MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

75th Annual

AMC 12 A

Wednesday, November 8, 2023

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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The problems and solutions for this AMC 12 A were prepared by the
MAA AMC 10/12 Editorial Board under the direction of
Gary Gordon and Carl Yerger, co-Editors-in-Chief.

1. Cities A and B are 45 miles apart. Alicia lives in A and Beth lives in B. Alicia bikes towards B at 18 miles per hour. Leaving at the same time, Beth bikes toward A at 12 miles per hour. How many miles from City A will they be when they meet?

   (A) 20 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27

   Answer (E): Let t be the number of hours they ride before meeting. Then $18t + 12t = 45$. This gives $t = 1.5$, so they each ride for 1.5 hours. Then Alicia has biked $18 \cdot 1.5 = 27$ miles, so their meeting point is 27 miles from A.

2. The weight of $\frac{1}{3}$ of a large pizza together with $3\frac{1}{2}$ cups of orange slices is the same as the weight of $\frac{3}{4}$ of a large pizza together with $\frac{1}{2}$ cup of orange slices. A cup of orange slices weighs $\frac{1}{4}$ of a pound. What is the weight, in pounds, of a large pizza?

   (A) $1\,\dfrac{4}{5}$ **(B)** 2 **(C)** $2\,\dfrac{2}{5}$ **(D)** 3 **(E)** $3\,\dfrac{3}{5}$

   Answer (A): Let p be the weight of a large pizza and q be the weight of a cup of orange slices, in pounds. Then

$$\frac{1}{3}p + \frac{7}{2}q = \frac{3}{4}p + \frac{1}{2}q,$$

   so $3q = \left(\frac{3}{4} - \frac{1}{3}\right) p$ and $p = \frac{36}{5}q$. Because a cup of orange slices weighs $\frac{1}{4}$ of a pound, the large pizza weighs

$$\frac{36}{5} \cdot \frac{1}{4} = \frac{9}{5} = 1\,\frac{4}{5} \text{ pounds.}$$

3. How many positive perfect squares less than 2023 are divisible by 5 ?

   (A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

   Answer (A): Because 5 is prime, a perfect square is divisible by 5 if and only if its square root is divisible by 5. Note that $40^2 = 1600 < 2023 < 2025 = 45^2$, so $5^2, 10^2, 15^2, 20^2, 25^2, 30^2, 35^2$, and 40^2 are the only positive perfect squares satisfying the given condition. There are 8 such numbers.

4. How many digits are in the base-ten representation of $8^5 \cdot 5^{10} \cdot 15^5$?

   (A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

   Answer (E): Because $8 = 2^3$ and $15 = 3 \cdot 5$, the prime factorization of the given number is $2^{15} \cdot 3^5 \cdot 5^{15}$, which is $3^5 \cdot 10^{15} = 243 \cdot 10^{15}$. The base-ten representation is 243 followed by 15 zeros, so it has 18 digits in all.

5. Janet rolls a standard 6-sided die 4 times and keeps a running total of the numbers she rolls. What is the probability that at some point, her running total will equal 3 ?

   (A) $\dfrac{2}{9}$ **(B)** $\dfrac{49}{216}$ **(C)** $\dfrac{25}{108}$ **(D)** $\dfrac{17}{72}$ **(E)** $\dfrac{13}{54}$

Answer (B): Janet rolls a total of 3 if she starts by rolling 111, 12, 21, or 3. These initial sequences of rolls occur with probability $\frac{1}{6^3}$, $\frac{1}{6^2}$, $\frac{1}{6^2}$, and $\frac{1}{6}$, respectively. Adding these probabilities gives

$$\frac{1}{6^3}+\frac{1}{6^2}+\frac{1}{6^2}+\frac{1}{6}=\frac{49}{216}.$$

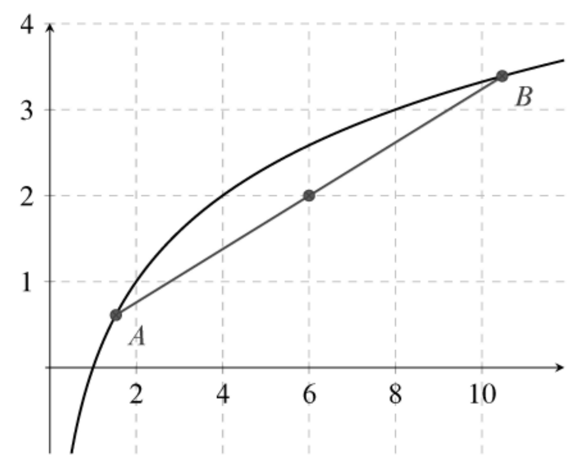
6. Points A and B lie on the graph of $y=\log_2 x$. The midpoint of $\overline{AB}$ is $(6,2)$. What is the positive difference between the x-coordinates of A and B?

(A) $2\sqrt{11}$ **(B)** $4\sqrt{3}$ **(C)** 8 **(D)** $4\sqrt{5}$ **(E)** 9

Answer (D): Let $(r,\log_2 r)$ and $(s,\log_2 s)$ be the coordinates of points A and B, respectively, and assume $r<s$. The midpoint of $\overline{AB}$ is

$$\left(\tfrac{r+s}{2},\tfrac{\log_2 r+\log_2 s}{2}\right)=\left(\tfrac{r+s}{2},\log_2\sqrt{rs}\right).$$

Setting this equal to $(6,2)$ yields the system of equations $r+s=12$ and $rs=16$. This means $r=6-2\sqrt{5}$ and $s=6+2\sqrt{5}$, so the requested difference is $4\sqrt{5}$.



7. A digital display shows the current date as an 8-digit integer consisting of a 4-digit year, followed by a 2-digit month, followed by a 2-digit date within the month. For example, Arbor Day this year is displayed as 20230428. For how many dates in 2023 does each digit appear an even number of times in the 8-digit display for that date?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (E): Because the first four digits of the display must be 2023, the last four digits must contain one 0, one 3, and two of the same digit. These digits cannot contain three 0s or three 3s, so the only possible repeated digits among them are 1 and 2.

If the repeated digit is 1, the possible dates are 01/13, 01/31, 03/11, 10/13, 10/31, 11/03, and 11/30. If the repeated digit is 2, the possible dates are 02/23 and 03/22. In all there are 9 dates in 2023 for which each digit appear an even number of times.

8. Maureen is keeping track of the mean of her quiz scores this semester. If Maureen scores an 11 on the next quiz, her mean will increase by 1. If she scores an 11 on each of the next three quizzes, her mean will increase by 2. What is the mean of her quiz scores currently?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer (D): Suppose Maureen has taken n quizzes so far, with mean m. Then

$$\frac{mn+11}{n+1} = m+1 \qquad \text{and} \qquad \frac{mn+3\cdot 11}{n+3} = m+2.$$

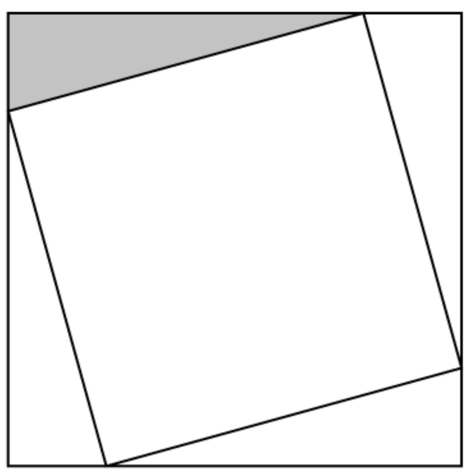
These two equations simplify to $11 = m+n+1$ and $33 = 3m+2n+6$, respectively. Subtracting twice the first equation from the second equation yields $11 = m+4$, so $m = 7$. Then $n = 3$, so Maureen has taken 3 quizzes so far, and their scores could, for example, be 6, 7, and 8. Then one score of 11 increases the mean to

$$\frac{6+7+8+11}{4} = \frac{32}{4} = 8,$$

and three scores of 11 increase the mean to

$$\frac{6+7+8+11+11+11}{6} = \frac{54}{6} = 9.$$

9. A square of area 2 is inscribed in a square of area 3, creating four congruent triangles, as shown below. What is the ratio of the shorter leg to the longer leg in the shaded right triangle?



(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $2-\sqrt{3}$ **(D)** $\sqrt{3}-\sqrt{2}$ **(E)** $\sqrt{2}-1$

Answer (C): Let x be the length of the shorter leg of the right triangle. Then the longer leg has length $\sqrt{3}-x$. The hypotenuse has length $\sqrt{2}$. By the Pythagorean Theorem,

$$x^2 + \left(\sqrt{3}-x\right)^2 = 2.$$

This equation simplifies to $2x^2 - 2\sqrt{3}x + 1 = 0$, and the Quadratic Formula gives the solutions $\frac{1}{2}\left(\sqrt{3} \pm 1\right)$. The requested ratio is

$$\frac{\frac{1}{2}\left(\sqrt{3}-1\right)}{\frac{1}{2}\left(\sqrt{3}+1\right)} = 2 - \sqrt{3}.$$

10. Positive real numbers x and y satisfy $y^3 = x^2$ and $(y - x)^2 = 4y^2$. What is $x + y$?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 36 **(E)** 42

Answer (D): The second equation can be written as the difference of squares: $(y - x)^2 - 4y^2 = 0$. Then $(y - x - 2y)(y - x + 2y) = (-x - y)(3y - x) = 0$. Because x and y are positive, the only solution to this equation is $x = 3y$. Substituting into the first equation gives $y^3 = 9y^2$, so $y = 9$ and $x = 27$. The requested sum is $27 + 9 = 36$.

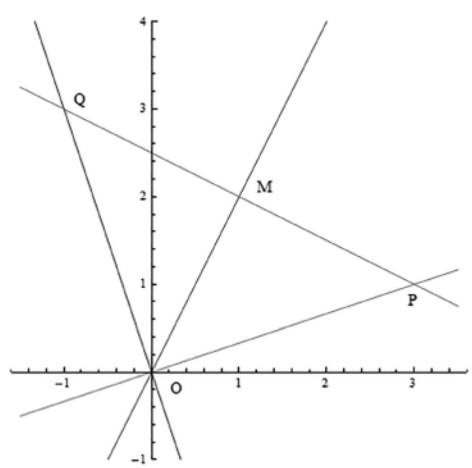
11. What is the degree measure of the acute angle formed by lines with slopes 2 and $\frac{1}{3}$?

(A) 30 **(B)** 37.5 **(C)** 45 **(D)** 52.5 **(E)** 60

Answer (C): Consider lines from the origin O to points $P(3, 1)$ and $Q(-1, 3)$. The lines are perpendicular because their slopes are negative reciprocals of each other. Points P and Q are equidistant from O, so their midpoint lies on the bisector of $\angle POQ$. The midpoint is

$$M = \left(\frac{3 + (-1)}{2}, \frac{1 + 3}{2}\right) = (1, 2),$$

and the line OM has slope 2. Angle $\angle MOP$ is formed by lines with slopes 2 and $\frac{1}{3}$ and is half of a right angle, so its measure is $45°$.



OR

Define points $O(0, 0)$, $M(1, 2)$, and $P(3, 1)$, so that lines OM and OP have the required slopes. Then $OM = \sqrt{1^2 + 2^2} = \sqrt{5}$, $MP = \sqrt{(3-1)^2 + (2-1)^2} = \sqrt{5}$, and $OP = \sqrt{3^2 + 1^2} = \sqrt{10}$.

Furthermore, $OP = \sqrt{2} \cdot OM$. It follows that $\triangle OMP$ is an isosceles right triangle with hypotenuse $\overline{OP}$, so $\angle MOP = 45^\circ$.

OR

The specified angle is the difference between angles α and β whose tangents are 2 and $\frac{1}{3}$, respectively. The Tangent Difference Formula gives

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta} = \frac{2 - \frac{1}{3}}{1 + \frac{2}{3}} = 1.$$

The acute angle whose tangent is 1 is 45°.

OR

Consider vectors $\boldsymbol{a} = \langle 3, 1 \rangle$ and $\boldsymbol{b} = \langle 1, 2 \rangle$. The cosine of the angle between them is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\sqrt{(\boldsymbol{a} \cdot \boldsymbol{a})\,(\boldsymbol{b} \cdot \boldsymbol{b})}} = \frac{5}{\sqrt{50}} = \frac{\sqrt{2}}{2}.$$

The acute angle whose cosine is $\frac{\sqrt{2}}{2}$ is 45°.

12. What is the value of

$$2^3 - 1^3 + 4^3 - 3^3 + 6^3 - 5^3 + \cdots + 18^3 - 17^3\,?$$

(A) 2023 **(B)** 2679 **(C)** 2941 **(D)** 3159 **(E)** 3235

Answer (D): The formula for the sum of the first n cubes is useful here:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Therefore

$$2^3 + 4^3 + 6^3 + \cdots + 18^3 = 8\left(1^3 + 2^3 + 3^3 + \cdots + 9^3\right) = 8 \cdot 45^2 = 16{,}200.$$

Also,

$$\begin{aligned} 1^3 + 3^3 + 5^3 + \cdots + 17^3 & \\ &= \left(1^3 + 2^3 + 3^3 + \cdots + 17^3\right) - \left(2^3 + 4^3 + 6^3 + \cdots + 16^3\right) \\ &= (17 \cdot 9)^2 - 8 \cdot 36^2 = 23{,}409 - 10{,}368 = 13{,}041. \end{aligned}$$

The requested value is $16{,}200 - 13{,}041 = 3159$.

OR

Recall the formulas for the sum of the first n positive integers and for the sum of the first n squares:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

and

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

In summation notation the given expression is

$$\begin{aligned}\sum_{n=1}^{9}\left((2n)^3-(2n-1)^3\right) &= \sum_{n=1}^{9}\left(8n^3-(8n^3-12n^2+6n-1)\right)\\ &= \sum_{n=1}^{9}\left(12n^2-6n+1\right)\\ &= 12\cdot\frac{9\cdot 10\cdot 19}{6}-6\cdot\frac{9\cdot 10}{2}+9\\ &= 3420-270+9=3159.\end{aligned}$$

13. In a table tennis tournament every participant played every other participant exactly once. Although there were twice as many right-handed players as left-handed players, the number of games won by left-handed players was 40% more than the number of games won by right-handed players. (There were no ties and no ambidextrous players.) What is the total number of games played?

(A) 15 **(B)** 36 **(C)** 45 **(D)** 48 **(E)** 66

Answer (B): Let ℓ denote the number of left-handed players, and let r denote the number of right-handed players. There were $\frac{\ell(\ell-1)}{2}$ games between two left-handed players, $\frac{r(r-1)}{2}$ games between two right-handed players, and $\ell\cdot r$ games involving one left-handed and one right-handed player. Let n denote the number of games won by right-handed players against left-handed players. Then right-handed players won a total of $\frac{r(r-1)}{2}+n$ games, and left-handed players won $\frac{\ell(\ell-1)}{2}+(\ell\cdot r-n)$ games. Because left-handed players won 40% more games than right-handed players,

$$\frac{\ell(\ell-1)}{2}+\ell\cdot r-n=1.4\left(\frac{r(r-1)}{2}+n\right).$$

Using $r=2\ell$ and simplifying this equation gives $\ell^2-3\ell+8n=0$. If $n\geq 1$, there are no real solutions for this quadratic equation. Therefore $n=0$, giving $\ell=3$ and $r=6$. The total number of games played is

$$\frac{\ell(\ell-1)}{2}+\frac{r(r-1)}{2}+\ell\cdot r=3+15+18=36.$$

Note that the $r=6$ right-handed players won a total of 15 games (all among themselves), and the $\ell=3$ left-handed players won a total of 21 games (3 games among themselves and all 18 games against right-handed players), and 21 is indeed 40% more than 15.

14. How many complex numbers satisfy the equation $z^5=\overline{z}$, where $\overline{z}$ is the conjugate of the complex number z?

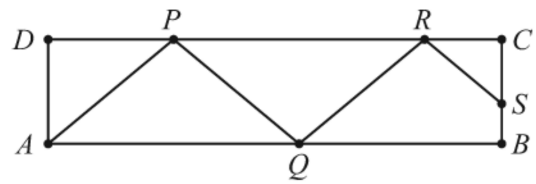
(A) 2 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 7

Answer (E): Taking the absolute value of each side of the equation gives

$$|z^5|=|z|^5=|\overline{z}|.$$

Because $|z|=|\overline{z}|$, this equation is satisfied only when $|z|=0$ or $|z|=1$. If $|z|=0$, then $z=0$ is the unique solution. If $|z|=1$, then multiplying the original equation by z gives $z^6=z\overline{z}=|z|^2=1$, so z must be a sixth root of unity. Each of the sixth roots of unity may be written as $e^{\frac{2\pi ki}{6}}$ (for $k=0,1,\ldots,5$), and each of these roots satisfies the original equation. Thus the total number of solutions is $1+6=7$.

15. Usain is walking for exercise by zigzagging across a 100-meter by 30-meter rectangular field, beginning at point A and ending on the segment $\overline{BC}$. He wants to increase the distance walked by zigzagging as shown in the figure below ($APQRS$). What angle $\theta = \angle PAB = \angle QPC = \angle RQB = \cdots$ will produce a length that is 120 meters? (The figure is not drawn to scale. Do not assume that the zigzag path has exactly four segments as shown; it could be more or fewer.)



(A) $\arccos \frac{5}{6}$ **(B)** $\arccos \frac{4}{5}$ **(C)** $\arccos \frac{3}{10}$ **(D)** $\arcsin \frac{4}{5}$ **(E)** $\arcsin \frac{5}{6}$

Answer (A): With the point labels shown in the figure, let P' be the foot of the perpendicular from P to $\overline{AB}$. Then

$$\cos\theta = \frac{AP'}{AP}.$$

Similar equations hold for other segments of the zigzag walk compared to the straight walk. Usain will achieve his objective if the reciprocal of this ratio is $120\% = \frac{6}{5}$. Therefore $\cos\theta = \frac{5}{6}$ and $\theta = \arccos\frac{5}{6}$, which is about 34°.

Note: The number of zigzags depends on the dimensions of the rectangle. When $AB = 100$ and $BC = 30$, the path consists of three segments, with lengths (in meters) of approximately 54.27, 54.27, and 11.46, summing to 120.

16. Consider the set of complex numbers z satisfying $\left|1 + z + z^2\right| = 4$. The maximum value of the imaginary part of z can be written in the form $\frac{\sqrt{m}}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

Answer (B): Completing the square gives

$$1 + z + z^2 = \tfrac{3}{4} + \left(\tfrac{1}{2} + z\right)^2.$$

Let $w = \frac{1}{2} + z$. Then $\left|w^2 + \frac{3}{4}\right| = 4$. Because $\operatorname{Im} w = \operatorname{Im} z$, it suffices to maximize $\operatorname{Im} w$. By the Triangle Inequality,

$$4 = \left|w^2 + \tfrac{3}{4}\right| \ge |w|^2 - \tfrac{3}{4} \ge (\operatorname{Im} w)^2 - \tfrac{3}{4}.$$

Solving yields $\operatorname{Im} w \le \frac{\sqrt{19}}{2}$, and the requested sum is $19 + 2 = 21$. Equality is achieved when $z = -\frac{1}{2} + \frac{\sqrt{19}}{2}i$.

17. Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$. What is the probability that Flora will eventually land at 10?

(A) $\frac{5}{512}$ **(B)** $\frac{45}{1024}$ **(C)** $\frac{127}{1024}$ **(D)** $\frac{511}{1024}$ **(E)** $\frac{1}{2}$

Answer (E): Suppose integers $a_1, a_2, a_3, \ldots, a_k$ satisfy $0 < a_1 < a_2 < a_3 < \cdots < a_k < 10$. Then Flora lands on the sequence of numbers $a_1, a_2, a_3, \ldots, a_k, 10$ with probability

$$\frac{1}{2^{a_1}} \cdot \frac{1}{2^{a_2-a_1}} \cdot \frac{1}{2^{a_3-a_2}} \cdots \frac{1}{2^{10-a_k}} = \frac{1}{2^{10}}.$$

Therefore every route Flora takes to get to 10 has the same probability. There is one such route for every subset of $\{1, 2, 3, \ldots, 9\}$, so there are 2^9 routes. It follows that the requested probability is

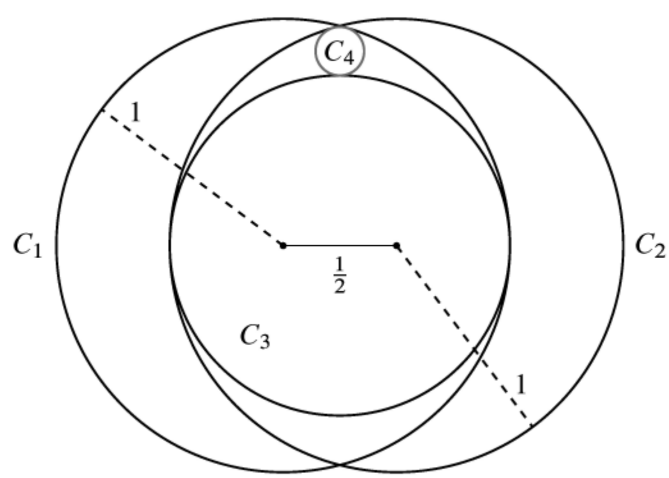
$$\frac{2^9}{2^{10}} = \frac{1}{2}.$$

Note: In fact, the probability Flora will eventually land on *any* specific positive integer distance from 0 is $\frac{1}{2}$.

This may be proved by induction. For every positive integer n, let p_n be the probability that Flora eventually lands on n, starting from 0. Then $p_1 = \frac{1}{2}$, because either she lands on 1 with probability $\frac{1}{2}$ on her very first jump, or she jumps over 1 and never returns to it. (This is the base case of the induction argument.) And, for any integer $k \geq 1$, if $p_1 = p_2 = p_3 = \cdots = p_k = \frac{1}{2}$, then

$$\begin{aligned} p_{k+1} &= \frac{1}{2^{k+1}} + p_1 \cdot \frac{1}{2^k} + p_2 \cdot \frac{1}{2^{k-1}} + \cdots + p_k \cdot \frac{1}{2} \\ &= \frac{1}{2^{k+1}} + \frac{1}{2}\left(\frac{1}{2^k} + \frac{1}{2^{k-1}} + \cdots + \frac{1}{2}\right) \\ &= \frac{1}{2^{k+1}} + \frac{1}{2}\left(1 - \frac{1}{2^k}\right) \\ &= \frac{1}{2}. \end{aligned}$$

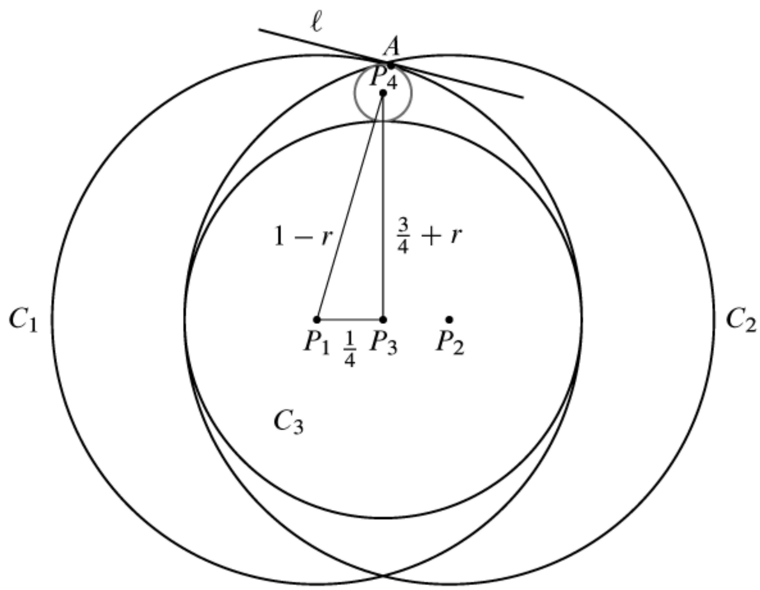
18. Circles C_1 and C_2 each have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externally tangent to C_3. What is the radius of C_4?



(A) $\frac{1}{14}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{10}$ **(D)** $\frac{3}{28}$ **(E)** $\frac{1}{9}$

Answer (D):

Let r be the radius of C_4. Let the point P_k be the center of C_k for $1 \le k \le 4$. Let A be the point of intersection of circles C_1 and C_4 and let ℓ be the tangent line to P_1 at A. By symmetry, P_3 is the midpoint of $\overline{P_1P_2}$, and $P_1P_3 = \frac{1}{4}$. Also by symmetry, P_4 lies on the perpendicular bisector of $\overline{P_1P_2}$. The radius $\overline{P_1A}$ is perpendicular to ℓ. Likewise the radius $\overline{P_4A}$ is perpendicular to ℓ, so P_1, P_4, and A all lie on the segment $\overline{P_1A}$ with P_4 between P_1 and A. Then $P_1P_4 = 1 - r$.



The radius of C_3 is $\frac{3}{4}$. Then $P_3P_4 = \frac{3}{4} + r$, and $\triangle P_1P_3P_4$ is a right triangle with $P_1P_3 = \frac{1}{4}$, $P_3P_4 = \frac{3}{4} + r$, and $P_1P_4 = 1 - r$. By the Pythagorean Theorem,

$$\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4} + r\right)^2 = (1 - r)^2.$$

Expanding gives

$$\frac{1}{16} + \frac{9}{16} + \frac{3r}{2} + r^2 = 1 - 2r + r^2,$$

which gives $r = \frac{3}{28}$.

19. What is the product of all solutions to the equation

$$\log_{7x} 2023 \cdot \log_{289x} 2023 = \log_{2023x} 2023\,?$$

(A) $(\log_{2023} 7 \cdot \log_{2023} 289)^2$ **(B)** $\log_{2023} 7 \cdot \log_{2023} 289$ **(C)** 1 **(D)** $\log_7 2023 \cdot \log_{289} 2023$ **(E)** $(\log_7 2023 \cdot \log_{289} 2023)^2$

Answer (C): Changing the logarithms to base $b = 2023 = 7 \cdot 289$ using the Change of Base Formula gives

$$\frac{1}{\log_b 7x} \cdot \frac{1}{\log_b 289x} = \frac{1}{\log_b bx}.$$

It follows that $\log_b 7x \cdot \log_b 289x = \log_b bx$. Let $y = \log_b x$. It is possible to rewrite this equation as

$$(y + \log_b 7)(y + \log_b 289) = y + 1.$$

Note that $\log_b 7 + \log_b 289 = 1$. Expanding the quadratic equation and simplifying gives

$$y^2 + y + (\log_b 7 \cdot \log_b 289) = y + 1.$$

Then $y^2 = 1 - \log_b 7 \cdot \log_b 289 > 0$, so there are two solutions for y, which can be denoted as y_1 and $y_2 = -y_1$. Setting $x_i = b^{y_i}$ (for $i = 1, 2$) gives $x_1 x_2 = b^{y_1+y_2} = b^0 = 1$. Hence the requested product is 1.

Note: The solutions to the given equation are approximately 943.8 and its reciprocal.

20. Rows 1, 2, 3, 4, and 5 of a triangular array of integers are shown below.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 5 & & 5 & & 1 & \\ 1 & & 7 & & 11 & & 7 & & 1 \end{array}$$

Each row after the first row is formed by placing a 1 at each end of the row, and each interior entry is 1 greater than the sum of the two numbers diagonally above it in the previous row. What is the units digit of the sum of the 2023 numbers in the 2023rd row?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Answer (C): The sums of the entries in the first 6 rows equal 1, 2, 5, 12, 27, and 58. Note that these sums equal $2^n - n$ for $1 \le n \le 6$. That this formula is valid for arbitrary n can be proved using mathematical induction, as follows. Let s_n be the sum of the entries in row n and assume the formula holds for row n, i.e., $s_n = 2^n - n$. Each of the $(n-2)$ interior entries k in row n (other than the end 1s) contributes $2k$ to s_{n+1}, and the plus 1 rule for generating the array contributes another $n - 1$ to s_{n+1}. The end 1s in row n contribute 2 to s_{n+1}, and the end 1s of row $n+1$ add another 2 to s_{n+1}. This gives

$$s_{n+1} = 2(s_n - 2) + (n - 1) + 2 + 2 = 2(2^n - n - 2) + n + 3 = 2^{n+1} - (n + 1),$$

and the proof is complete. Thus the sum of the entries in the 2023rd row is $2^{2023} - 2023$.

It remains to compute the units digit of $2^{2023} - 2023$. Note that the units digits of successive powers of 2 form the sequence $2, 4, 8, 6, 2, 4, 8, 6, \ldots$. Because 2023 leaves a remainder of 3 when divided by 4, the units digit of 2^{2023} is 8. The requested units digit is therefore $8 - 3 = 5$.

OR

Let $a_{n,k}$ be the kth entry in the nth row of the array, and let $b_{n,k} = a_{n,k} + 1$. Then

$$\begin{aligned} b_{n,k} &= a_{n,k} + 1 \\ &= (a_{n-1,k-1} + a_{n-1,k} + 1) + 1 \\ &= (a_{n-1,k-1} + 1) + (a_{n-1,k} + 1) \\ &= b_{n-1,k-1} + b_{n-1,k} \end{aligned}$$

for $1 \le k \le n-1$. Because $b_{n,k}$ satisfies this Pascal Identity together with the conditions $b_{n,0} = b_{n,n} = 2$, these values must be 2 times the binomial coefficients: $b_{n,k} = 2\binom{n-1}{k}$. In other words, the resulting triangle is 2 times the ordinary Pascal Triangle.

$$\begin{array}{ccccccc} & & & 2 & & & \\ & & 2 & & 2 & & \\ & 2 & & 4 & & 2 & \\ 2 & & 6 & & 6 & & 2 \end{array}$$

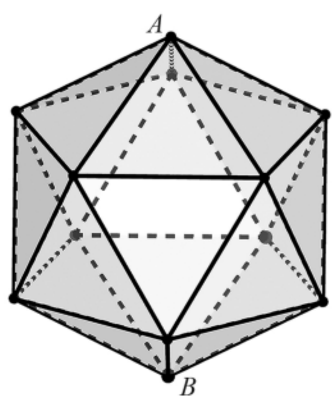
Therefore the sum of the 2023 entries in row 2023 of this array is $2 \cdot 2^{2022}$, and the sum of the 2023 entries in row 2023 of the original array is $2 \cdot 2^{2022} - 2023$, as above.

Note: The sequence whose nth term is $2^n - n$ arises in many different contexts; see A000325 in the On-Line Encyclopedia of Integer Sequences.

21. If A and B are vertices of a polyhedron, define the *distance* $d(A, B)$ to be the minimum number of edges of the polyhedron one must traverse in order to connect A and B. For example, if $\overline{AB}$ is an edge of the polyhedron, then $d(A, B) = 1$, but if $\overline{AC}$ and $\overline{CB}$ are edges and $\overline{AB}$ is not an edge, then $d(A, B) = 2$. Let Q, R, and S be randomly chosen distinct vertices of a regular icosahedron (regular polyhedron made up of 20 equilateral triangles). What is the probability that $d(Q, R) > d(R, S)$?

(A) $\frac{7}{22}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{5}{12}$ **(E)** $\frac{1}{2}$

Answer (A): One way to envision a regular icosahedron is with pentagonal pyramids on top and bottom, joined by a pentagonal antiprism. Let vertex A be at the top, as illustrated.



There are five vertices on the base of the pyramid with apex A, so they are all at distance 1 from A. There are five vertices on the base of the pyramid with apex B, which are all at distance 1 from the previous five vertices, so they are at distance 2 from A. Finally, only vertex B is at distance 3 from A. Thus from any vertex, there are 5 vertices at distance 1, 5 at distance 2, and 1 at distance 3. Randomly select Q, R, and S from the 12 vertices. Let $\mathbb{P}\big[E\big]$ denote the probability of event E, and let $\mathbb{P}\big[E \mid F\big]$ denote the probability of event E given event F. By symmetry

$$\mathbb{P}\big[d(Q,R) > d(R,S)\big] = \mathbb{P}\big[d(Q,R) < d(R,S)\big],$$

so it suffices to compute $\mathbb{P}\big[d(Q,R) = d(R,S)\big]$ and divide its complement by 2 to find $\mathbb{P}\big[d(Q,R) > d(R,S)\big]$. Note that $\mathbb{P}\big[d(Q,R) = 1\big] = \frac{5}{11}$ and

$$\mathbb{P}\big[d(R,S) = 1 \mid d(Q,R) = 1\big] = \frac{4}{10} = \frac{2}{5},$$

because vertex Q is not available. Therefore

$$\mathbb{P}\big[d(Q,R) = d(R,S) = 1\big] = \frac{5}{11} \cdot \frac{2}{5} = \frac{2}{11}.$$

Because the number of vertices that are at distance 2 from Q is also 5, $\mathbb{P}\big[d(Q,R) = d(R,S) = 2\big]$ is also $\frac{2}{11}$.

Note that $d(Q,R) = d(R,S) = 3$ is impossible, because Q and S are distinct. Therefore

$$\mathbb{P}\big[d(Q,R) > d(R,S)\big] = \frac{1}{2}\left(1 - \frac{2}{11} - \frac{2}{11}\right) = \frac{7}{22}.$$

OR

Use the same setting and notation as in the first solution. If $d(Q,R) = 1$, then $d(Q,R) \not> d(R,S)$. If $d(Q,R) = 3$, then $d(Q,R) > d(R,S)$, and this happens with probability $\frac{1}{11}$. If $d(Q,R) = 2$, then $d(Q,R) > d(R,S)$ exactly when $d(R,S) = 1$, which happens for 5 vertices of the remaining 10 from which to choose. Because $\mathbb{P}\big[d(Q,R) = 2\big] = \frac{5}{11}$,

$$\mathbb{P}\big[d(R,S) = 1 \,|\, d(Q,R) = 2\big] = \frac{5}{10} = \frac{1}{2},$$

so

$$\mathbb{P}\big[d(Q,R) > d(R,S) \text{ and } d(Q,R) = 2\big] = \frac{5}{11} \cdot \frac{1}{2} = \frac{5}{22}.$$

The two ways found are mutually exclusive, so $\mathbb{P}\big[d(Q,R) > d(R,S)\big] = \frac{1}{11} + \frac{5}{22} = \frac{7}{22}$.

22. Let f be the unique function defined on the positive integers such that

$$\sum_{d\,|\,n} d \cdot f\left(\frac{n}{d}\right) = 1$$

for all positive integers n, where the sum is taken over all positive divisors of n. What is $f(2023)$?

(A) -1536 **(B)** 96 **(C)** 108 **(D)** 116 **(E)** 144

Answer (B): The number 1 has 1 divisor, so $1 \cdot f(1) = 1$, and hence $f(1) = 1$. If n is a prime number, say $n = p$, then n has 2 divisors, namely 1 and p, and so $1 = 1 \cdot f(p) + p \cdot f(1) = f(p) + p$, and hence $f(p) = 1 - p$ for all primes p. If p is prime and $n = p^2$, then n has 3 divisors, namely 1, p, and p^2, so

$$1 = 1 \cdot f\left(p^2\right) + pf(p) + p^2 f(1) = f\left(p^2\right) + p(1-p) + p^2,$$

which says that $f\left(p^2\right) = 1 - p$. If p and q are distinct primes and $n = pq$, then n has 4 divisors, and

$$1 = 1 \cdot f(pq) + pf(q) + qf(p) + pqf(1) = f(pq) + p(1-q) + q(1-p) + pq,$$

so $f(pq) = (1-p)(1-q)$. If p and q are distinct primes, then pq^2 has 6 divisors, and

$$\begin{aligned} 1 &= 1 \cdot f\left(pq^2\right) + pf\left(q^2\right) + qf(pq) + pqf(q) + q^2 f(p) + pq^2 f(1) \\ &= f\left(pq^2\right) + p(1-q) + q(1-p)(1-q) + pq(1-q) + q^2(1-p) + pq^2, \end{aligned}$$

which gives $f\left(pq^2\right) = (1-p)(1-q)$. Because $2023 = 7 \cdot 17^2$, it follows that $f(2023) = (-6)(-16) = 96$.

23. How many ordered pairs of positive real numbers (a, b) satisfy the equation

$$(1 + 2a)(2 + 2b)(2a + b) = 32ab\,?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** an infinite number

Answer (B): Divide each side of the given equation by ab. The equation can then be expanded and regrouped as follows:

$$\begin{aligned}(1+2a)(2+2b)\left(\frac{2}{b}+\frac{1}{a}\right) &= 32\\ (2+4a+2b+4ab)\left(\frac{2}{b}+\frac{1}{a}\right) &= 32\\ \frac{4}{b}+\frac{8a}{b}+4+8a+\frac{2}{a}+4+\frac{2b}{a}+4b &= 32\\ 8+4\left(2a+\frac{1}{2a}\right)+4\left(b+\frac{1}{b}\right)+4\left(\frac{2a}{b}+\frac{b}{2a}\right) &= 32.\end{aligned}$$

Recall that $x + \frac{1}{x} \geq 2$ for all positive real numbers x, with equality if and only if $x = 1$. Therefore the left-hand side of this equation is bounded below by $8 + 8 + 8 + 8$, and the only solution occurs when $2a = 1$ and $b = 1$.

OR

Applying the Arithmetic Mean-Geometric Mean Inequality gives the following statements:

- $1 + 2a = \frac{1}{2} + \frac{1}{2} + a + a \geq 4\sqrt[4]{\frac{a^2}{4}}$; equality holds if and only if $a = \frac{1}{2}$;
- $2 + 2b = 1 + 1 + b + b \geq 4\sqrt[4]{b^2}$; equality holds if and only if $b = 1$;
- $2a + b = a + a + \frac{b}{2} + \frac{b}{2} \geq 4\sqrt[4]{\frac{a^2b^2}{4}}$; equality holds if and only if $b = 2a$.

Multiplying the three inequalities above gives $(1+2a)(2+2b)(2a+b) \geq 32ab$, and equality holds if and only if $a = \frac{1}{2}$ and $b = 1$.

24. Let K be the number of sequences $A_1, A_2, \ldots, A_n$ such that n is a positive integer less than or equal to 10, each A_i is a subset of $\{1, 2, 3, \ldots, 10\}$, and A_{i-1} is a subset of A_i for each i between 2 and n, inclusive. For example, $\{\}$, $\{5, 7\}$, $\{2, 5, 7\}$, $\{2, 5, 7\}$, $\{2, 5, 6, 7, 9\}$ is one such sequence, with $n = 5$. What is the remainder when K is divided by 10 ?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Answer (C): Such a sequence can be described by a value of n together with a function f from $\{1, 2, 3, \ldots, 10\}$ to $\{1, 2, \ldots, n, n + 1\}$ such that $f(j)$ is the least i such that $j \in A_i$ if $j \in A_n$ and $f(j) = n+1$ if $j \notin A_n$. The number of such functions is $(n + 1)^{10}$, because there are $n + 1$ choices for where to send each element of the domain. Therefore $K = 2^{10} + 3^{10} + 4^{10} + \cdots + 11^{10}$. It remains to determine the remainders when each of the terms in this sum is divided by 10.

- Powers of 2: The remainders of powers of 2 have the pattern $2, 4, 8, 6, 2, 4, 8, 6, 2, 4$, so 2^{10} contributes 4 to the sum.

- Powers of 3: The remainders of powers of 3 have the pattern $3, 9, 7, 1, 3, 9, 7, 1, 3, 9$, so 3^{10} contributes 9.
- Powers of 4: The remainders of powers of 4 have the pattern $4, 6, 4, 6, 4, 6, 4, 6, 4, 6$, so 4^{10} contributes 6.
- Powers of 5: All powers of 5 have remainder 5.
- Powers of 6: All powers of 6 have remainder 6.
- Powers of 7: The remainders of powers of 7 have the pattern $7, 9, 3, 1, 7, 9, 3, 1, 7, 9$, so 7^{10} contributes 9.
- Powers of 8: The remainders of powers of 8 have the pattern $8, 4, 2, 6, 8, 4, 2, 6, 8, 4$, so 8^{10} contributes 4.
- Powers of 9: The remainders of powers of 9 are alternately 9 and 1, so that term contributes 1.

Finally, 10^{10} contributes 0 and 11^{10} contributes 1. Therefore the remainder when K is divided by 10 is the same as the remainder when $4 + 9 + 6 + 5 + 6 + 9 + 4 + 1 + 0 + 1 = 45$ is divided by 10, namely 5.

OR

As above, it suffices to compute the remainder when $2^{10} + 3^{10} + 4^{10} + \cdots + 11^{10}$ is divided by 10. Note that $m^{10} \equiv m^2 \pmod{10}$ for all m. Then modulo 10,

$$K = 2^{10} + 3^{10} + \cdots + 11^{10} \equiv 2^2 + 3^2 + \cdots + 11^2 = \frac{11 \cdot (11+1) \cdot (2 \cdot 11 + 1)}{6} - 1 = 505 \equiv 5.$$

Note: In fact, $K = 40{,}851{,}766{,}525$.

25. There is a unique sequence of integers $a_1, a_2, a_3, \ldots, a_{2023}$ such that

$$\tan 2023x = \frac{a_1 \tan x + a_3 \tan^3 x + a_5 \tan^5 x + \cdots + a_{2023} \tan^{2023} x}{1 + a_2 \tan^2 x + a_4 \tan^4 x + \cdots + a_{2022} \tan^{2022} x}$$

whenever $\tan 2023x$ is defined. What is a_{2023} ?

(A) -2023 **(B)** -2022 **(C)** -1 **(D)** 1 **(E)** 2023

Answer (C): By de Moivre's Formula and the Binomial Theorem,

$$\begin{aligned}\cos nx + i \sin nx &= (\cos x + i \sin x)^n \\ &= \cos^n x + i\binom{n}{1}\cos^{n-1} x \cdot \sin x + i^2\binom{n}{2}\cos^{n-2} x \cdot \sin^2 x \\ &\quad + i^3\binom{n}{3}\cos^{n-3} x \cdot \sin^3 x + \cdots + i^n \sin^n x.\end{aligned}$$

By equating the real and imaginary parts of the respective sides above, it follows that for odd n,

$$\cos nx = \cos^n x - \binom{n}{2}\cos^{n-2} x \cdot \sin^2 x + \binom{n}{4}\cos^{n-4} x \cdot \sin^4 x - \cdots \pm \binom{n}{n-1}\cos x \cdot \sin^{n-1} x$$

$$= C_n(\cos x, \sin x)$$

and

$$\begin{aligned}\sin nx &= \binom{n}{1}\cos^{n-1} x \cdot \sin x - \binom{n}{3}\cos^{n-3} x \cdot \sin^3 x + \cdots \pm \sin^n x \\ &= S_n(\cos x, \sin x),\end{aligned}$$

where $C_n(u, v)$ and $S_n(u, v)$ are homogeneous polynomials of degree n. (A homogeneous polynomial is a polynomial for which all terms have the same total degree.) Hence

$$\tan nx = \frac{S_n(\cos x, \sin x)}{C_n(\cos x, \sin x)}.$$

By dividing the numerator and denominator above by $\cos^n x$, it follows that

$$\tan nx = \frac{S_n(1, \tan x)}{C_n(1, \tan x)}.$$

The general term in $S_n(u, v)$ has the form

$$(-1)^k \binom{n}{2k+1} u^{n-2k-1} v^{2k+1},$$

so with $n = 2023$, the last term occurs when $k = 1011$. Hence the last term of $S_n(1, \tan x)$ is $-1\binom{2023}{2023}v^{2023}$, and thus $a_{2023} = -1$.

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