Introduction

The *Massachusetts Curriculum Framework for Mathematics* builds on the *Common Core State Standards for Mathematics*. The standards in this *Framework* are the culmination of an extended, broad-based effort to fulfill the charge issued by the states to create the next generation of pre-kindergarten through grade 12 standards in order to help ensure that all students are college and career ready in mathematics no later than the end of high school.

In 2008 the Massachusetts Department of Elementary and Secondary Education convened a team of educators to revise the existing *Massachusetts Mathematics Curriculum Framework* and, when the Council of Chief State School Officers (CCSSO) and the National Governors Association Center for Best Practice (NGA) began a multi-state standards development initiative in 2009, the two efforts merged. The *Common Core State Standards for Mathematics* were adopted by the Massachusetts Board of Elementary and Secondary Education on July 21, 2010.

In their design and content, refined through successive drafts and numerous rounds of feedback, the standards in this document represent a synthesis of the best elements of standards-related work to date and an important advance over that previous work. As specified by CCSSO and NGA, the standards are (1) research- and evidence-based, (2) aligned with college and work expectations, (3) rigorous, and (4) internationally benchmarked. A particular standard was included in the document only when the best available evidence indicated that its mastery was essential for college and career readiness in a twenty-first-century, globally competitive society. The standards are intended to be a living work: as new and better evidence emerges, the standards will be revised accordingly.

Unique Massachusetts Standards and Features

The *Massachusetts Curriculum Framework for Mathematics* incorporates the *Common Core State Standards* and a select number of additional standards unique to Massachusetts (coded with an initial "MA" preceding the standard number), as well as additional features unique to Massachusetts that add further clarity and coherence to the *Common Core* standards. These unique Massachusetts elements include standards for pre-kindergartners; Guiding Principles for mathematics programs; expansions of the *Common Core's* glossary and bibliography; and an adaptation of the high school model courses from the *Common Core State Standards for Mathematics Appendix A: Designing High School Mathematics Courses Based on the Common Core State Standards*.

Staff at the Massachusetts Department of Elementary and Secondary Education worked closely with the Common Core writing team to ensure that the standards are comprehensive and organized in ways to make them useful for teachers. The pre-kindergarten standards were adopted by the Massachusetts Board of Early Education and Care on December 14, 2010.

Toward Greater Focus and Coherence

For over a decade, research studies conducted on mathematics education in high-performing countries have pointed to the conclusion that the mathematics curriculum in the United States must become substantially more focused and coherent in order to improve mathematics achievement in this country. To deliver on the promise of common standards, the standards must address the problem of a curriculum that is "a mile wide and an inch deep." The standards in this *Framework* are a substantial answer to that challenge and aim for clarity and specificity.

William Schmidt and Richard Houang (2002) have said that content standards and curricula are coherent if they are:

> articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives. That is, what and how students are taught should reflect not only the topics that fall within a certain academic discipline, **but also the key ideas** that determine how knowledge is organized and generated within that discipline. This implies that to be coherent, a set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, including simple math facts and routine computational procedures associated with whole numbers and fractions) to deeper structures inherent in the discipline. These deeper structures then serve as a

> means for connecting the particulars (such as an understanding of the rational number system and its properties). (emphasis added)

The development of these standards began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skills, and understanding develop over time.

The standards do not dictate curriculum or teaching methods. In fact, standards from different domains and clusters are sometimes closely related. For example, just because topic A appears before topic B in the standards for a given grade, it does not necessarily mean that topic A must be taught before topic B. A teacher might prefer to teach topic B before topic A, or might choose to highlight connections by teaching topic A and topic B at the same time. Or, a teacher might prefer to teach a topic of his or her own choosing that leads, as a byproduct, to students reaching the standards for topics A and B.

What students can learn at any particular grade level depends upon what they have learned before. Ideally then, each standard in this document might have been phrased in the form, "Students who already know … should next come to learn …." But at present this approach is unrealistic—not least because existing education research cannot specify all such learning pathways. Of necessity therefore, grade placements for specific topics have been made on the basis of state and international comparisons and the collective experience and collective professional judgment of educators, researchers and mathematicians. One promise of common state standards is that over time they will allow research on learning progressions to inform and improve the design of standards to a much greater extent than is possible today. Learning opportunities will continue to vary across schools and school systems, and educators should make every effort to meet the needs of individual students based on their current understanding.

These standards are not intended to be new names for old ways of doing business. They are a call to take the next step. It is time for states to work together to build on lessons learned from two decades of standards based reforms. It is time to recognize that standards are not just promises to our children, but promises we intend to keep.

Highlights of the 2011 *Massachusetts Curriculum Framework for Mathematics*

- Guiding Principles for Mathematics Programs, revised from the past Massachusetts *Mathematics Framework,* now show a strong connection to the Standards for Mathematical Practice.
- New Standards for Mathematical Practice describe mathematically proficient students, and should be a part of the instructional program along with the content standards.
- In contrast to earlier Massachusetts mathematics content standards, which were grouped by grade spans, the pre-kindergarten to grade 8 content standards in this document are written for individual grades.
    - The introduction at each grade level articulates a small number of critical mathematical areas that should be the focus for that grade.
    - A stronger middle school progression includes new and rigorous grade 8 standards that encompass some standards covered in the 2000 Algebra I course.
    - These pre-kindergarten through grade 8 mathematics standards present a coherent progression and a strong foundation that will prepare students for the 2011 Model Algebra I course. Students will need to progress through the grade 8 mathematics standards in order to be prepared for the 2011 Model Algebra I course.
- At the high school level, standards are grouped into six conceptual categories, each of which is further divided into domain groupings.
    - In response to many educators' requests to provide models for how standards can be configured into high school courses, this Massachusetts *Framework* also presents eight model courses for high school standards, featuring two primary pathways:
        - Traditional Pathway (Algebra I, Geometry, Algebra II);
        - Integrated Pathway (Mathematics I, Mathematics II, Mathematics III); and
        - Also included are two additional advanced model courses (Precalculus, Advanced Quantitative Reasoning).

- The following supplementary resources are included in this *Framework*.
    - Application of Common Core State Standards for English Language Learners (from the *Common Core State Standards*);
    - Application of Common Core State Standards for Students with Disabilities (from the *Common Core State Standards*);
    - An updated Glossary of Mathematical Terms; and
    - Sample of Works Consulted.

Document Organization

Six **Guiding Principles for Mathematical Programs in Massachusetts** follow this introductory section. The Guiding Principles are philosophical statements that underlie the standards and resources in this *Curriculum Framework*.

Following the Guiding Principles are the eight **Standards for Mathematical Practice** that describe the varieties of expertise that all mathematics educators at all levels should seek to develop in their students.

The **Standards for Mathematical Content** (learning standards) are next in the document, and are presented in three sections:

- Pre-kindergarten through grade 8 content standards are presented by *grade level*;
- High school content standards are presented by *conceptual category;* and
- High school content standards are also presented through *model high school courses*—six model courses outlined in two pathways (Traditional and Integrated) and two model advanced courses.

The **supplementary resources** that follow the learning standards address how to apply the standards for English language learners and students with disabilities. The glossary and list of references from the *Common Core State Standards* are also included and expanded with Massachusetts additions.

The Standards for Mathematical Practice

The Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on the following two sets of important "processes and proficiencies," each of which has longstanding importance in mathematics education:

- The NCTM process standards
  - problem solving
  - reasoning and proof
  - communication
  - representation
  - connections
- The strands of mathematical proficiency specified in the National Research Council's report "Adding It Up"
  - adaptive reasoning
  - strategic competence
  - conceptual understanding (comprehension of mathematical concepts, operations, and relations)
  - procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently, and appropriately)
  - productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy)

The Standards for Mathematical Practice

1. **Make sense of problems and persevere in solving them.**
   Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.**
   Mathematically proficient students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically, and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meanings of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.**
   Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is

a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square, and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. **Look for and express regularity in repeated reasoning.**
   Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Kindergarten

Introduction

In kindergarten, instructional time should focus on two critical areas: (1) representing, relating, and operating on whole numbers, initially with sets of objects; and (2) describing shapes and space. More learning time in kindergarten should be devoted to number than to other topics.

(1) Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects, or eventually with equations such as 5 + 2 = 7 and 7 – 2 = 5. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

(2) Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Overview

Counting and Cardinality

- Know number names and the count sequence.
- Count to tell the number of objects.
- Compare numbers.

Operations and Algebraic Thinking

- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

Number and Operations in Base Ten

- Work with numbers 11–19 to gain foundations for place value.

Measurement and Data

- Describe and compare measurable attributes.
- Classify objects and count the number of objects in each category.

Geometry

- Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).
- Analyze, compare, create, and compose shapes.

STANDARDS FOR MATHEMATICAL PRACTICE

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for an express regularity in repeated reasoning.

Kindergarten

Content Standards

Counting and Cardinality — K.CC

Know number names and the count sequence.

1. Count to 100 by ones and by tens.
2. Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
3. Write numbers from 0 to 20. Represent a number of objects with a written numeral 0–20 (with 0 representing a count of no objects).

Count to tell the number of objects.

4. Understand the relationship between numbers and quantities; connect counting to cardinality.
   a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.
   b. Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.
   c. Understand that each successive number name refers to a quantity that is one larger.
5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

Compare numbers.

6. Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.[5]
7. Compare two numbers between 1 and 10 presented as written numerals.

Operations and Algebraic Thinking — K.OA

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

1. Represent addition and subtraction with objects, fingers, mental images, drawings[6], sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.
2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.
3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).
4. For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.
5. Fluently add and subtract within 5.

Number and Operations in Base Ten — K.NBT

Work with numbers 11–19 to gain foundations for place value.

1. Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

[5] Include groups with up to ten objects.
[6] Drawings need not show details, but should show the mathematics in the problem. (This applies wherever drawings are mentioned in the standards.)

Measurement and Data K.MD

Describe and compare measurable attributes.

1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. *For example, directly compare the heights of two children and describe one child as taller/shorter.*

Classify objects and count the number of objects in each category.

3. Classify objects into given categories; count the numbers of objects in each category and sort the categories by count.[7]

Geometry K.G

Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).

1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.
2. Correctly name shapes regardless of their orientations or overall size.
3. Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Analyze, compare, create, and compose shapes.

4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).
5. Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.
6. Compose simple shapes to form larger shapes. *For example, "Can you join these two triangles with full sides touching to make a rectangle?"*

[7] Limit category counts to be less than or equal to 10.