $\log_3 16$ is between 2 and 3.

- Activity 8: Critical Thinking Writing

The decibel scale measures the relative intensity of a sound. One formula for the decibel level, D, of sound is $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity level in watts per square meter and I_0 is the intensity of barely audible sound.

(1) If the intensity level of a jet is 10^{14} watts per square meter times the intensity of barely audible sound ($10^{14}I_0$), what is the decibel level of a jet take-off.
(2) The decibel level of loud music with amplifiers is 120. How many times more intense is this sound than a barely audible sound?
(3) Compare the decibel levels of jets and loud music.
(4) Are there any ordinances in your town about the acceptable decibel level of sound?

Solutions: (1) 140 decibels, (2) $10^{12}I_0$

- Activity 12: Critical Thinking Writing

In 1990, statistical data estimated the world population at 5.3 billion with a growth rate of approximately 1.9% each year.

(1) Let 1990 be time 0 and determine the equation that best models population growth.
(2) What will the population be in the year 2010?
(3) What was the population in 1980?
(4) In what year will the population be 10 billion?
(5) Discuss the validity of using the data to predict the future.

Solution: (1) $A = 5.3e^{.019t}$, (2) 7.8 billion, (3) 4.4 billion, (4) 2023

- Activity 13: Evaluate the Money in the Bank Research Project (see activity) using the following rubric:

Grading Rubric for Money in the Bank Research Project

10 pts. – Information sheet: Name of bank, name of person you spoke to, bank address and phone number or the URL if online, types of accounts, interest rates, and how funds are compounded (source and date of data)

10 pts. – Compound interest equation for each situation; account value for both accounts at the end of high school, college, and when you retire in 50 years (show all your work)
10 pts. – Solution showing your work of how long it will take you to double your money in each account
10 pts. – Discussion of where you will put your money and why
10 pts. – Poster - neatness, completeness, readability
10 pts. – Class presentation