K

Kendall Hunt

IM K-5 MATH™ beta pilot

Kindergarten

COURSE GUIDE

Illustrative Mathematics CERTIFIED™

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KINDERGARTEN

Teacher Course Guide

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Teacher Course Guide

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Kindergarten

About This Curriculum

These materials were created by Illustrative Mathematics. They were piloted and revised in the 2019-2020 school year.

Each grade level contains 8 or 9 units. Units contain between 8 and 25 lesson plans. Each unit, depending on the grade level, has pre-unit practice problems in the first section, checkpoints or checklists after each section, and an end-of-unit assessment. In addition to lessons and assessments, units have aligned center activities to support the unit content and ongoing procedural fluency.

The time estimates in these materials refer to instructional time. Each lesson plan is designed to fit within a class period that is at least 60 minutes long. Some units contain optional lessons and some lessons contain optional activities that provide additional student practice for teachers to use at their discretion.

Teachers can access the teacher materials either in print or in a browser as a digital PDF. When possible, lesson materials should be projected so all students can see them.

Many activities require blackline masters of recording sheets, game boards, or cards that teachers need to photocopy or cut up ahead of time. Teachers might stock up on two sizes of resealable plastic bags: sandwich size and gallon size. For a given activity, one set of cards can go in each small bag, and then the small bags for one class can be placed in a large bag. If these are labeled and stored in an organized manner, it can facilitate preparing ahead of time and reusing card sets for multiple activities.

Scope and Sequence for Kindergarten

The big ideas in kindergarten include: representing and comparing whole numbers, initially with sets of objects; understanding and applying addition and subtraction; and describing shapes and space. More time in kindergarten is devoted to numbers than to other topics.

The mathematical work for kindergarten is partitioned into 8 units:

1. Math in Our World
2. Numbers 1–10
3. Flat Shapes All Around Us
4. Add and Subtract within 10
5. Compose and Decompose Numbers to 10
6. Numbers within 20
7. Solid Shapes All Around Us
8. Putting it All Together

Unit 1: Math in Our World

Students come into kindergarten with a range of counting experiences, concepts, and skills. In Unit 1, students explore and use mathematical tools while teachers gather information through observations and questions about students' counting knowledge and skill. Throughout the unit, students create norms for and reflect on their mathematical community. The lessons at the beginning of the unit are intentionally short to allow teachers to teach routines while building a mathematical community.

In Section A, students are introduced to a variety of centers that relate to math tools. This allows them to engage in Geometry and Measurement and Data work, although these domains are not the primary focus of the section. Every lesson in the course includes time for center activities so everyone can learn the structures and routines for center time. The materials used in the center activities should be organized for students to use throughout the unit.

In the first 3 sections, students engage in mathematical activities that do not require counting. Students may choose to count, but the activities are designed to be accessible to all learners, regardless of their experience before kindergarten. Throughout these sections, teachers use a checklist and a more detailed interview assessment to uncover their students' understanding and skill in counting.

In the last section, students count collections of objects. Optional activities are provided with explicit focus on aspects of counting (the count sequence, one-to-one tagging, organizing objects to count.) By the end of the unit, students are expected to count a group of up to 10 objects to prepare for the next unit that focuses more deeply on numbers 1–10. Students may be more engaged in counting the collections if they are encouraged to bring in collections of objects to count from home.

Estimated Days: 17 - 18

Standards addressed in this unit

Addressing	K.CC, K.CC.A.1, K.CC.B, K.CC.B.4, K.CC.B.4.a, K.CC.B.5, K.CC.C.6, K.G, K.G.B

Unit 2: Numbers 1–10

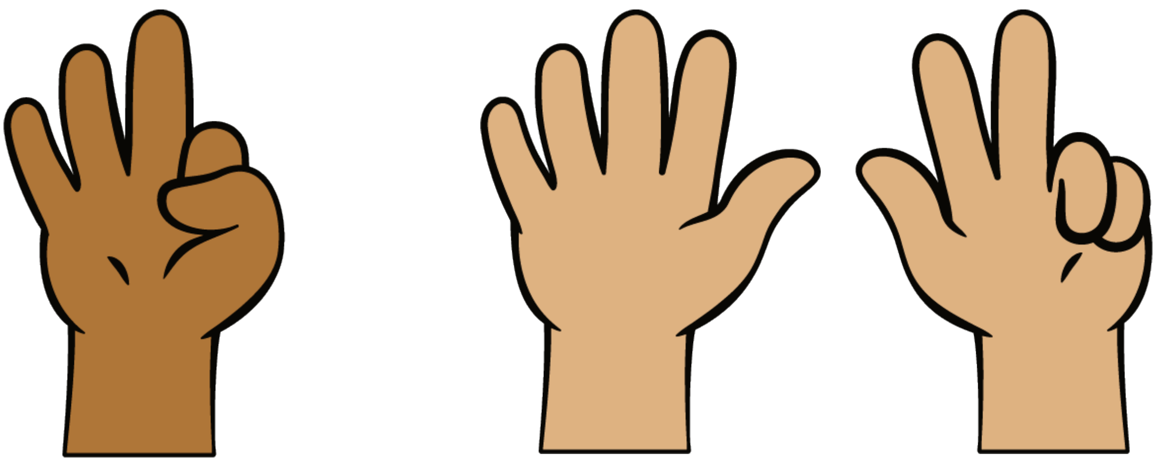
In the previous unit, students subitized to answer “how many” questions, answered “are there enough” questions, and counted groups of up to 10 objects. This unit provides more opportunity for students to develop counting concepts and skills, including comparing, while also introducing number writing.

As in the previous unit, activity structures repeat throughout the unit. This allows students to focus on the mathematical concept because they already know the structure of the activity. This supports students working at different paces and provides opportunities to practice. Students may make a connection the second or third time that they have participated in an activity that they did not make initially.

Students count and compare the number of objects and then move on to counting groups of images in a variety of organized arrangements. Students count and compare the number of images arranged in lines, arrays, number cube patterns, and on 5-frames. The variety of representations allow students to associate different representations with the same number. There is an emphasis on representing numbers with fingers and 5-frames to support students in understanding the numbers $6 - 10$ as $5 + n$. Because of the emphasis on the $5 + n$ structure of numbers, students use a 5-frame rather than a 10-frame, which will be introduced in a later unit.

Throughout the unit, students use language to compare both the number of objects/images in a group and, eventually, written numbers. In the first two sections, “fewer” and “more” are used because the number of things in a group are being compared. In the last section, as students compare written numbers, the term “less” is introduced. In general, “less” is used to compare numerals, and “fewer” is used to compare groups of objects. Students may use these terms interchangeably at first, but will develop proficiency with each term over time.

Fingers are helpful for counting and representing quantities because they are always available to students. Sometimes students may be embarrassed about using their fingers. Students should be encouraged to use their fingers whenever they find them helpful. In later units, students may find using their fingers helpful with topics such as counting on and addition and subtraction. The structure of fingers also encourages students to notice how numbers are related to 5 and 10. Students may choose to represent quantities on their fingers in many different ways. Throughout the kindergarten materials, quantities represented with fingers will be shown beginning with the left pinky.



Numbers 6–10 continue with the thumb on the right hand. When demonstrating numbers on fingers for

students, begin with the right pinky so that, from students' perspective, they see the fingers being held up from left to right. Students can represent numbers with their own fingers however they want, as long as they show the correct number of fingers. It may be helpful to students to hold their fingers down on the table or on their lap to represent numbers 8 and 9.

The first two sections do not require students to read or write numbers in order to emphasize the conceptual aspect of counting and comparing. Students trace numbers during the centers in these sections and begin writing and reading numbers in the third section. Number writing practice can also happen during other parts of the day, such as during writing or handwriting activities. Crayons, colored pencils, markers, glue sticks, and paint brushes with water can be used to trace and write numbers to increase student engagement. Reversals are common when students begin writing numbers, so the emphasis is on writing a number that is recognizable to others with practice.

In the next unit, students will connect counting work to the operation of addition as they make meaning of addition and subtraction.

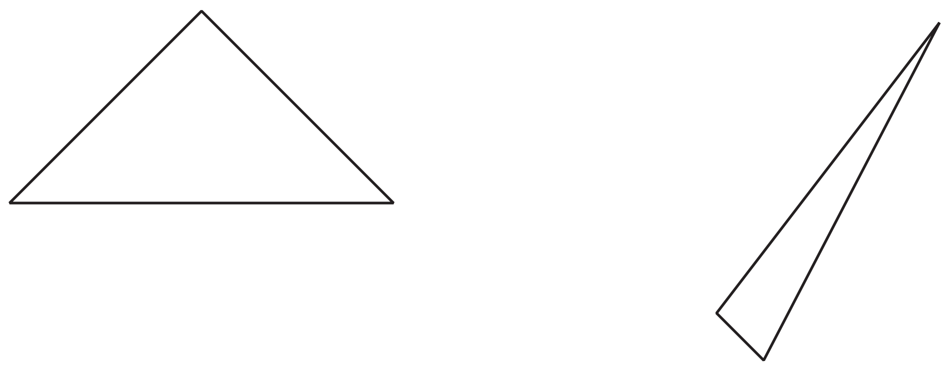
Estimated Days: 21 - 22

Standards addressed in this unit

Addressing	K.CC, K.CC.A.1, K.CC.A.3, K.CC.B, K.CC.B.4, K.CC.B.4.b, K.CC.B.4.c, K.CC.B.5, K.CC.C.6, K.CC.C.7, K.G.B

Unit 3: Flat Shapes All Around Us

In this unit, students are introduced to the foundational concepts of geometry. The focus is exclusively on two-dimensional shapes to help students develop concepts and geometric language with more familiar shapes. Students initially understand shapes only visually—a rectangle is a rectangle because it looks like a door. Because young children are often shown only one or two examples of a shape, they may have a very limited image of shapes when they come to kindergarten even if they know the shape name. For example, a student may know that an isosceles triangle sitting on its base is a triangle but not know that a scalene triangle balanced on a vertex is also a triangle.



Students need to explore many different examples of shapes to broaden their understanding and allow them to perceive shapes in their environment. Students learn that congruent shapes are still the "same" even if they are in different orientations, a crucial understanding for recognizing and describing shapes.

In this unit, students use informal language to describe, compare, and sort shapes. Students learn and use the names of a limited set of shapes (circle, triangle, rectangle, and square.) Students do not need to describe what makes a triangle a triangle until Grade 1.

As the unit continues, students use pattern blocks to make larger shapes. They have an opportunity to continue to develop counting and comparing concepts and skills as they count the pattern blocks they use to create larger shapes. They also use positional words (above, below, next to, beside) to describe shapes they create.

In a later unit, students will be introduced to solid (three-dimensional) shapes and describe, sort, and count them. They will also work with measurable attributes of three-dimensional shapes -- weight and capacity.

Estimated Days: 14 - 15

Standards addressed in this unit

Addressing	K.CC, K.CC.A.1, K.CC.A.3, K.CC.B, K.CC.B.4, K.CC.B.4.c, K.CC.B.5, K.CC.C, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.G.B.5, K.G.B.6, K.MD.A.2, K.MD.B.3

Unit 4: Add and Subtract Within 10

In previous units and in center activities, students have developed counting skills that are foundational for work with addition and subtraction. Students have represented quantities with their fingers, objects, drawings, and numbers, which they will do in this unit in the context of representing and solving addition and subtraction story problems.

Addition is an extension of counting as students count two groups of objects and find the total. Students represent the actions of addition and subtraction with physical objects and count to determine the total or difference. The word “total” is used instead of “sum” to reduce student confusion with the word “some,” which indicates a part of a whole. It is not important that students use this language consistently, as they are learning the concept but they should be encouraged to use more precise language as the year goes on. Students then work with story problems, beginning with story problems without questions to allow students to make sense of stories before they need to solve them. Students solve Add To and Take From, Result Unknown story problems and may use different ways to represent the problem. Students represent addition and subtraction story problems by acting them out and by using objects, drawings, or numbers to represent them. Some students may need to use objects to represent the story problems all the way through the unit, so connecting cubes or counters should be available to students in every activity, including cool-downs.

Students are introduced to expressions, a symbolic way to represent addition and subtraction. In the first sections, the teacher records expressions that correspond to student work. Only in the last section are students expected to connect expressions to story problems and pictures and fill in expressions. Students aren't expected to "read" expressions and may use imprecise language, such as "minussing" to describe the act of subtracting. Teachers use precise language and read expressions to students as "5 and 4" or "5 plus 4" or "5 minus 1." Students find the value of addition and subtraction expressions within 10.

Students are not expected to read story problems independently. All story problems should be read aloud by the teacher. Students may need to hear story problems read aloud multiple times. They should be displayed to develop concepts of print and so that students can refer to them, if they wish. The contexts for the story problems were chosen to be accessible to the widest range of students, however, they can be adapted to a context that is more relevant to students.

In the next unit, students will compose and decompose numbers up to 10 and solve Put Together, Total Unknown and Put Together, Both Addends Unknown story problems. This unit provides the foundation for the major work of grades 1 and 2 which includes developing fluency with addition and subtraction within 20 as well as solving all types of story problems. As students develop confidence in their count, they are able to progress from the Level 1 counting all strategy to Level 2 counting on strategies where they do not need to recount the original quantity.

Estimated Days: 19

Standards addressed in this unit

Addressing	K.CC, K.CC.A.1, K.CC.A.2, K.CC.A.3, K.CC.B, K.CC.B.4.c, K.CC.B.5, K.CC.C.6, K.G.A.1, K.G.B.5, K.G.B.6, K.OA.A.1, K.OA.A.2

Unit 5: Compose and Decompose Numbers to 10

Students compose and decompose numbers within 10.

- Compose and decompose numbers up to 9 in more than one way.
- For any number from 1 to 9, find the number that makes 10 when added to the given number.
- Solve addition and subtraction word problems

Standards Addressed: K.OA.A.2, K.OA.A.3, K.OA.4, K.OA.A.5

Estimated Days: 20

Unit 6: Numbers within 20

Students answer how many questions and count out groups within 20. They understand that numbers 11 to 19 are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones. They write numbers within 20.

- Count groups of up to 20 objects and images.
- Count out a number of objects up to 20
- Understand numbers 11-19 as 10 ones and some more ones
- Represent counts with a written number.
- Count to 100 by 1.

Standards Addressed: K.CC.A.2, K.CC.A.3, K.CC.B.4.a, K.CC.B.4.b, K.CC.B.5, K.NBT.A.1, K.OA.A.1

Estimated Days: 20

Unit 7: Solid Shapes All Around Us

Students identify, describe, analyze, compare and compose two- and three- dimensional shapes. Students sort and count shapes.

- Compose shapes from smaller shapes.
- Describe and compare three-dimensional shapes.
- Count to 100 by 10.

Standards Addressed: K.G.A.1, K.G.A.2, K.G.A.3, K.G.B.4, K.G.B.5, K.G.B.6, K.MD.B.3

Estimated Days: 15

Unit 8: Putting It All Together

Students consolidate and solidify their understanding of various concepts and skills on major work of the grade. They also continue to work toward fluency goals of the grade.

- Fluently add and subtract numbers within 5.
- Apply strategies to solve addition and subtraction word problems and compose and decompose numbers within 10.

Standards Addressed: K.CC.A.1, K.OA.A.2, K.OA.A.3, K.OA.A.4, K.OA.A.5
Estimated Days: 15

Center Overview

Center Activities

Center activities are intended to give students time to practice skills and concepts that are developed across the year. Some activities provide an opportunity to go deeper with specific math content and some help develop grade-level fluency.

Center activities are aligned by Common Core standards to the section goals within each unit. As a result, every section of a unit has centers. Each center builds across multiple stages that may span several grades. For example, The Product Game, a center that helps students develop fluency with multiplication, has 4 stages that span grades 3–5. Note: Early center stages in kindergarten may be building toward the aligned kindergarten grade-level standards.

Structure of Center Time

In kindergarten, center time is built into each lesson so that students have a chance to spend more time on topics that require more time to develop understanding. New centers are introduced during this time and then students are given a choice to work on previously introduced centers.

In grades 1 and 2, there is an optional center day at the end of each section of each unit. In these lessons, new centers are introduced and students also have time to choose between previously introduced centers that reinforce content from the unit or build grade-level fluencies.

In grades 3–5, center time is in addition to regular class time, as desired by the teacher. Optional center day lessons are included occasionally in a unit to introduce a center to students, but in general centers are provided as an extra resource for teachers.

Centers can be used in a variety of additional ways. Students can work on centers if a lesson is completed and there is class time remaining. Entire class sessions can also be dedicated to centers for students to practice or solidify the mathematical ideas of a unit. Students can work on center activities during morning work time, or any other free periods throughout the day. Centers can also be used as support for students when practice with prior grade-level standards is needed.

Center: Connecting Cubes

Students work with connecting cubes.

Stage 1: Exploring

Aligned Sections Kindergarten.1.A

Stage Description

Students have free exploration time with connecting cubes.

Standards Alignments

Addressing 1.G, K.CC.B, K.G.B, K.MD

Materials to Gather

Connecting cubes

Stage 2: Build to Match

Aligned Sections Kindergarten.1.A

Stage Description

Students look at images of figures made of connecting cubes and make a figure to match.

Standards Alignments

Addressing K.CC.B, K.G.B

Materials to Gather

Connecting cubes

Materials to Copy

Connecting Cubes.Build to Match Stage 2 (groups of 1)

Stage 3: Get and Build

Aligned Sections Kindergarten.1.D

Stage Description

Students use a specified number of each color of connecting cube to build a figure of their choice.

Standards Alignments

Addressing K.CC, K.CC.B.4, K.G.B

Materials to Gather

Connecting cubes

Materials to Copy

Connecting Cubes.Get and Build Stage 3 (groups of 2)

Center: Pattern Blocks

Students work with pattern blocks.

Stage 1: Exploring

Aligned Sections Kindergarten.1.A

Stage Description

Students have free exploration time with pattern blocks.

Standards Alignments

Addressing 1.G, K.CC.B, K.G, K.MD.B.3

Materials to Gather

Pattern blocks

Stage 2: Puzzles

Aligned Sections Kindergarten.1.A

Stage Description

Students use pattern blocks to fill in puzzles where the edges of each shape do not touch.

Standards Alignments

Addressing K.G

Materials to Gather

Pattern blocks

Materials to Copy

Pattern Blocks Puzzles Stage 2 (groups of 1)

Stage 3: Get and Build

Aligned Sections Kindergarten.1.D

Stage Description

Students use a specified number of each pattern block to build a creation of their choice.

Standards Alignments

Addressing K.CC, K.CC.B.4, K.G.B

Materials to Gather

Pattern blocks

Materials to Copy

Pattern Blocks.Get and Build Stage 3 (groups of 2)

Center: Geoblocks

Students work with geoblocks.

Stage 1: Exploring

Aligned Sections Kindergarten.1.A

Stage Description

Students have free exploration time with geoblocks.

Standards Alignments

Addressing K.G

Materials to Gather

Geoblocks

Stage 2: Build It

Aligned Sections Kindergarten.1.A

Stage Description

Students use geoblocks to build objects pictured on cards.

Standards Alignments

Addressing K.G

Materials to Gather

Geoblocks

Materials to Copy

Geoblocks Build It Stage 2 (groups of 2)

Center: Picture Books

Students work with picture books.

Stage 1: Exploring

Aligned Sections Kindergarten.1.B

Stage Description

Students look at picture books and identify groups of objects. They may recognize small quantities or count to figure out how many.

Standards Alignments

Addressing K.CC.B.4

Materials to Gather

Picture books

Stage 2: Creating

Aligned Sections Kindergarten.1.B

Stage Description

Students create their own picture book representing different numbers.

Standards Alignments

Addressing K.CC.B.4

Materials to Gather

Colored pencils or crayons

Materials to Copy

Picture Books Creating Stage 2 (groups of 1)

Key Structures in This Course

Student Journal Prompts

Opportunities for communication, in particular classroom discourse, are foundational to the problem-based structure of the IM K–5 Math curriculum. NCTM's Principles and Standards for School Mathematics (NCTM 2000, p. 60) states, "Students who have opportunities, encouragement, and support for speaking, writing, reading, and listening in mathematics classes reap dual benefits: they communicate to learn mathematics, and they learn to communicate mathematically." Opportunities for each of these areas are intentionally embedded directly into the curriculum materials through the student task structures and supported by the accompanying teacher directions.

One highly visible form of discourse is student discussion during the course of a lesson. Another, not as highly visible form of discourse is writing. While this is often only seen as the written responses in a student workbook, journal writing can provide an additional opportunity to support each student in their learning of mathematics.

Writing can be a useful catalyst in learning mathematics because it not only supplies students with an opportunity to describe their feelings, thinking, and ideas clearly, but it also serves as a means of communicating with other people (Baxter, Woodward, Olson & Robyns, 2002; Liedtke & Sales, 2001; NCTM, 2000). NCTM (1989) suggests that writing about mathematics can help students clarify their ideas and develop a deeper understanding of the mathematics at hand.

To encourage the use of journal-writing in math class, we have provided a list of journal prompts that can be used at any point in time during a unit and across the year. These prompts are divided into two overarching categories: Reflections on Content and Reflection on Beliefs and Feelings.

Reflections on content focus on the students' learning or specific learning objectives in each lesson. We first ask students to reflect on the mathematical content because, in general, the act of writing is viewed as requiring a deliberate analysis that encourages an explicit association between current and new knowledge that becomes part of a deliberate web of meaning (Vygotsky, 1987). For example, when students are asked to write about ways in which the math they learned in class that day was connected to something they knew from an earlier unit or grade, they are explicitly connecting their prior and new understandings.

John Dewey asserted that students make sense of the world through metacognition, making connections between their lived experiences and knowledge base, and argued that education should provide students with opportunities to make connections between school and their lived experiences in the world. This belief alongside one of Ladson-Billings' principles of CRT that states teachers must help students effectively connect their culturally- and community-based knowledge to the learning experiences taking place in the classroom supports the need for students to continually reflect not only on the mathematics, but on their own beliefs and experiences as well. Reflections on beliefs are more metacognitive and focus on students' feelings, mindset, and thinking around using mathematics. Writing about these things promotes metacognitive frameworks that extend students' reflection and analysis (Pugalee, 2001, 2004). For example, as students describe something they found challenging during a lesson, they have the chance to reflect on the factors that made it a challenge.

Since the prompts, regardless of the category, can be used at any point during the year, they live in the Course Guide. We imagine these prompts could be used in a variety of ways. In the early grades, they might be used as discussion prompts between partners or students may be asked to respond to a prompt in the form of a drawing or example from their work of the day. In later grades, students can establish a math journal at the beginning of the year and record their reflections at the beginning, in the middle, or at the end of lesson, depending on the prompt. For schools or districts who require homework, the prompts may serve as a nice way for students to reflect on their learning of the day or ask questions they may not have asked during the class period.

Journal writing not only encourages explicit connections between current and new knowledge and promotes metacognitive frameworks to extend ideas but it also provides opportunities for teachers to learn more about each student's identity and math experiences. We believe that writing in mathematics can offer a means for teachers to forge connections with students who typically drift or run rapidly away from mathematics and offer students to continually relate mathematical ideas to their own lives (Baxter, Woodward, and Olson, 2005, p. 132). Writing prompts and journaling work well because students who may not advocate well for themselves when they are struggling get their voices heard in a different way, and thus their needs met (Miller, "Writing to Learn").

It is our hope that through the use of these questions and prompts, students will communicate to learn mathematics as well as learn to communicate mathematically.

Reflecting on Content and Practices

- What math did you learn and do today that connected to something you knew from an earlier unit or grade?
- Describe a time you used the math you learned today outside of school.
- How did your thinking change about something in math today?
- How did any predictions you made in class today work out? Why do you think that happened?
- What questions do you still have about the math today? What new questions do you have?
- Describe the way you solved a problem in class today that you are proud of.
- Where do you see the math you did in class today outside of school? (MP4)
- What math tool did you find most helpful today? Why? (MP5)
- What patterns did you notice in the mathematics today? Why did that pattern happen? (MP7, MP8)
- Starter prompts:
    - The most important thing I learned today is...
    - Today, I struggled and worked through a problem when...(MP1)
    - I could use what I learned today in math in my life when I...
    - At the end of this unit, I want to be able to...
    - I knew one of my answers was right today when...
    - Another strategy I could have used to solve a problem today is...
    - The most important thing to remember when doing the problems like we did today is...

Reflecting on Learning and Feelings about Math

- Describe something you really understand well after today's lesson or describe something that was confusing or challenging.

- In math class, it’s important to be able to explain your thinking. Describe a time when you were able to explain your ideas to other people in your class. (MP3)
- In math class, it’s important to listen to other people’s ideas. Describe a time when you learned something by listening carefully to someone in your class. (MP3)
- What does it mean to be good at math?
- Describe a time when you asked a question about math you were working on. How did asking a question help you?
- If you could change anything about math class, what would it be? Why?
- Starter prompts
    - I learned from a mistake today in math when…
    - When it comes to math, I find it difficult to…
    - I love math because…
    - I felt heard during class today when…
    - I felt my ideas were valued during class today when…
    - The most helpful thing that happened today was…

Teacher Learning

Teaching mathematics is complex work. It requires teachers to plan lessons that offer each student access, elicit students' ideas during these lessons, find ways in which to respond to those ideas, and build a classroom community where students feel known, heard, and seen. Teachers must always be flexible and timely in decision making in order to engage students in rich mathematical discussions. Within each decision lies the opportunity to orient students to one another's ideas and the mathematical goal, and position each student as a competent learner and doer of mathematics. One of the biggest challenges to learning from the work of teaching is that the majority, if not all, of a teacher's learning, planning, and decision-making happens in isolation.

Professional learning communities, [PLCs] are spaces in which teachers can work together around planning and teaching. PLCs include any time teachers or coaches work collaboratively in recurring cycles of collective inquiry and action research to achieve better results for the students they serve. Professional learning communities operate under the assumption that the key to improved learning for students is continuous job-embedded learning for educators. (DuFour, R., Dufour, R., Eaker, R., & Many, T, 2006).

To support teacher collaboration around planning and teaching, we have identified an activity in every unit section as a PLC activity. We also organized a set of activities for teachers to use as they work together in professional learning communities. This activity was chosen because it is either a key mathematical idea of the section or requires a more complex facilitation.

The suggested activities are categorized within a pre-, during-, and post-lesson structure that offers teachers the opportunities to experiment with instruction during both planning and the classroom enactment by collectively discussing instructional decisions in the moment (Gibbons, Kazemi, Hintz, & Hartmann, 2017). These suggested activities are meant to provide guidance for a professional learning community of teachers and coaches that meet to plan for upcoming lessons. While using all of the activities in the given structure is ideal, they are flexible enough to adapt to fit any teacher's given schedule and context.

Suggested activities before a professional learning community meeting

- Read the upcoming lesson that is the focus of the meeting.
- Review student cool-downs from previous lessons.
- Discuss:
    - current student understandings
    - ways in which these understandings build toward the PLC problem

Suggested activities during a professional learning community meeting

- Do the math of the PLC problem individually.
- Read the CCSS and learning goal addressed by the problem.
- Discuss how the standard and learning goal are reflected in what the problem is asking students to do. Think about:
    - Are students conceptually explaining a new, or developing, understanding?
    - Are students making connections between a conceptual understanding and a procedure or process?

- Based on students' previous lesson cool-downs, discuss 1–2 ways students might solve the problem.
- Discuss:
    - How might student responses reflect the CCSS and lesson learning goal?
    - What unfinished learning might students have?
- Based on these discussions, make a plan for:
    - look-fors as you monitor students during their work time
    - questions to ask that assess and advance student thinking
    - the sharing of work and student discussion during the activity synthesis

Suggested activities after a professional learning community meeting

- Record observations as students work.
- Review student cool-downs in relation to the learning goal of the lesson.

Teacher Moves to Support a Math Community

As stated in our design principles, within a problem-based curriculum “students learn mathematics by doing mathematics.” Given the nature of math classrooms, however, students come with differing math identities, which means some students are more prone to see themselves as doers of mathematics than others. Furthermore, apparent inequities in math instruction suggest that some students have opportunities to bring their voice into the classroom, and others do not. In order to extend the invitation to all students to do mathematics, we must work to explicitly develop the math learning community.

According to Horn (2012) and Webel (2010):

Classroom environments that foster a sense of community that allows students to express their mathematical ideas—together with norms that expect students to communicate their mathematical thinking to their peers and teacher, both orally and in writing, using the language of mathematics—positively affect participation and engagement among all students.

To support teachers to develop math learning communities in their classrooms, the first six lessons of each course embed structures to collectively identify what it looks like and sounds like to do math together, create, and reflect on classroom norms that support those actions.

Beyond the first six days, teachers should revisit these norms at least once a week to sustain the math learning community. Consistently returning to these ideas shows students that we value the math learning community as much as we value the math content. Students should also be provided with opportunities to reflect on the norms by stating which ones are the most challenging for them and why. Teacher reflection questions periodically remind teachers of points in a unit where it may be helpful to reflect on these norms.

Additional teaching moves can be used to support the development of math learning communities throughout the school year. The section below highlights teaching moves, put forth by Phil Daro and the SERP Institute, that are intended to support students' engagement in the mathematical practices. A solid math learning community exists when all students display these observable actions, called student vital actions.

Teaching Moves to Support Math Community

student vital actions	teacher moves
All students participate.	• Assign rotating roles, and provide routines for collaboration so that every student is actively engaged in each task, and has experience in all roles over time. • When students are confused ask them to show where they got lost or ask a question that can help them move forward (more than “I don’t get it” or “How do you do it?”). • Check to see if there are recognizable patterns between participation and prior achievement or social groups (for example, ELL, race/ethnicity, or

student vital actions	teacher moves
	gender).

Students say a second sentence.	• Ask and encourage students to ask:
	  ○ Can you tell me more about that?
	  ○ Why do you think that?
	  ○ What changed and what stayed the same?
	  ○ Is that an answer that makes sense for this problem? How do you know?
	  ○ How did you get that answer? Why did you (reference student work)?
	  ○ Is it always true? Sometimes true?

Students talk about each other's thinking.	• Show and discuss work generated by students when working with mathematics concepts. Questions that may be used to prompt students:
	  ○ Did anyone approach the problem a different way?
	  ○ How is your thinking different from theirs?
	  ○ What does their way of thinking help you understand?
	  ○ Do you think their method would work with this kind of problem? Why or why not?
	• Try only responding to questions from groups when no one in the group can answer the question and everyone in the group can ask it.

Students revise their thinking.	• If a student is presenting an explanation, play the role of not understanding and say "Could you help me make sense of your thinking? Could you revise your explanation?" • Have a student quote a classmate's statement that inspired them to revise. • Have students confer in small groups after whole-class presentations to revise and refine their way of thinking.
Students engage and persevere.	• Ask a student who has given a wrong answer additional questions to explore his or her thinking. Demonstrate curiosity about that thinking. • Have students share their thinking and attempts even when they have not found a viable solution. • When some groups are "finished" earlier than others, ask them to analyze their work and seek places to revise their explanation so more students will understand it, or look for an alternative approach.

student vital actions	teacher moves
Students use general and discipline-specific academic language.	• Before beginning small group work, give students sentence frames and probing questions that feature important terms. • Accept students' everyday way of talking as a starting point for joining the math conversation. • Teachers can refer to student statements using some student language while strategically incorporating more precise academic language with the addition of a key word or phrase.
English learners produce language.	• For everyday words that have precise mathematical meaning, provide multiple contexts where the word is useful and have students explain what it refers to in that context. Ask them to use the word to make connections between the different representations. • Encourage students to use language to construct meaning from representations with prompts such as: ○ "Explain where you see...(length, ten, oranges) in the...(figure, equation, table). How do you know it represents the same thing?" • Every student speaks, listens, reads, and writes.

For more details and a full list of teaching moves, visit the SERP Institute site: https://www.serpinstitute.org/5x8-card/vital-student-actions

Design Principles

It is our intent to create a problem-based curriculum that fosters the development of mathematics learning communities in classrooms, gives students access to the mathematics through a coherent progression, and provides teachers the opportunity to deepen their knowledge of mathematics, student thinking, and their own teaching practice. In order to design curriculum and professional learning materials that support student and teacher learning, we need to be explicit about the principles that guide our understanding of mathematics teaching and learning. This document outlines how the components of the curriculum are designed to support teaching and learning aligning with this belief.

Students as Capable Learners of Mathematics

Students enter the mathematics learning community as capable learners of meaningful mathematics, each with unique knowledge and needs. Mathematics instruction that supports students in viewing themselves as capable and competent must be grounded in equitable structures and practices that provide students with access to grade-level content and provide teachers with necessary guidance to listen, learn, and support each student. The curriculum materials include classroom structures that support students in taking risks, engaging in mathematical discourse, productively struggling through problems, and participating in ways that their ideas are visible. Through these structures, teachers have daily opportunities to learn about their students' understandings and experiences and ways in which to position each student as a capable learner of mathematics.

Learning Mathematics by Doing Mathematics

In a mathematics learning community, students learn mathematics by doing mathematics. Doing mathematics can be defined as learning mathematical concepts and procedures while engaging in the mathematical practices—making sense of problems, reasoning abstractly and quantitatively, making arguments and critiquing the reasoning of others, modeling with mathematics, making appropriate use of tools, attending to precision in their use of language, looking for and making use of structure, and expressing regularity in repeated reasoning. This is in contrast to students either watching someone else engage in the mathematical practices or taking notes about mathematics that has already been done. By engaging in the mathematical practices with their peers, students have the opportunity to see themselves as mathematical thinkers with worthwhile ideas and perspectives.

Problem-based Lesson Structure

"Students learn mathematics as a result of solving problems. Mathematical ideas are the outcomes of the problem-solving experience rather than the elements that must be taught before problem solving." (Hiebert et al, 1996) A problem-based instructional framework supports teachers in structuring lessons so students are the ones doing the problem solving to learn the mathematics. The activities and routines are designed to give teachers opportunities to see what students can notice and figure out before having concepts and procedures explained to them. The teacher has many roles in this framework: listener, facilitator, questioner, synthesizer, and more. In all these roles, teachers must listen to and make use of student thinking, be mindful about who participates, and continuously be aware of how students are positioned in terms of status inside and outside the classroom. Teachers also guide students in understanding the problem they are being asked to solve, ask questions to advance students' thinking in productive ways, provide structure for how students share their work, orchestrate discussions so students have the opportunity to understand and take a position on others' ideas, and synthesize the learning with students at the end of activities and lessons.

Balancing Rigor

There are three aspects of rigor essential to mathematics: conceptual understanding, procedural fluency, and the ability to apply these concepts and skills to mathematical problems with and without real-world contexts. These aspects are interconnected in ways that support student understanding. For example, in order to be successful in applying mathematics, students must both conceptually understand and procedurally be able to do the mathematics. (Principles to Action, NCTM, pg. 42)

The materials support the development of the three aspects of rigor by offering students opportunities to access new mathematics, engage in rigorous tasks, and connect new representations and mathematical language to prior learning. This positions them to retrieve and apply their knowledge to novel problems, which often require both conceptual understanding and procedural fluency. Specific grade-level expectations for procedural fluency are supported by the warm-ups, centers, and practice problems. Continual opportunities for students to apply their understandings to mathematical situations give them practice with new material and a review of concepts, skills, and applications from earlier lessons and units.

Coherent Progression

The basic architecture of the materials supports all learners through a coherent progression of the mathematics based both on the standards and on research-based learning trajectories. Each activity and lesson is part of a mathematical story across units and grade levels. This coherence allows students to view mathematics as a connected set of ideas that makes sense.

To support students in making connections to prior understandings and upcoming grade-level work, it is important for teachers to understand the progressions in the materials. Grade-level, unit, lesson, and activity narratives describe decisions about the flow of the mathematics, connections to prior and upcoming grade-level work, and the purpose of each lesson and activity. When appropriate, the narratives explain whether a decision about the scope and sequence is required by the standards or a choice made by the authors.

Each unit, lesson, and activity has the same overarching design structure: an invitation to the mathematics, followed by a deep study of concepts and procedures, and concludes with an opportunity to consolidate understanding of mathematical ideas. The invitation to the mathematics is particularly important in offering students access to the mathematics, as it builds on prior knowledge and encourages the use of their own language to make sense of ideas before formal language is introduced, both of which are consistent with the principles of Universal Design for Learning.

The overarching design structure at each level is as follows:

- Each unit starts with an invitation to the mathematics. The first few lessons provide an accessible entry point for all students and offer teachers the opportunity to observe students' prior understandings.
- Each lesson starts with a warm-up to activate prior knowledge and set up the work of the day. This is followed by instructional activities in which students are introduced to new concepts, procedures, contexts, or representations, or make connections between them. The lesson ends with a synthesis to consolidate understanding and make the learning goals of the lesson explicit, followed by a cool-down to apply what was learned.

- Each activity starts with a launch that gives all students access to the task. This is followed by independent work time that allows them to grapple with problems individually before working in small groups. The activity ends with a synthesis to ensure students have an opportunity to consolidate their learning by making connections between their work and the mathematical goals. In each of the activities, care has been taken to choose contexts and numbers that support the coherent sequence of learning goals in the lesson.

Community Building

To support students in developing a productive disposition about mathematics and engage in the mathematical practices, it is important for teachers from the start of the school year to build a mathematical community that allows all students to express their mathematical ideas and discuss them with others. Students learn math by doing math both individually and collectively. Community is central to learning and identity development (Vygotsky, 1978) within this collective learning.

The materials foster conversation so that students voice their thinking around mathematical ideas, and the teacher is supported to make use of those ideas to meet the mathematical goals of the lessons. Additionally, the first unit in each grade level provides lesson structures which establish a mathematical community, establish norms, and invite students into the mathematics with accessible content. Each lesson offers opportunities for the teacher and students to learn more about one another, develop mathematical language, and become increasingly familiar with the curriculum routines. To maintain this community, materials provide ideas for ongoing support to revisit and highlight the mathematical community norms in meaningful ways.

Instructional Routines

Instructional routines create structures so that all students can engage and contribute to mathematical conversations. Instructional routines have a predictable structure and flow. They are enacted in classrooms to structure the relationship between the teacher and the students around content in ways that consistently maintain high expectations of student learning while adapting to the contingencies of particular instructional interactions (Kazemi, Franke, & Lampert, 2009).

In the materials, we chose to use a small set of instructional routines to ensure they are used frequently enough to become truly routine. The focused number of routines benefits teachers as well as students. Consistently using a small set of carefully chosen routines is just one way that we attempt to lower the cognitive load for teachers. Teachers are free to focus the energy that would be used on structuring an activity on other things, such as student thinking and how mathematical ideas are playing out.

While each routine serves a different specific purpose, they all have the general purpose of supporting students in accessing the mathematics and they all require students to think and communicate mathematically. Throughout the curriculum, routines are introduced in a purposeful way to build a collective understanding of their structure, and are selected for activities based on their alignment with the unit, lesson, or activity learning goals. The teacher course guide explains the purpose and use of the instructional routines in the curriculum. To help teachers identify when a particular routine appears in the curriculum, each activity is tagged with the name of the routine so teachers are able to search for upcoming opportunities to try out or focus on a particular instructional routine. Professional learning for the curriculum materials includes video of the routines in classrooms so teachers understand what the routines look like when they are enacted. Teachers also have opportunities in curriculum workshops and PLCs to practice and reflect on their own enactment of the routines.

Using the *5 Practices for Orchestrating Productive Discussions*

Promoting productive and meaningful conversations between students and teachers is essential to the success in a problem-based classroom. The teacher course guide describes the framework presented in *5 Practices for Orchestrating Productive Mathematical Discussions* (Smith & Stein, 2011) and points teachers to the book for further reading. In all lessons, teachers are supported in the practices of anticipating, monitoring, and selecting student work to share during whole-group discussions. In lessons in which there are opportunities for students to make connections between representations, strategies, concepts, and procedures, the lesson and activity narratives provide support for teachers to also use the practices of sequencing and connecting, and the lesson is tagged so teachers can easily identify these opportunities. Teachers have opportunities in curriculum workshops and PLCs to practice and reflect on their own enactment of the 5 Practices.

Task Complexity

Mathematical tasks can be complex in different ways, with the source of complexity varying based on students' prior understandings, backgrounds, and experiences. In curriculum activities, careful attention is given to the complexity of contexts, numbers, and required computation with few assumptions as to students' familiarity with contexts and representations. Activities are designed to focus students' productive struggle on specific mathematical ideas without layering additional complexities that distract from the intended learning goal. To help students gain familiarity with complexities such as contexts or representations, there are intentional warm-ups and activity launches with teacher-facing narratives that provide guidance on adapting the context of instructional tasks for student relevance without losing the intended mathematics of the task.

In addition to tasks that provide access to the mathematics for all students, the materials provide guidance for teachers on how to ensure that during the tasks, all students are provided the opportunity to engage in the mathematical practices. Specifically, teacher reflection questions ask teachers to think carefully about who is participating and why. Teachers are supported in considering the assumptions they make about their students' understanding and mathematical ideas so they can work to leverage all student thinking in their classroom.

Purposeful Representations

In the materials, mathematical representations are used for two main purposes: to help students develop an understanding of mathematical concepts and procedures, or to help them solve problems. For example, in third grade, equal-groups drawings are used to introduce students to the concept of multiplication. Later on, students make equal-groups drawings to find the total number of objects in situations involving equal groups.

Curriculum representations and the grade levels at which they are used are determined by their usefulness for particular mathematical learning goals. Across lessons and units, students are systematically introduced to representations and encouraged to use representations that make sense to them. As their learning progresses, students are given opportunities to make connections between different representations and the concepts and procedures they represent. Over time, they will see and understand more efficient methods of representing and solving problems, which supports the development of procedural fluency.

Representations that are more concrete are introduced before those that are more abstract. For example, in kindergarten, students begin by counting and moving objects before they represent these objects in 5- and

10-frames to lay the foundation for understanding the base-ten system. In later grades, these familiar representations are extended so that as students encounter larger numbers, they are able to use place-value diagrams and more symbolic methods, such as equations, to represent their understanding.

The teacher course guide makes explicit the selection of a representation when appropriate, so that teachers understand the reasoning behind certain representation choices in the materials.

Equitable Teaching Structures and Practices

For each and every student to have access to mathematical learning opportunities, teachers must first believe that each and every student can learn mathematics with appropriate instruction and understand it is their responsibility to position students in a way that supports that learning. Equitable instruction leverages the teacher's knowledge of mathematics and the socio-cultural contexts of the students in the classroom to deepen learning for all students.

While the problem-based lesson structure and the mathematical community aspects of the curriculum support equity and access, there are five additional ways in which the materials support teachers in equitable teaching practices:

- authentic use of contexts
- suggested launch adaptations
- advancing student thinking questions
- response to student thinking
- teacher reflection questions

Authentic use of contexts and suggested launch adaptations

The use of authentic contexts and adaptations provide students opportunities to bring their own experiences to the lesson activities and see themselves in the materials and mathematics. When academic knowledge and skills are taught within the lived experiences and students' frames of reference, they are more personally meaningful, have higher interest appeal, and are learned more easily and thoroughly (Gay, 2010). By design, unit lessons include contexts that provide opportunities for students to see themselves in the activities or learn more about others' cultures and experiences. In places where there are opportunities to adapt a context to open the space more for students to bring in themselves and their experiences, we have provided suggested prompts to elicit these ideas.

There are two sections within each lesson plan that support teachers in learning more about what each student knows and that provide guidance on ways in which to respond to students' understandings and ideas.

- **Advancing Student Thinking**
  Effective teaching requires being able to support students as they work on challenging tasks without taking over the process of thinking for them (Stein, Smith, Henningsen & Silver, 2000). As teachers monitor during the course of an activity, they gain insight into what students know and are able to do. Based on these insights, the advancing student thinking section provides teachers questions that advance student understanding of mathematical concepts, strategies, or connections between representations.
- **Responding to Student Thinking**
  Each lesson ends with a cool-down problem to formatively assess student thinking in relation to the

learning goal of that day's lesson. If students demonstrate unfinished learning on the cool-down problem, the materials offer guidance on next steps. This guidance falls into one of two categories, next-day support or prior-unit support, based on the anticipated student response. The purpose of this guidance is to allow teachers to continue teaching grade-level content with appropriate and aligned practice and support for students. These suggestions range from providing students with more concrete representations in the next day's lessons to recommended center activities from prior grade level units.

Teacher Reflection Questions

To ensure that all students have access to an equitable mathematics program, educators need to identify, acknowledge, and discuss the mindsets and beliefs they have about students' abilities (NCTM PtA, 64). To support teachers in identifying and acknowledging their own mindsets and beliefs, we have created a set of teacher reflection questions within the category we call Beliefs and Positioning. These questions prompt teachers to reflect on, and challenge, the assumptions they make—about mathematics, learners of mathematics, and the communication of mathematics in their classrooms.

Teacher Learning Through Curriculum Materials

Getting better at teaching requires teachers to plan at the course, unit, and lesson level, and to reflect on, and improve on, each day's instruction. Educative materials that support teacher learning from teaching must leverage and enhance the teacher's knowledge of mathematics and pedagogical practices, their students, and the socio-cultural contexts of each student in the classroom. There are three key components of the curriculum materials that support teachers in this learning: Unit, Lesson, and Activity Narratives, Teacher Reflection Questions, and Professional Learning Community (PLC) activities and structure.

Unit, Lesson, and Activity Narratives

The narratives included in the materials provide teachers with a deeper understanding of the mathematics and its progression within the materials.

Teacher Reflection Question

To encourage teachers to reflect on the teaching and learning in their classroom, each lesson concludes with a teacher-directed reflection question on the mathematical work or pedagogical practices of the lesson. The questions are drawn from four categories: mathematical content, pedagogy, student thinking, or beliefs and positioning. The questions are designed to be used by individuals, grade-level teams, coaches, or anyone who supports teachers.

Professional Learning Communities (PLC)

Teaching mathematics requires continual learning. Teachers must be adept at moment-to-moment decision making, in order to engage students in rich discussions of mathematical content (O'Connor & Snow, 2018). We believe this learning should be embedded within a teacher's daily work and be a collective experience within professional learning communities. To support teachers and coaches in this collective work, each unit section has an activity identified as a PLC activity. This activity either highlights an important mathematical idea in the unit or has a complex facilitation that would benefit from teachers planning and rehearsing the activity together. We have also included a structure for the learning community included in the Professional Learning Community section of the Course Guide.

Model with Mathematics K–5

In K–5, modeling with mathematics is problem solving. It is problem-solving that provides opportunities for students to notice, wonder, estimate, pose problems, create representations, assess reasonableness, and continually make revisions as needed. In the early grades, these opportunities involve various precursor modeling skills that support students in being flexible about the way they solve problems. In upper elementary, these precursor skills become various stages of the modeling process that students will experience in grades 6–12. In addition to the precursor skills and modeling stages that appear across lessons, each unit culminates with a lesson that explicitly addresses these modeling skills and stages while pulling together the mathematical work of the unit.

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Representations in the Curriculum

"The power of a representation can . . . be described as its capacity, in the hands of a learner, to connect matters that, on the surface, seem quite separate. This is especially crucial in mathematics" (Bruner, 1966, p. 48)

Mathematical representations can be used for two main purposes: to help students develop an understanding of mathematical concepts and procedures or to help them solve problems. The materials make thoughtful use of representations in both ways.

Curriculum representations and the grade levels at which they are used are determined by their usefulness for particular mathematical learning goals. Across lessons and units, students are systematically introduced to representations and encouraged to use representations that make sense to them. As their learning progresses, students are given opportunities to make connections between different representations and the concepts and procedures they represent. Over time, they will see and understand more efficient methods of representing and solving problems, which support the development of procedural fluency.

In general, more concrete representations are introduced before those that are more abstract. There are a couple of key progressions of representations that occur across grade bands in different domains.

Linear Measurement

objects of equal length → rulers → number line diagrams

Numbers and Operations in Base Ten

5- and 10-frames → connecting cubes in towers of 10 → base-ten blocks → base-ten diagrams

Number and Operations – Fractions

rectangles (partitioned into equal parts) → fraction strips → tape diagrams → number line diagrams

Operations and Algebraic Thinking: Addition and Subtraction

objects (fingers, counters, cubes) → drawings → tape diagrams

Operations and Algebraic Thinking: Multiplication and Division

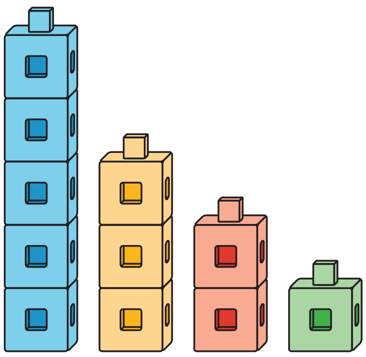
drawings → arrays (objects and images) → inch tiles → area diagrams (gridded and ungridded)

These progressions, as well as the descriptions below can be helpful in providing support for students who have unfinished learning and would benefit from more concrete representations to make sense of mathematical concepts.

Two-color Counters (K-1)

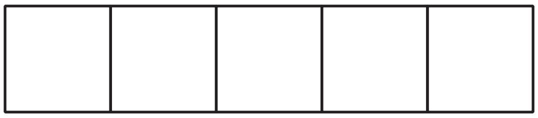
Counters of one color are used frequently to represent quantities in the early grades. Students use the two-color counters to support their work in comparing, counting, combining, and decomposing quantities. In later grades, the counters can be used to visually represent properties of operations.

Connecting Cubes (K-5)



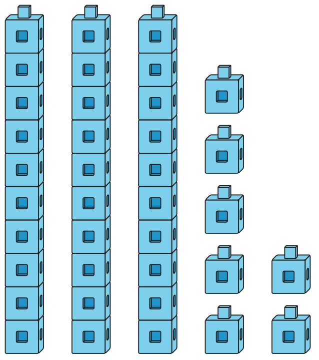
Like counters, cubes can be used in the early grades for comparing, counting, combining, and decomposing numbers. In later grades, they are used to represent multiplication and division, and in grade 5, to study volume. Teachers of grade 5 should use cubes that connect on multiple sides to develop understanding of volume.

5-Frame and 10-Frame (K-2)



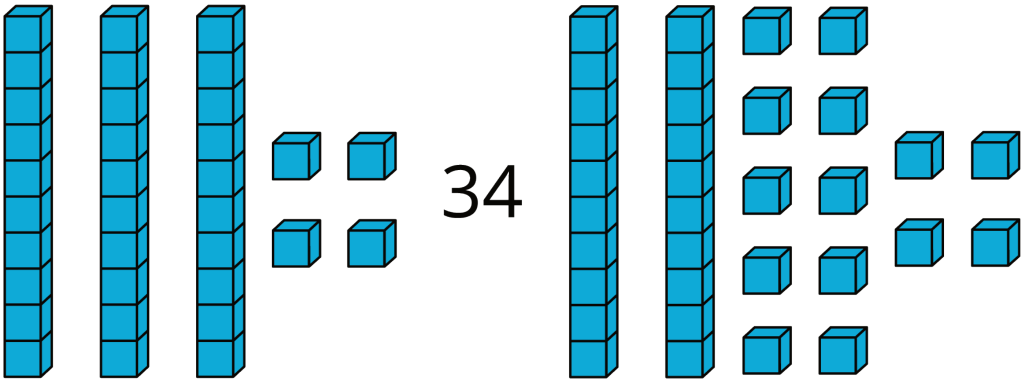
5- and 10-frames provide students with a way of seeing the numbers 5 and 10 as units and also combinations that make these units. Because we use a base-ten number system, it is critical for students to have a robust mental representation of the numbers 5 and 10. Students learn that when the frame is full of ten individual counters, we have what we call a ten, and when we cannot fill another full ten, the "extra" counters are ones, supporting a foundational understanding of the base ten number system. The use of multiple 10-frames supports students in extending the base 10 number system to larger numbers.

Connecting Cubes in Towers of 10 (1-2)



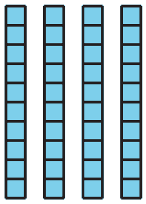
Cubes that are in towers of 10 support students in using place value structure for adding, subtracting, and comparing numbers. Connecting cubes have the advantage that students can physically compose and decompose numbers, unlike place value blocks or Cuisenaire rods. The cubes are a helpful physical representation as students begin to unitize. For example, students can understand that 10 of the single cubes are the same as 1 ten and 10 of the tens are the same as 1 hundred.

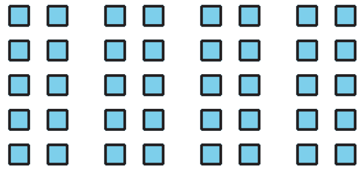
Base-ten Blocks (2-5)



Base-ten blocks are used after students have had the physical experience of composing and decomposing towers of 10 cubes. The blocks offer students a way to physically represent concepts of place value and operations of whole numbers and decimals. Because the blocks cannot be broken apart, as the connecting cube towers can, students must focus on the unit. As students regroup, or trade, the blocks, they are able to develop a visual representation of the algorithms. The size relationships among the place value blocks and the continuous nature of the larger blocks allow students to investigate number concepts more deeply. The blocks are used to represent whole numbers and, in grade 4 and 5, decimals by defining different size blocks as the whole.

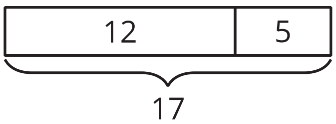
Base-ten Diagram (1-5)

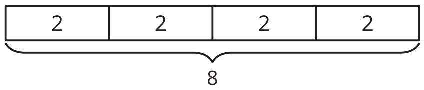




Base-ten diagrams offer students a way to represent base-ten blocks after they no longer need concrete representations. Although individual units might be shown, the advantage of place value diagrams is that they can serve as a “quick sketch” of representing numbers and operations.

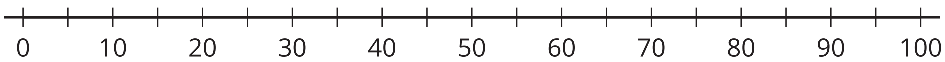
Tape Diagram (2-5)

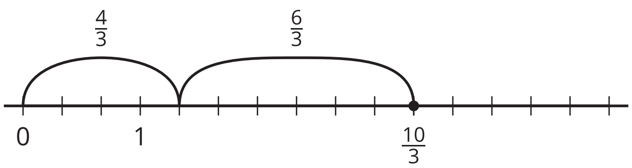




Tape diagrams, resembling a segment of tape, are primarily used to represent the operations of adding, subtracting, multiplying, and dividing. Students use them first with whole numbers and later with fractions and decimal numbers to emphasize the idea that the meaning and properties of operations are true as the number system expands. They can help students represent problems, visualize relationships between quantities, and solve mathematical problems.

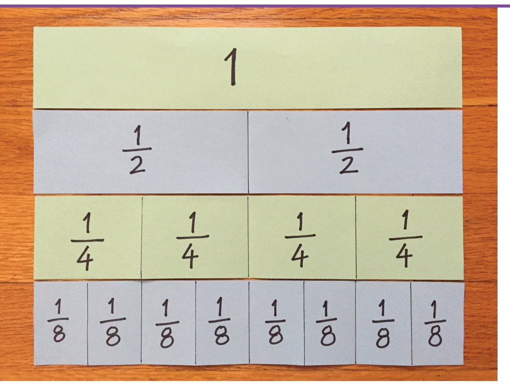
Number Line Diagram (2-5)





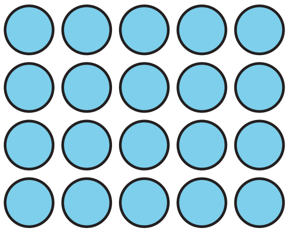
Number line diagrams are used to represent and compare numbers, and can also be used to represent operations. Understanding of number line diagrams are built on students’ grade 2 experience with rulers. Students begin by working with number lines with tick marks to represent the whole numbers. Then, they work with number lines where tick marks correspond to multiples of 10, 100 or 1,000 to develop an understanding of place value and relative magnitude. In later grades, students understand that there are numbers between the whole numbers. They extend their work with whole number operations on the number line to include fractions and decimals.

Fraction Strips (3-4)



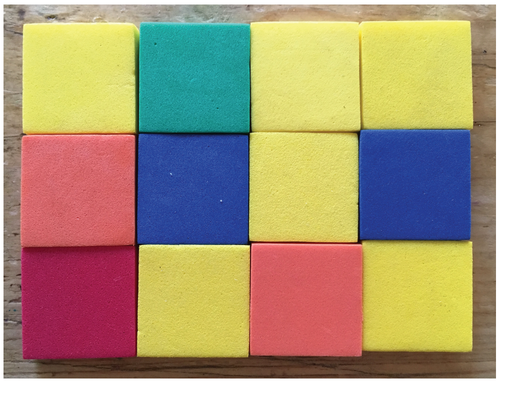
Fraction strips are rectangular pieces of paper or cardboard used to represent different parts of the same whole. They help students concretely visualize and explore fraction relationships. As students manipulate the parts of the same whole, they develop understanding of fractional-sized pieces and equivalency, and learn to compare and order fractions. This representation builds on the number line.

Array (2-3)



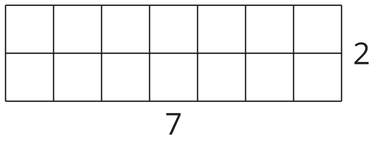
An array is an arrangement of objects or images in rows and columns that can be used to represent multiplication and division. Each column must contain the same number of objects as the other columns, and each row must have the same number of objects as the other rows.

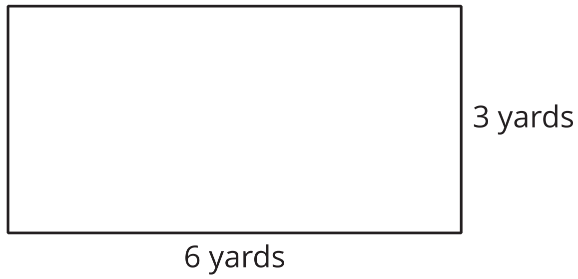
Inch Tiles (2-4)

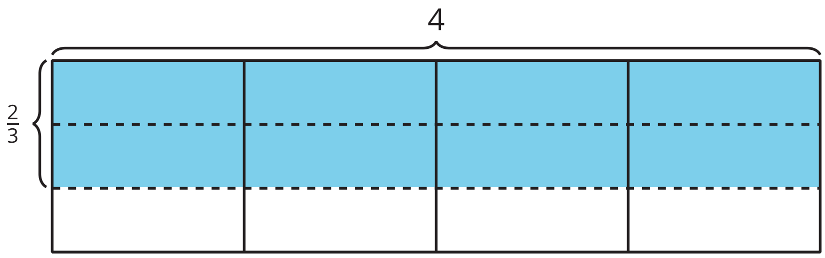


Inch tiles offer students a way to create physical representations of two-dimensional figures that have a certain area and to cover a two-dimensional figure with square units to determine its area. Students organize inch tiles into rows and columns to connect the area of rectangles to multiplication and division.

Area Diagram (3-5)

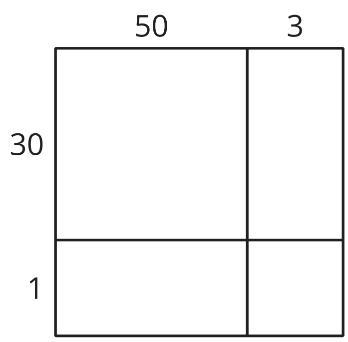






An area diagram is a rectangular diagram that can be used to represent multiplication and division of whole numbers, fractions, and decimals. The area diagram may be overlayed with a grid to show individual units. As students move from working with an area diagram overlayed with a grid to one without, they move from a more concrete understanding of area to a more abstract one. In an area diagram without a grid, the unit squares are not explicitly represented, which makes this diagram useful when working with larger numbers or fractions and making connections to the distributive property and algorithms.

As the numbers in products become larger, area diagrams are difficult to read if the ones, tens, and eventually hundreds are shown accurately. This diagram shows a way to visualize the product 53×31.



It shows how to decompose the product into 4 parts, represented in the diagram as smaller rectangles. The size of each smaller rectangle in the diagram does not represent its actual size since the segment labeled 30 is not 30 times as long as the segment labeled 1. Even though the small rectangles do not have the correct relative size, the diagram can still be used to correctly decompose the product 53×31,

$$53 \times 31 = (50 \times 30)(50 \times 1)(3 \times 30) + (3 \times 1)$$

The diagram helps visualize geometrically why the equation is true.

References

- Bruner, J. (1966). *Towards a Theory of Instruction*. Cambridge, MA: Harvard University Press.

A Typical Lesson

A note about optional activities: A relatively small number of activities throughout the course have been marked "optional." Some common reasons an activity might be optional include:

- The activity addresses a concept or skill that goes beyond the requirements of a standard. The activity is nice to do if there is time, but students won’t miss anything important if the activity is skipped.
- The activity provides an opportunity for additional practice on a concept or skill that we know many students (but not necessarily all students) need. Teachers should use their judgment about whether class time is needed for such an activity.

A typical lesson has four phases:

1. a warm-up
2. one or more instructional activities
3. the lesson synthesis
4. a cool-down

In kindergarten, lessons do not include cool-downs and since activities are shorter, each lesson includes 20–30 minutes of time for center activities.

In grade 1, some lessons do not have cool-downs and instead have 15–25 minutes for center activities. Over the course of the year, cool-downs appear in lessons more often.

Warm-up
The first event in every lesson is a warm-up. Every warm-up is an instructional routine. The warm-up invites all students to engage in the mathematics of each lesson. The warm-ups provide opportunities for students to bring their personal experiences as well as their mathematical knowledge to problems and discussions. They place value on students’ voices as they communicate their developing ideas, ask questions, justify their responses, and critique the reasoning of others.

A warm-up either:

- helps students get ready for the day’s lesson, or
- gives students an opportunity to strengthen their number sense or procedural fluency

A warm-up that helps students get ready for today’s lesson might serve to remind them of a context they have seen before, get them thinking about where the previous lesson left off, or preview a context or idea that will come up in the lesson so that it doesn't get in the way of learning new mathematics.

A warm-up that is meant to strengthen number sense or procedural fluency asks students to do mental arithmetic or reason numerically or algebraically. It gives them a chance to make deeper connections or become more flexible in their thinking.

In addition to the mathematical purposes, these routines serve the additional purpose of strengthening students’ skills in listening and speaking about mathematics.
Once students and teachers become used to the routine, warm-ups should take 5–10 minutes. If warm-ups

frequently take much longer than that, the teacher should work on concrete moves to more efficiently accomplish the goal of the warm-up.

At the beginning of the year, consider establishing a small, discreet hand signal students can display to indicate they have an answer they can support with reasoning. This signal could be a thumbs up, or students could show the number of fingers that indicates the number of responses they have for the problem. This is a quick way to see if students have had enough time to think about the problem and keeps them from being distracted or rushed by classmates' raised hands.

Classroom Activities
After the warm-up, lessons consist of a sequence of one to three classroom activities. The activities are the heart of the mathematical experience and make up the majority of the time spent in class.

An activity can serve one or more of many purposes.

- Provide experience with a new context.
- Introduce a new concept and associated language.
- Introduce a new representation.
- Formalize a definition of a term for an idea previously encountered informally.
- Identify and resolve common mistakes and misconceptions that people make.
- Practice using mathematical language.
- Work toward mastery of a concept or procedure.
- Provide an opportunity to apply mathematics to a modeling or other application problem.

The purpose of each activity is described in its narrative. Read more about how activities serve these different purposes in the section on design principles.

Lesson Synthesis
After the activities for the day are done, students should take time to synthesize what they have learned. This portion of class should take 5–10 minutes before students start working on the cool-down. Each lesson includes a lesson synthesis that assists the teacher with ways to help students incorporate new insights gained during the activities into their big-picture understanding. Teachers can use this time in any number of ways, including posing questions verbally and calling on volunteers to respond, asking students to respond to prompts in a written journal, asking students to add on to a graphic organizer or concept map, or adding a new component to a persistent display like a word wall.

Cool-down
The cool-down task is to be given to students at the end of the lesson. Students are meant to work on the cool-down for about 5 minutes independently and turn it in. The cool-down serves as a brief formative assessment to determine whether students understood the lesson. Students' responses to the cool-down can be used to make adjustments to further instruction.

The response to student thinking provides guidance on how teachers might make adjustments based on specific student responses to a cool-down. Next day supports, such as providing students access to specific manipulatives or having students discuss their reasoning with a partner, are recommended for cool-down responses that should be addressed while continuing on to the next lesson. Teachers are directed to appropriate prior grade-level support for cool-down responses that may need more attention.

How to Use the Materials

Each Lesson and Unit Tells a Story

The story of each grade is told in eight or nine units. Each unit has a narrative that describes the mathematical work that will unfold in that unit. Each lesson in the unit also has a narrative. Lesson narratives explain:

- the mathematical content of the lesson and its place in the learning sequence
- the meaning of any new terms introduced in the lesson
- how the mathematical practices come into play, as appropriate

Activities within lessons also have narratives, which explain:

- the mathematical purpose of the activity and its place in the learning sequence
- what students are doing during the activity
- what the teacher needs to look for while students are working on an activity to orchestrate an effective synthesis
- connections to the mathematical practices, when appropriate

Launch - Work - Synthesize

Each classroom activity has three phases.

Launch
During the launch, the teacher makes sure that students understand the context (if there is one) and what the problem is asking them to do. This is not the same as making sure the students know how to do the problem—part of the work that students should be doing for themselves is figuring out how to solve the problem. The launch invites students into the lesson and helps them connect to contexts that may be unfamiliar.

Student Work Time
The launch for an activity frequently includes suggestions for grouping students. This gives students the opportunity to work individually, with a partner, or in small groups.

Activity Synthesis
During the activity synthesis, the teacher orchestrates some time for students to synthesize what they have learned. This time is used to ensure that all students have an opportunity to understand the mathematical punch line of the activity and situate the new learning within students' previous understanding.

Practice Problems

Each section in a unit includes an associated set of practice problems. There are 3 types of practice problems: pre-unit, lesson, and exploration. Teachers may decide to assign practice problems for homework or for extra practice in class. They may decide to collect and score it or to provide students with answers ahead of time for self-assessment. It is up to teachers to decide which problems to assign (including assigning none at all).

Pre-unit Problems
The practice problem set associated with the first section of each unit also includes several prior grade-level questions. These questions can be used to review prerequisite material from the previous grade or as a pre-unit assessment, if desired.

Lesson Practice Problems
The practice problem set associated with each section typically includes one question for each lesson in the section.

Exploration Problems
Each practice problem set also includes exploration questions that provide an opportunity for differentiation for students ready for more of a challenge. There are two types of exploration questions. One type is a hands-on activity that students can do directly related to the material of the unit, either in class if they have free time, or at home. The second type of exploration is more open-ended and challenging. These problems go deeper into grade-level mathematics. They are not routine or procedural, and they are not just “the same thing again but with harder numbers”.

Exploration questions are intended to be used on an opt-in basis by students if they finish a main class activity early or want to do more mathematics on their own. It is not expected that an entire class engages in exploration problems, and it is not expected that any student works on all of them. Exploration problems may also be good fodder for a Problem of the Week or similar structure.

Instructional Routines

Instructional Routines

Instructional Routines are designs for interaction that invite all students to engage in the mathematics of each lesson. They provide opportunities for students to bring their personal experiences as well as their mathematical knowledge to problems and discussions. They place value on students' voices as they communicate their developing ideas, ask questions, justify their responses, and critique the reasoning of others.

Instructional routines have a predictable structure and flow. They are enacted in classrooms to structure the relationship between the teacher and the students around content in ways that consistently maintain high expectations of student learning while adapting to the contingencies of particular instructional interactions (Kazemi, Franke, & Lampert, 2009). A finite set of routines support the pacing of lessons as they become familiar and save time in classroom choreography, so students can spend less time learning how to execute lesson directions, and more time on learning mathematics.

There are two types of Instructional Routines used in the materials: Warm-up Routines and Lesson Activity Routines. A list of the routines within each type is outlined in this table.

Warm-up Routines	Lesson Activity Routines
Act It Out	MLR1: Stronger and Clearer Each Time
Choral Count	MLR2: Collect and Display
Estimation Exploration	MLR3: Critique, Correct, and Clarify
How Many Do You See?	MLR4: Information Gap
Notice and Wonder	MLR5: Co-craft Questions
Number Talk	MLR6: Three Reads
Questions About Us	MLR7: Compare and Connect
True or False?	MLR8: Discussion Supports
What Do You Know About _____?	5 Practices
Which One Doesn't Belong?	Card Sort
	Counting Collections

Each lesson begins with a Warm-up Routine intentionally designed to elicit student discussions around the mathematical goal of the lesson. The Lesson Activity Routines embed structures within the tasks of the lessons that allow students to engage in the content, and collaborate in ways that support the development of student thinking and precision with language. Math Language Routines (MLRs) are Lesson Activity Routines that provide additional structures in order to support English learners. MLRs are written into each lesson, either as an embedded structure of a lesson activity in which all students engage, or as a suggested optional support specifically for English learners.

Below is a list of each routine with a brief description of its purpose.

Warm-up Routines

Act it Out

Act It Out is a K–1 routine that allows students to represent story problems (MP4). Students listen to a story problem and act it out, connecting language to mathematical representations. This routine provides an opportunity for students to connect with the storytelling tradition, typically found in ethnically diverse cultures.

Choral Count

While Choral Counting offers students the opportunity to practice verbal counting, the recorded count is the primary focus of the routine. As students reflect on the recorded count, they make observations, predict upcoming numbers in the count, and justify their reasoning (MP7 and MP3).

How Many Do You See?

How Many Do You See helps early math learners develop an understanding of counting and quantity through subitizing and combining parts of sets to find the total in a whole collection. In later grades, this routine encourages students to use operations and groupings that make finding the total number of dots faster. Through these recorded strategies, students look for relationships between the operations and their properties (MP7).

Notice and Wonder

Notice and Wonder invites all students into a mathematical task with two low-stakes prompts: “What do you notice? What do you wonder?” By thinking about things they notice and wonder, students gain entry into the context and might have their curiosity piqued. Students learn to make sense of problems (MP1) by taking steps to become familiar with a context and the mathematics that might be involved. Note: Notice and Wonder and I Notice/I Wonder are trademarks of NCTM and the Math Forum and are used in these materials with permission.

Questions About Us

Questions About Us is a kindergarten routine that allows students to consider number concepts in a familiar context. Students analyze data collected about the students, and answer questions such as: “What do you notice? What do you wonder?” Using data that represents students helps them to see math in the world around them.

What Do You Know About _____?

The What Do You Know About _____? routine elicits students’ ideas of numbers, place value, operations, and groupings through visuals of quantity, expressions, and other representations. It is an invitational prompt that could include such things as understanding where students see numbers embedded in various contexts or how students compare and order numbers.

Which One Doesn't Belong?

Which One Doesn’t Belong fosters a need for students to identify defining attributes and use language precisely in order to compare and contrast a carefully chosen group of geometric figures, images, or other mathematical representations (MP3 and MP6).

Other Instructional Routines

5 Practices

Lessons that include this routine are designed to allow students to solve problems in ways that make sense to them. During the activity, students engage in a problem in meaningful ways and teachers monitor to uncover and nurture conceptual understandings. During the activity synthesis, students collectively reveal multiple approaches to a problem and make connections between these approaches (MP3).

MLR7 Compare and Connect

Fosters students’ meta-awareness as they identify, compare, and contrast different mathematical approaches, representations, and language. *Embedded in grades K–5.*

Notice and Wonder

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Supports for Diverse Learners

Mathematical Language Development for All Students and Advancing English Learners in IM K–5 Math

Overview

Embedded within the curriculum are instructional routines and supports to help teachers address the specialized academic language demands when planning and delivering lessons, including the demands of reading, writing, speaking, listening, conversing, and representing in math (Aguirre & Bunch, 2012). While these instructional routines and supports can and should be used to support all students learning mathematics, they are particularly well-suited to meet the needs of linguistically and culturally diverse students who are learning mathematics while simultaneously acquiring English.

Design Principles For Promoting Mathematical Language Use And Development

The following design principles and related routines work to make language development an integral part of planning and delivering instruction while guiding teachers to amplify the most important language that students are expected to bring to bear on the central mathematics of each unit.

Principle 1: SUPPORT SENSE-MAKING

Scaffold tasks and amplify language so students can make their own meaning. Students do not need to understand a language completely before they can engage with academic content in that language. Language learners of all levels can and should engage with grade-level content that is appropriately scaffolded. Students need multiple opportunities to talk about their mathematical thinking, negotiate meaning with others, and collaboratively solve problems with targeted guidance from the teacher.

Teachers can make language more accessible for students by amplifying rather than simplifying speech or text. Simplifying includes avoiding the use of challenging words or phrases. Amplifying means anticipating where students might need support in understanding concepts or mathematical terms, and providing multiple ways to access them. Providing visuals or manipulatives, demonstrating problem-solving, engaging in think-alouds, and creating analogies, synonyms, or context are all ways to amplify language so that students are supported in taking an active role in their own sense-making of mathematical relationships, processes, concepts, and terms.

Principle 2: OPTIMIZE OUTPUT

Strengthen opportunities and supports for students to describe their mathematical thinking to others, orally, visually, and in writing. Linguistic output is the language that students use to communicate their ideas to others (oral, written, visual, etc.), and refers to all forms of student linguistic expressions except those that include significant back-and-forth negotiation of ideas. (That type of conversational language is addressed in the third principle.) All students benefit from repeated, strategically optimized, and supported opportunities to articulate mathematical ideas into linguistic expression.

Opportunities for students to produce output should be strategically optimized for both (a) important concepts of the unit or course, and (b) important disciplinary language functions (for example, explaining reasoning, critiquing the reasoning of others, making generalizations, and comparing approaches and representations). The focus for optimization must be determined, in part, by how students are currently using language to engage with important disciplinary concepts.

Principle 3: CULTIVATE CONVERSATION

Strengthen opportunities for constructive mathematical conversations (pairs, groups, and whole class). Conversations are back-and-forth interactions with multiple turns that build up ideas about math. Conversations act as scaffolds for students developing mathematical language because they provide opportunities to simultaneously make meaning, communicate that meaning, and refine the way content understandings are communicated.

When students have a purpose for talking and listening to each other, communication is more authentic. During effective discussions, students pose and answer questions, clarify what is being asked and what is happening in a problem, build common understandings, and share experiences relevant to the topic. Meaningful conversations depend on the teacher using activities and routines as opportunities to build a classroom culture that motivates and values efforts to communicate.

Principle 4: MAXIMIZE META-AWARENESS

Strengthen the meta-connections and distinctions between mathematical ideas, reasoning, and language. Language is a tool that not only allows students to communicate their math understanding to others, but also to organize their own experiences, ideas, and learning for themselves. Meta-awareness is consciously thinking about one's own thought processes or language use. Meta-awareness develops when students and teachers engage in classroom activities or discussions that bring explicit attention to what students need to do to improve communication and reasoning about mathematical concepts. When students are using language in ways that are purposeful and meaningful for themselves, in their efforts to understand—and be understood by—each other, they are motivated to attend to ways in which language can be both clarified and clarifying.

Meta-awareness in students can be strengthened when, for example, teachers ask students to explain to each other the strategies they brought to bear to solve a challenging problem. They might be asked, “How does yesterday’s method connect with the method we are learning today?” or, “What ideas are still confusing to you?” These questions are metacognitive because they help students to reflect on their own and others’ learning. Students can also reflect on their expanding use of language—for example, by comparing the language they used to clarify a mathematical concept with the language used by their peers in a similar situation. This is called metalinguistic awareness because students reflect on English as a language, their own growing use of that language, and the particular ways ideas are communicated in mathematics. Students learning English benefit from being aware of how language choices are related to the purpose of the task and the intended audience, especially if oral or written work is required. Both metacognitive and metalinguistic awareness are powerful tools to help students self-regulate their academic learning and language acquisition.

These four principles motivate the use of mathematical language routines, described in detail below. The eight routines included in this curriculum are:

- MLR 1: Stronger and Clearer Each Time
- MLR 2: Collect and Display
- MLR 3: Clarify, Critique, Correct
- MLR 4: Information Gap
- MLR 5: Co-Craft Questions
- MLR 6: Three Reads
- MLR 7: Compare and Connect
- MLR 8: Discussion Supports

Mathematical language routines (MLRs) are instructional routines that leverage a focus on language to foster deep conceptual understanding of mathematics. MLRs are included in each unit to provide all students with explicit opportunities to develop mathematical and academic language. English learners in particular will benefit from this approach.

Advancing English Learners in IM K–5 Math

Mathematical language routines are also included in each lesson's Support for English learners, to provide teachers with additional language strategies to meet the individual needs of their students. Teachers can use the suggested MLRs as appropriate to provide students with access to an activity without reducing the mathematical demand of the task. When selecting from these supports, teachers should take into account the language demands of the specific activity and the language needed to engage the content more broadly, in relation to their students' current ways of using language to communicate ideas as well as their students' English language proficiency. Using these supports can help maintain student engagement in mathematical discourse and ensure that struggle remains productive. All of the supports are designed to be used as needed, and use should be faded out as students develop understanding and fluency with the English language.

Mathematical Language Routines

A mathematical language routine is a structured but adaptable format for amplifying, assessing, and developing students' language. The mathematical language routines were selected because they are effective and practical for simultaneously learning mathematical practices, content, and language. These routines can be adapted and incorporated across lessons in each unit to fit the mathematical work wherever there are productive opportunities to support students in using and improving their English and disciplinary language use.

These routines facilitate attention to student language in ways that support in-the-moment teacher, peer, and self-assessment. The feedback enabled by these routines will help students revise and refine not only the way they organize and communicate their own ideas, but also ask questions to clarify their understandings of others' ideas.

Mathematical Language Routine 1: Stronger and Clearer Each Time

Adapted from Zwiers (2014)

Purpose

To provide a structured and interactive opportunity for students to revise and refine both their ideas and their verbal and written output (Zwiers, 2014). This routine also provides a purpose for student conversation through the use of a discussion-worthy and iteration-worthy prompt. The main idea is to have students think and write individually about a question, use a structured pairing strategy to have multiple opportunities to refine and clarify their response through conversation, and finally revise their original written response. Subsequent conversations and second drafts should naturally show evidence of incorporating or addressing new ideas and language. They should also show evidence of refinement in precision, communication, expression, examples, and reasoning about mathematical concepts.

How it Happens

Prompt
This routine begins by providing a thought-provoking question or prompt. The prompt should guide students to think about a concept or big idea connected to the content goal of the lesson, and should be answerable in a format that is connected with the activity's primary disciplinary language function.

Response - First Draft
Students draft an initial response to the prompt by writing or drawing their thoughts in a first draft. It is not

necessary that students finish this draft before moving to the structured pair meetings step. However, students should be encouraged to write or draw something before meeting with a partner. This encouragement can come over time as class culture is developed, strategies and supports for getting started are shared, and students become more comfortable sharing their ideas with others. (2–3 min)

Structured Pair Meetings
Next, use a structured pairing strategy to facilitate students having 2–3 meetings with different partners. Each meeting gives each partner an opportunity to be the speaker and an opportunity to be the listener. As the speaker, each student shares their ideas. As a listener, each student should (a) ask questions for clarity and reasoning, (b) press for details and examples, and (c) give feedback that is relevant (1–2 min each meeting).

Response - Second Draft
Finally, after meeting with 2–3 different partners, students write a second draft. This draft should naturally reflect borrowed ideas from partners, as well as refinement of initial ideas through repeated communication with partners. This second draft will be stronger (with more or better evidence of mathematical content understanding) and clearer (more precision, organization, and features of disciplinary language function). After students are finished, their first and second drafts can be compared. (2–3 min)

Mathematical Language Routine 2: Collect and Display

Purpose
To capture a variety of students' oral words and phrases into a stable, collective reference. The intent of this routine is to stabilize the varied and fleeting language in use during mathematical work, in order for students' own output to become a reference in developing mathematical language. The teacher listens for, and scribes, the language students use during partner, small group, or whole class discussions using written words, diagrams and pictures. This collected output can be organized, revoiced, or explicitly connected to other language in a display that all students can refer to, build on, or make connections with during future discussion or writing. Throughout the course of a unit (and beyond), teachers can reference the displayed language as a model, update and revise the display as student language changes, and make bridges between prior student language and new disciplinary language (Dieckman, 2017). This routine provides feedback for students in a way that supports sense-making while simultaneously increasing meta-awareness of language.

How it Happens

Collect
During this routine, circulate and listen to student talk during paired, group, or as a whole-class discussion. Jot down the words, phrases, drawings, or writing students use. Capture a variety of uses of language that can be connected to the lesson content goals, as well as the relevant disciplinary language function(s). Collection can happen on a word wall, anchor chart, digitally, or with a clipboard, or directly onto poster paper. Capturing on a whiteboard is not recommended due to risk of erasure.

Display
Display the language collected for the whole class to use as a reference during further discussions throughout the lesson and unit. Encourage students to suggest revisions, updates, and connections to be added to the display as they develop—over time—both new mathematical ideas and new ways of communicating ideas. The display provides an opportunity to showcase connections between student ideas and new vocabulary. It also provides an opportunity to highlight examples of students using disciplinary language functions, beyond just vocabulary words.

Mathematical Language Routine 3: Clarify, Critique, Correct

Purpose
To give students a piece of mathematical writing that is not their own to analyze, reflect on, and develop. The intent is to prompt student reflection with an incorrect, incomplete, or ambiguous written mathematical statement, and for students to improve upon the written work by correcting errors and clarifying meaning. Teachers can demonstrate how to effectively and respectfully critique the work of others with meta-think-alouds and press for details when necessary. This routine fortifies output and engages students in meta-awareness. More than just error analysis, this routine purposefully engages students in considering both the author's mathematical thinking as well as the features of their communication.

How it Happens

Original Statement
Students are presented with a written mathematical statement that intentionally includes conceptual (or common) mathematical reasoning errors as well as ambiguities in language. (1–2 min)

Discussion with Partner
Next, students discuss the original statement in pairs, and respond to the following questions: "What do you think the author means?", "Is anything unclear?", or "Are there any reasoning errors?" In addition to these general guiding questions, 1–2 questions can be added that specifically address the content or language goals of the activity. (2–3 min)

Improved Statement
Students individually revise the original statement, drawing on conversations with their partners, to create an "improved statement." In addition to resolving any mathematical errors or misconceptions and clarifying ambiguous language, other requirements can be added as parameters for the improved response. (3–5 min).

Mathematical Language Routine 4: Information Gap

Adapted from Zwiers 2004

Purpose
To create a need for students to communicate (Gibbons, 2002). This routine allows teachers to facilitate meaningful interactions by positioning some students as holders of information that is needed by other students. The information is needed to accomplish a goal, such as solving a problem or winning a game. With an information gap, students need to orally (or visually) share ideas and information in order to bridge a gap and accomplish something that they could not have done alone. Teachers should demonstrate how to ask for and share information, how to justify a request for information, and how to clarify and elaborate on information. This routine cultivates conversation.

How it Happens

Problem/Data Cards
Students are paired into Partner A and Partner B. Partner A is given a card with a problem that must be solved, and Partner B has the information needed to solve it on a "data card." Data cards can also contain diagrams, tables, graphs, etc. Neither partner should read nor show their cards to their partners. Partner A determines what information they need, and prepares to ask Partner B for that specific information. Partner B should not share information unless Partner A specifically asks for it and justifies the need for the information. Because partners don't have the same information, Partner A must work to produce clear and

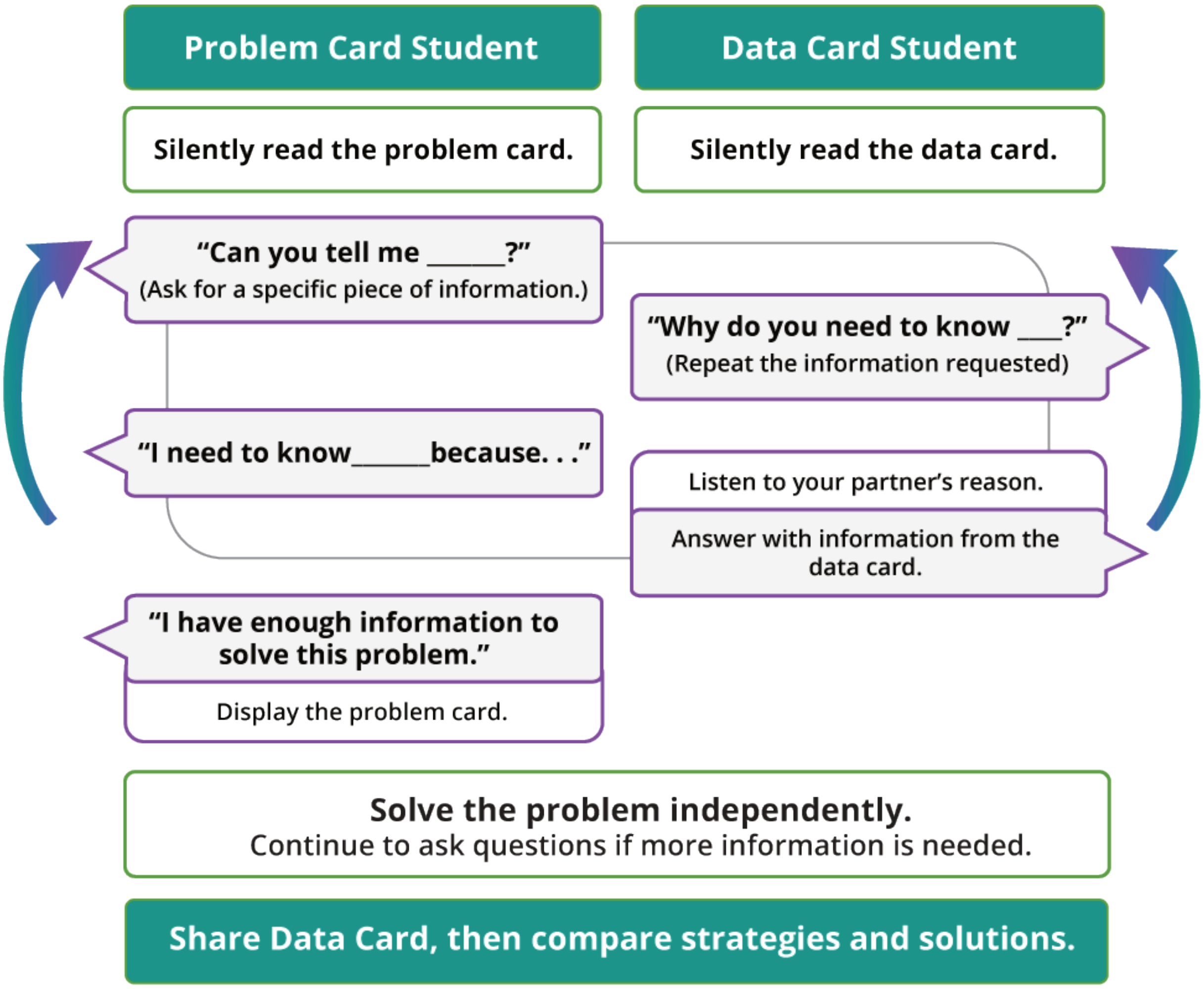
specific requests, and Partner B must work to understand more about the problem through Partner A's requests and justifications.

Bridging the Gap

- Partner B asks "What specific information do you need?" Partner A asks for specific information from Partner B.
- Before sharing the requested information, Partner B asks Partner A for a justification: "Why do you need that information?"
- Partner A explains how they plan to use the information.
- Partner B asks clarifying questions as needed, and then provides the information.
- These four steps are repeated until Partner A is satisfied that they have the information they need to solve the problem.

Solving the Problem

- Partner A shares the problem card with Partner B. Partner B does not share the data card.
- Both students solve the problem independently, then discuss their strategies. Partner B can share the data card after discussing their independent strategies.



Mathematical Language Routine 5: Co-craft Questions

Purpose

To allow students to get inside of a context before feeling pressure to produce answers, to create space for students to produce the language of mathematical questions themselves, and to provide opportunities for students to analyze how different mathematical forms and symbols can represent different situations. Through this routine, students are able to use conversation skills to generate, choose, and improve questions and situations as well as develop meta-awareness of the language used in mathematical questions and problems.

How it Happens

Hook

Begin by presenting students with a hook—a context or a stem for a problem, with or without values included. The hook can also be a picture, video, or list of interesting facts.

Students Generate Questions

Next, students generate possible mathematical questions that might be asked about the situation. It is preferable that students write down these questions, however, if students are still developing their writing skills they can state their questions orally or discuss them with a partner. These should be questions that students think are answerable by doing math and could be questions about the situation, information that appears to be missing, and even about assumptions that they think are important. (1–2 minutes)

Students Compare Questions
Students compare with a partner the questions they generated (1–2 minutes) or another pair of students if they initially shared with a peer before sharing questions with the whole class. As students share with the class, document their questions for all to see. Demonstrate (or ask students to demonstrate) identifying specific questions that are aligned to the content or language goals of the lesson. (2–3 minutes)

Actual Question(s) Revealed/Identified
Finally, the actual questions students are expected to work on are revealed or selected from the list that students generated.

Mathematical Language Routine 6: Three Reads

Purpose
To ensure that students know what they are being asked to do, create opportunities for students to reflect on the ways mathematical questions are presented, and equip students with tools used to actively make sense of mathematical situations and information (Kelemanik, Lucenta, & Creighton, 2016). This routine supports reading comprehension, sense-making, and meta-awareness of mathematical language.

How it Happens
In this routine, students are supported in reading and interpreting a mathematical text, situation, word problem, or graph three times, each with a particular focus. At times, the intended question or main prompt may be intentionally withheld until the third read so that students can concentrate on making sense of what is happening before rushing to a solution or method.

Read #1: "What is this situation about?"
The first read focuses on the situation, context, or main idea of the text. After a shared reading, ask students, "What is this situation about?" This is the time to identify and resolve any challenges with any non-mathematical vocabulary. (1 minute)

Read #2: "What can be counted or measured?"
After the second read, students list any quantities that can be counted or measured. Students are encouraged not to focus on specific values. Instead they focus on naming what is countable or measurable in the situation. It is not necessary to discuss the relevance of the quantities, just to be specific about them (examples: "number of people in a room" rather than "people," "number of markers after" instead of "markers"). Some of the quantities will be explicit (example: 32 apples) while others are implicit (example: the time it takes to brush one tooth). Record the quantities as a reference to use when solving the problem after the third read. (3–5 minutes)

Read #3: "What are different ways or strategies we can use to solve this problem?"
If initially withheld, the final question or prompt is revealed for the third read. Students discuss possible solution strategies, referencing the relevant quantities recorded after the second read. It may be helpful for students to create diagrams to represent the relationships among quantities identified in the second read, or to represent the situation with a picture (Asturias, 2014). (1–2 minutes)

Mathematical Language Routine 7: Compare and Connect

Purpose
To foster students' meta-awareness as they identify, compare, and contrast different mathematical

approaches and representations. This routine leverages the powerful mix of disciplinary representations available in mathematics as a resource for language development. In this routine, students make sense of mathematical strategies other than their own by relating and connecting other approaches to their own. Students should be prompted to reflect on, and linguistically respond to, these comparisons (for example, exploring why or when one might do or say something a certain way, or by identifying and explaining correspondences between different mathematical representations or methods).

How it Happens

Students Prepare Displays of their Work
Students are given a problem that can be approached and solved using multiple strategies, or a situation that can be modeled using multiple representations. Students prepare a visual display of their work, paying attention to the language and details they include that will allow others to make sense of their approach and reasoning. Variation is encouraged and supported among the representations that different students use to show what makes sense.

Compare
Students investigate each others' work by taking a tour of the visual displays or reviewing a range of different approaches that all can see. Students can examine each other's work in a self-guided format, or the teacher can act as "docent" by providing questions for students to ask of each other, pointing out important mathematical features, and facilitating comparisons. Comparisons should focus on the typical structures, purposes, and affordances of the different approaches or representations: what worked well in this or that approach, or what is especially clear in this or that representation. During this discussion, listen for and amplify any comments about what might make this or that approach or representation more complete or easy to understand.

Connect
The discussion then turns to identifying correspondences between different representations. Students are prompted to find correspondences in how specific mathematical relationships, operations, quantities, or values appear in each approach or representation. Guide students to refer to each other's thinking by asking them to make connections between specific features of expressions, tables, graphs, diagrams, words, and other representations of the same mathematical situation. During the discussion, amplify language students use to communicate about mathematical features that are important for solving the problem or modeling the situation. Call attention to the similarities and differences between the ways those features appear.

Mathematical Language Routine 8: Discussion Supports

Purpose
To support rich and inclusive discussions about mathematical ideas, representations, contexts, and strategies (Chapin, O'Connor, & Anderson, 2009). Rather than another structured format, the examples provided in this routine are instructional moves that can be combined and used together with any of the other routines. They include multimodal strategies for helping students make sense of complex language, ideas, and classroom communication. The examples can be used to invite and incentivize more student participation, conversation, and meta-awareness of language. Eventually, as teachers continue to demonstrate, students should begin using these strategies themselves to prompt each other to engage more deeply in discussions.

How it Happens

- Unlike the other routines, this support is a collection of strategies and moves that can be combined and used to support discussion during almost any activity.
- Examples of possible strategies:
- Revoice student ideas to demonstrate mathematical language use by restating a statement as a question in order to clarify, apply appropriate language, and involve more students.
- Press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or give an example.
- Show central concepts multi-modally by using different types of sensory inputs: acting out scenarios or inviting students to do so, showing videos or images, using gestures, and talking about the context of what is happening.
- Practice phrases or words through choral response.
- Think aloud by talking through thinking about a mathematical concept while solving a related problem or doing a task.
- Demonstrate uses of disciplinary language functions such as detailing steps, describing and justifying reasoning, and questioning strategies.
- Give students time to make sure that everyone in the group can explain or justify each step or part of the problem. Then make sure to vary who is called on to represent the work of the group so students get accustomed to preparing each other to fill that role.
- Prompt students to think about different possible audiences for the statement, and about the level of specificity or formality needed for a classmate vs. a mathematician, for example. [Convince Yourself, Convince a Friend, Convince a Skeptic (Mason, Burton, & Stacey, 2010)]

References

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Access for Students with Disabilities

These materials empower all students with activities that capitalize on their existing strengths and abilities to ensure that all learners can participate meaningfully in rigorous mathematical content. Lessons support a flexible approach to instruction and provide teachers with options for additional support to address the needs of a diverse group of students.

Curriculum Features that Support Access

Purposeful design elements that support all learners, but that are especially helpful for students with disabilities include:

Lesson Structures are Consistent
The structure of every lesson is the same: warm-up, activities, synthesis, cool-down (with centers in kindergarten and grade 1). By keeping the components of each lesson similar from day to day, the flow of work in class becomes predictable for students. This reduces cognitive demand and enables students to focus on the mathematics at hand rather than the mechanics of the lesson.

Concepts Develop from Concrete to Abstract
Mathematical concepts are introduced simply, concretely, and repeatedly, with complexity and abstraction developing over time. Students begin with concrete examples, and transition to drawings and diagrams before relying on symbols to represent the mathematics they encounter.

Individual to Pair, or Small Group to Whole Class Progression
Providing students with time to think through a situation or question independently before engaging with others allows students to carry the weight of learning, with supports arriving just in time from the community of learners. This progression allows students to first activate what they already know, and continue to build from this base with others.

Instructional strategies that support access

The following general instructional strategies can be used to help eliminate unnecessary barriers and make activities accessible to all students:

Processing Time
Increased time engaged in thinking and learning leads to mastery of grade level content for all students, including students with disabilities. Some students may need additional time, which should be provided as required.

Manipulatives
Physical manipulatives help students make connections between concrete ideas and abstract representations. Often, students with disabilities benefit from hands-on activities, which allow them to make sense of the problem at hand and communicate their own mathematical ideas and solutions.

Visual Aids
Visual aids such as images, diagrams, vocabulary charts, color coding, or physical demonstrations support conceptual processing and language development. Providing continued access to visual aids, either by keeping them posted or providing students with individual copies, can support independence and working or short-term memory.

Graphic Organizers
Word webs, Venn diagrams, tables, and other metacognitive visual supports provide structures that illustrate relationships between mathematical facts, concepts, words, or ideas. Graphic organizers can be used to support students with organizing thoughts and ideas, planning problem-solving approaches, visualizing ideas, sequencing information, and comparing and contrasting ideas.

Brain Breaks
Brain breaks are short, structured, 2–3 minute movement breaks taken between activities, or to break up a longer activity (approximately every 20–30 minutes during a class period). Brain breaks are a quick, effective way of refocusing and re-energizing the physical and mental state of students during a lesson. Brain breaks have also been shown to positively impact student concentration and stress levels, resulting in more time spent engaged in mathematical problem solving.

Timers
Timers are an excellent classroom tool that can help students develop independence and time management. They can also be used to ease transitions between activities. Songs that are familiar to all students, especially those that have become a part of a daily classroom routine, can also be used. Older students will benefit from knowing the allotted amount of time for a particular task or activity, and visual timers can be used for students who have not yet learned how to read a clock. At all grade levels, students will benefit from being able to see the timer.

Access for Students with Disabilities

In line with the Universal Design for Learning Guidelines (http://udlguidelines.cast.org), each lesson includes additional strategies to help teachers increase access and eliminate barriers. These supports provide teachers with additional ways students can access activities, engage in content, and communicate their learning. Designed with students with disabilities in mind, they are appropriate for any students who need additional support to access rigorous, grade-level content. Each support aligns to one of the three principles of UDL: engagement, representation, and action and expression.

Engagement
Students' attitudes, interests, and values help to determine the ways in which they are most engaged and motivated to learn. Supports that provide students with multiple means of engagement include suggestions that:

- leverage curiosity and students' existing interests
- leverage choice around perceived challenge
- encourage and support opportunities for peer collaboration
- provide structures that help students maintain sustained effort and persistence during a task
- provide tools and strategies designed to help students self-motivate and become more independent

Representation
Teachers can reduce barriers and leverage students' individual strengths by inviting students to engage with the same content in different ways. Supports provide students with multiple means of representation including suggestions that:

- offer alternatives for the ways information is presented or displayed
- help develop students' understanding and use of mathematical language and symbols

- illustrate connections between and across mathematical representations using color and annotations
- identify opportunities to activate or supply background knowledge
- describe organizational methods and approaches designed to help students internalize learning

Action and Expression
Throughout the curriculum, students are invited to share both their understanding and their reasoning about mathematical ideas with others. Supports that provide students with multiple means of action and expression include suggestions that:

- encourage flexibility and choice with the ways students demonstrate their understanding
- support discourse with sentence frames or accompany writing prompts
- use appropriate tools, templates, and assistive technologies
- support the development of organizational skills in problem-solving
- enable students to monitor their own progress with checklists

Understanding students' individual strengths and challenges is important when it comes to selecting instructional supports. To help teachers identify and select appropriate supports, each support is tagged with the areas of cognitive functioning it is designed to address. The following areas of cognitive functioning are integral to learning mathematics (Addressing Accessibility Project, Brodesky et al., 2002).

- *Conceptual Processing* includes perceptual reasoning, problem solving, and metacognition.
- *Language* includes auditory and visual language processing and expression.
- *Visual-Spatial Processing* includes processing visual information and understanding relation in space of visual mathematical representations and geometric concepts.
- *Organization* includes organizational skills, attention, and focus.
- *Memory* includes working memory and short-term memory.
- *Attention* includes paying attention to details, maintaining focus, and filtering out extraneous information.
- *Social-Emotional Functioning* includes interpersonal skills and the cognitive comfort and safety required in order to take risks and make mistakes.
- *Fine-motor Skills* include tasks that require small muscle movement and coordination such as manipulating objects (graphing, cutting with scissors, writing).

For additional information about the Universal Design for Learning framework, or to learn more about supporting students with disabilities, visit the Center for Applied Special Technology (CAST) at www.cast.org/udl.

References

- Brodesky, A., Parker, C., Murray, E., & Katzman, L. (2002). Accessibility strategies toolkit for mathematics. Education Development Center. Retrieved from http://www2.edc.org/accessmath/resources/strategiesToolkit.pdf
- CAST (2018). Universal Design for Learning Guidelines version 2.2. Retrieved from http://udlguidelines.cast.org

Assessment Guidance

Learning Goals

Teacher-facing learning goals appear at the top of lesson plans. They describe, for a teacher audience, the mathematical and pedagogical goals of the lesson.

Student-facing learning goals appear in student materials at the beginning of each lesson and start with the word "Let's." They are intended to invite students into the work of that day without giving away too much and spoiling the problem-based instruction. They are suitable for writing on the board before class begins.

How to Assess Progress

The materials contain many opportunities and tools for both formative and summative assessment. Some things are purely formative, but the tools that can be used for summative assessment can also be used formatively.

Formative Assessment Opportunities

- The practice problems for Section A in each unit, for grades 1 to 5, have several items designated as pre-assessment. These target concepts and skills that are prerequisite to the unit. While many of them are based on earlier grade level material, later units often include problems addressing important work of the grade.
- Each instructional task is accompanied by commentary about expected student responses and opportunities to advance student thinking so that teachers can adjust their instruction depending on what students are doing in response to the task. Often there are suggested questions to help teachers better understand students' thinking.
- Each lesson, in grades 2-5, includes a cool-down (analogous to an exit slip or exit ticket) to assess whether students understood the work of that day's lesson. In Grade 1, cool-downs are included in the lessons with greater frequency throughout the year.
- One or more practice problems is provided for each lesson (starting in Kindergarten, Unit 4). These can be used for in-class practice, homework, or as a means to assess certain learning on a particular concept. Each section contains two or more explorations, designed to engage students in thinking creatively about the mathematics of the unit at school or at home.
- Each section in grades 2-5 has a 3–4 problem Checkpoint to assess the section learning goals. These can be used for extra practice or can be used to check student understanding before the end of the unit. Each section in kindergarten and Grade 1 has a checklist of indicators that students are meeting the section goals.

Summative Assessment Opportunity

- Each unit (starting in Kindergarten, Unit 2) includes an end-of-unit written assessment that is intended for students to complete individually to assess what they have learned at the conclusion of the unit. In K-2, the assessment may be read aloud to students, as needed.

Pre-Unit Practice Problems

These problems address prerequisite concepts and skills for the unit. Teachers can use these problems to identify unfinished learning that can be carefully addressed during the unit.

What if a large number of students can't do the same pre-unit problem? Teachers are encouraged to address prerequisite skills while continuing to work through on-grade tasks and concepts of each unit, instead of abandoning the current work in favor of material that only addresses prerequisite skills. Look for opportunities within the upcoming unit where the target skill or concept could be addressed in context or with a center activity. For example, an upcoming activity might require adding or subtracting within 100 to compare length measurements. Some strategies might include:

- ask a student who can add and subtract reliably to present their method
- add additional questions to the warm-up with the purpose of revisiting the skill
- add to the activity launch a few related addition or subtraction problems to solve, before students need to do this in the context of measurement
- pause the class while working on the activity to focus on the portion that requires addition or subtraction

Then, attend carefully to students as they work through the activity. If difficulty persists, add more opportunities to practice arithmetic, by adapting tasks or practice problems, including practice problems from a previous unit.

Cool-Downs

Each lesson includes a cool-down (also known as an exit slip or exit ticket) to be given to students at the end of the lesson. This activity serves as a brief check in to determine whether students understood the main concepts of that lesson. Teachers can use this as a formative assessment to plan further instruction.

When appropriate, guidance for unfinished learning, evidenced by the cool-down, is provided in two categories: next-day support and prior-unit support. This guidance is meant to provide teachers ways in which to continue grade level content while also giving students the additional support they may need.

End-of-Unit Assessments

At the end of each unit is the end-of-unit assessment. These assessments are intended to gauge students' understanding of the key concepts of the unit while also preparing students for new-generation standardized exams. Problem types include multiple-choice, multiple response, short answer, restricted constructed response, and extended response. Problems vary in difficulty and depth of knowledge.

Teachers may choose to grade these assessments in a standardized fashion, but may also choose to grade more formatively by asking students to show and explain their work on all problems. Teachers may also decide to make changes to the provided assessments to better suit their needs. If making changes, teachers are encouraged to keep the format of problem types provided, and to include problems of different types and different levels of difficulty.

All summative assessment problems include a complete solution and standard alignment. Multiple-choice and multiple response problems often include a reason for each potential error a student might make.

Unlike formative assessments, problems on summative assessments generally do not prescribe a method of solution.

Design Principles for Summative Assessments

Students should get the correct answer on assessment problems for the right reasons, and get incorrect answers for the right reasons. To help with this, our assessment problems are targeted and short, use consistent, positive wording, and have clear, undebatable correct responses.

In multiple choice problems, distractors are common errors and misconceptions directly relating to what is being assessed, since problems are intended to test whether the student has proficiency with a specific skill. The distractors serve as a diagnostic, giving teachers the chance to quickly see which of the most common errors are being made. There are no “trick” questions and in earlier grades students are told how many answers to select on multiple select problems.

When a multiple response prompt does not give the number of correct responses, it always includes the phrase “select all” to clearly indicate their type. Each part of a multiple response problem addresses a different piece of the same overall skill, again serving as a diagnostic for teachers to understand which common errors students are making.

Short answer, restricted constructed response, and extended response problems are careful to avoid the “double whammy” effect, where a part of the problem asks for students to use correct work from a previous part. This choice is made to ensure that students have all possible opportunities to show proficiency on assessments.

When possible, extended response problems provide multiple ways for students to demonstrate understanding of the content being assessed, through some combination of words, diagrams, and equations.

Lessons Listed by Standards Alignment

Standards	Aligned Lessons
K.CC	K.1.1, K.1.2, K.1.3, K.1.4, K.1.5, K.1.6, K.1.7, K.1.8, K.1.9, K.1.10, K.1.11, K.1.12, K.2.1, K.2.2, K.2.3, K.2.4, K.2.5, K.2.6, K.2.12, K.2.13, K.2.16, K.2.18, K.3.10, K.3.12, K.3.13, K.4.5
K.CC.A.1	K.1.13, K.1.14, K.1.15, K.1.16, K.1.17, K.2.1, K.2.2, K.2.3, K.2.14, K.2.15, K.3.4, K.3.7, K.4.3, K.4.12, K.4.18
K.CC.A.2	K.4.14, K.4.18
K.CC.A.3	K.2.3, K.2.4, K.2.6, K.2.7, K.2.8, K.2.10, K.2.11, K.2.12, K.2.13, K.2.15, K.2.16, K.2.17, K.2.18, K.2.19, K.2.20, K.2.21, K.2.22, K.3.5, K.3.6, K.3.7, K.3.8, K.3.9, K.3.11, K.3.12, K.3.13, K.3.14, K.4.1, K.4.3, K.4.4, K.4.5, K.4.6, K.4.7, K.4.8, K.4.9, K.4.10, K.4.11, K.4.12, K.4.13, K.4.14, K.4.15, K.4.16, K.4.17, K.4.18
K.CC.B	K.1.14, K.1.15, K.1.16, K.1.17, K.2.7, K.2.8, K.2.9, K.2.10, K.2.11, K.2.12, K.3.1, K.3.2, K.3.3, K.3.4, K.3.5, K.3.6, K.3.7, K.3.8, K.3.9, K.4.9, K.4.10, K.4.11, K.4.12, K.4.13
K.CC.B.4	K.1.8, K.1.9, K.1.10, K.1.11, K.1.12, K.1.13, K.1.14, K.1.15, K.1.16, K.1.17, K.2.2, K.2.3, K.2.7, K.2.12, K.2.13, K.2.17, K.2.18, K.2.19, K.3.1, K.3.2, K.3.3, K.3.4, K.3.5, K.3.6, K.3.7, K.3.8, K.3.9
K.CC.B.4.a	K.1.13, K.1.14, K.1.15, K.1.16, K.1.17
K.CC.B.4.b	K.2.2, K.2.3, K.2.7
K.CC.B.4.c	K.2.17, K.2.18, K.3.11, K.4.17
K.CC.B.5	K.1.18, K.2.2, K.2.3, K.2.7, K.2.8, K.2.9, K.2.10, K.2.11, K.2.12, K.2.13, K.2.14, K.2.15, K.2.16, K.2.17, K.2.18, K.2.19, K.2.20, K.2.21, K.2.22, K.3.1, K.3.2, K.3.3, K.3.4, K.3.5, K.3.6, K.3.7, K.3.8, K.3.9, K.3.10, K.3.11, K.3.12, K.3.13, K.3.14, K.3.15, K.4.1, K.4.2, K.4.3, K.4.4, K.4.5, K.4.6, K.4.7, K.4.8, K.4.9, K.4.10, K.4.11, K.4.12, K.4.13, K.4.19
K.CC.C	K.3.10, K.3.11, K.3.12, K.3.13, K.3.14
K.CC.C.6	K.1.18, K.2.3, K.2.4, K.2.5, K.2.6, K.2.8, K.2.9, K.2.10, K.2.11, K.2.13, K.2.14, K.2.15, K.2.16, K.2.17, K.2.18, K.2.19, K.2.20, K.2.21, K.2.22, K.3.1, K.3.2, K.3.3, K.3.4, K.3.5, K.3.6, K.3.7, K.3.8, K.3.9, K.3.12, K.3.15, K.4.19
K.CC.C.7	K.2.19, K.2.20, K.2.21, K.2.22

Standards	Aligned Lessons
K.G	K.1.4, K.1.5, K.1.6, K.1.7, K.1.8, K.1.9, K.1.10, K.1.11, K.1.12, K.1.13, K.1.14, K.1.15, K.1.16, K.1.17, K.3.1, K.3.2, K.3.3, K.3.4, K.3.5, K.3.6, K.3.7, K.3.8, K.3.9, K.3.10, K.3.11, K.3.12, K.3.13, K.3.14
K.G.A.1	K.3.2, K.3.4, K.3.9, K.3.13, K.3.14, K.3.15, K.4.1, K.4.3, K.4.4, K.4.5, K.4.6, K.4.7, K.4.8, K.4.9, K.4.10, K.4.11, K.4.12, K.4.13
K.G.A.2	K.3.2, K.3.5, K.3.6, K.3.7, K.3.8, K.3.9, K.3.11, K.3.14, K.3.15
K.G.B	K.1.5, K.2.1
K.G.B.4	K.3.1, K.3.2, K.3.3, K.3.4, K.3.5, K.3.6, K.3.7, K.3.8, K.3.9, K.3.10, K.3.11, K.3.12, K.3.13, K.3.14
K.G.B.5	K.3.7, K.3.8, K.3.9, K.3.10, K.3.11, K.3.12, K.3.13, K.3.14, K.4.6, K.4.7, K.4.8, K.4.9, K.4.10, K.4.11, K.4.12, K.4.13
K.G.B.6	K.3.10, K.3.11, K.3.12, K.3.13, K.3.14, K.3.15, K.4.1, K.4.3, K.4.4, K.4.5
K.MD.A.2	K.3.6, K.3.7
K.MD.B.3	K.3.1, K.3.2, K.3.3, K.3.4, K.3.5, K.3.6, K.3.7, K.3.8, K.3.9
K.OA.A.1	K.4.1, K.4.2, K.4.4, K.4.5, K.4.6, K.4.7, K.4.8, K.4.9, K.4.10, K.4.11, K.4.12, K.4.13, K.4.14, K.4.15, K.4.16, K.4.17, K.4.18, K.4.19
K.OA.A.2	K.4.6, K.4.7, K.4.8, K.4.9, K.4.10, K.4.11, K.4.12, K.4.13, K.4.14, K.4.15, K.4.16, K.4.17, K.4.18, K.4.19

Standards Addressed Listed by Lesson

Lesson	Standards Addressed
Unit 1, Lesson 1	K.CC, K.CC.B, K.G.B, K.MD, K.MD.B.3
Unit 1, Lesson 2	K.CC, K.CC.B, K.G, K.MD.B.3
Unit 1, Lesson 3	K.CC, K.CC.B, K.MD.B.3
Unit 1, Lesson 4	K.CC, K.CC.B, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 5	K.CC, K.CC.B, K.G, K.G.B, K.MD.B.3
Unit 1, Lesson 6	K.CC, K.CC.B, K.CC.B.4, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 7	K.CC, K.CC.B, K.CC.B.4, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 8	K.CC, K.CC.B, K.CC.B.4, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 9	K.CC, K.CC.B, K.CC.B.4, K.CC.C.6, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 10	K.CC, K.CC.B, K.CC.B.4, K.CC.C.6, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 11	K.CC, K.CC.B, K.CC.B.4, K.CC.C.6, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 12	K.CC, K.CC.B, K.CC.B.4, K.CC.C.6, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 13	K.CC.A.1, K.CC.B, K.CC.B.4, K.CC.B.4.a, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 14	K.CC.A.1, K.CC.B, K.CC.B.4, K.CC.B.4.a, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 15	K.CC.A.1, K.CC.B, K.CC.B.4, K.CC.B.4.a, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 16	K.CC.A.1, K.CC.B, K.CC.B.4, K.CC.B.4.a, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 17	K.CC.A.1, K.CC.B, K.CC.B.4, K.CC.B.4.a, K.G, K.MD, K.MD.B.3
Unit 1, Lesson 18	K.CC.B.5, K.CC.C.6
Unit 2, Lesson 1	K.CC, K.CC.A.1, K.CC.B, K.G.B
Unit 2, Lesson 2	K.CC, K.CC.A.1, K.CC.B.4, K.CC.B.4.b, K.CC.B.5
Unit 2, Lesson 3	K.CC, K.CC.A.1, K.CC.A.3, K.CC.B.4, K.CC.B.4.b, K.CC.B.5, K.CC.C.6

Lesson	Standards Addressed
Unit 2, Lesson 4	K.CC, K.CC.A.3, K.CC.C.6
Unit 2, Lesson 5	K.CC, K.CC.C.6
Unit 2, Lesson 6	K.CC, K.CC.A.3, K.CC.C.6
Unit 2, Lesson 7	K.CC.A.3, K.CC.B, K.CC.B.4, K.CC.B.4.b, K.CC.B.5
Unit 2, Lesson 8	K.CC.A.3, K.CC.B, K.CC.B.5, K.CC.C.6
Unit 2, Lesson 9	K.CC.B, K.CC.B.5, K.CC.C.6
Unit 2, Lesson 10	K.CC.A.3, K.CC.B, K.CC.B.5, K.CC.C.6
Unit 2, Lesson 11	K.CC.A.3, K.CC.B, K.CC.B.5, K.CC.C.6
Unit 2, Lesson 12	K.CC, K.CC.A.3, K.CC.B, K.CC.B.4, K.CC.B.5
Unit 2, Lesson 13	K.CC, K.CC.A.3, K.CC.B.4, K.CC.B.5, K.CC.C.6
Unit 2, Lesson 14	K.CC.A.1, K.CC.B.5, K.CC.C.6
Unit 2, Lesson 15	K.CC.A.1, K.CC.A.3, K.CC.B.5, K.CC.C.6
Unit 2, Lesson 16	K.CC, K.CC.A.3, K.CC.B.5, K.CC.C.6
Unit 2, Lesson 17	K.CC.A.3, K.CC.B.4, K.CC.B.4.c, K.CC.B.5, K.CC.C.6, K.CC.C.7
Unit 2, Lesson 18	K.CC, K.CC.A.3, K.CC.B.4, K.CC.B.4.c, K.CC.B.5, K.CC.C.6, K.CC.C.7
Unit 2, Lesson 19	K.CC.A.3, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.CC.C.7
Unit 2, Lesson 20	K.CC.A.3, K.CC.B.5, K.CC.C.6, K.CC.C.7
Unit 2, Lesson 21	K.CC.A.3, K.CC.B.5, K.CC.C.6, K.CC.C.7
Unit 2, Lesson 22	K.CC.A.3, K.CC.B.5, K.CC.C.6, K.CC.C.7
Unit 3, Lesson 1	K.CC.B, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.MD.B.3
Unit 3, Lesson 2	K.CC.B, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.MD.B.3
Unit 3, Lesson 3	K.CC.B, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.MD.B.3
Unit 3, Lesson 4	K.CC.A.1, K.CC.B, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.MD.B.3

Lesson	Standards Addressed
Unit 3, Lesson 5	K.CC.A.3, K.CC.B, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.MD.B.3
Unit 3, Lesson 6	K.CC.A.3, K.CC.B, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.MD.A.2, K.MD.B.3
Unit 3, Lesson 7	K.CC.A.1, K.CC.A.3, K.CC.B, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.G.B.5, K.MD.A.2, K.MD.B.3
Unit 3, Lesson 8	K.CC.A.3, K.CC.B, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.G.B.5, K.MD.B.3
Unit 3, Lesson 9	K.CC.A.3, K.CC.B, K.CC.B.4, K.CC.B.5, K.CC.C.6, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.G.B.5, K.MD.B.3
Unit 3, Lesson 10	K.CC, K.CC.B.5, K.CC.C, K.G, K.G.B.4, K.G.B.5, K.G.B.6
Unit 3, Lesson 11	K.CC.A.3, K.CC.B.4.c, K.CC.B.5, K.CC.C, K.G, K.G.A.2, K.G.B.4, K.G.B.5, K.G.B.6
Unit 3, Lesson 12	K.CC, K.CC.A.3, K.CC.B.5, K.CC.C, K.CC.C.6, K.G, K.G.B.4, K.G.B.5, K.G.B.6
Unit 3, Lesson 13	K.CC, K.CC.A.3, K.CC.B.5, K.CC.C, K.G, K.G.A.1, K.G.B.4, K.G.B.5, K.G.B.6
Unit 3, Lesson 14	K.CC.A.3, K.CC.B.5, K.CC.C, K.G, K.G.A.1, K.G.A.2, K.G.B.4, K.G.B.5, K.G.B.6
Unit 3, Lesson 15	K.CC.B.5, K.CC.C.6, K.G.A.1, K.G.A.2, K.G.B.6
Unit 4, Lesson 1	K.CC.A.3, K.CC.B.5, K.G.A.1, K.G.B.6, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 2	K.CC.B.5, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 3	K.CC.A.1, K.CC.A.3, K.CC.B.5, K.G.A.1, K.G.B.6, K.OA.A.2
Unit 4, Lesson 4	K.CC.A.3, K.CC.B.5, K.G.A.1, K.G.B.6, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 5	K.CC, K.CC.A.3, K.CC.B.5, K.G.A.1, K.G.B.6, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 6	K.CC.A.3, K.CC.B.5, K.G.A.1, K.G.B.5, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 7	K.CC.A.3, K.CC.B.5, K.G.A.1, K.G.B.5, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 8	K.CC.A.3, K.CC.B.5, K.G.A.1, K.G.B.5, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 9	K.CC.A.3, K.CC.B, K.CC.B.5, K.G.A.1, K.G.B.5, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 10	K.CC.A.3, K.CC.B, K.CC.B.5, K.G.A.1, K.G.B.5, K.OA.A.1, K.OA.A.2

Lesson	Standards Addressed
Unit 4, Lesson 11	K.CC.A.3, K.CC.B, K.CC.B.5, K.G.A.1, K.G.B.5, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 12	K.CC.A.1, K.CC.A.3, K.CC.B, K.CC.B.5, K.G.A.1, K.G.B.5, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 13	K.CC.A.3, K.CC.B, K.CC.B.5, K.G.A.1, K.G.B.5, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 14	K.CC.A.2, K.CC.A.3, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 15	K.CC.A.3, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 16	K.CC.A.3, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 17	K.CC.A.3, K.CC.B.4.c, K.OA.A.1, K.OA.A.2, K.OA.A.5
Unit 4, Lesson 18	K.CC.A.1, K.CC.A.2, K.CC.A.3, K.OA.A.1, K.OA.A.2
Unit 4, Lesson 19	K.CC.B.5, K.CC.C.6, K.OA.A.1, K.OA.A.2

Contributors

Writing Team

Alex Clayton, Kindergarten Lead
Brianne Durst, Grades K-2
Brigitte Lahme
Chepina Rumsey
Chloe Moryl
Claudia Martinez Garcia
Cynthia Santos
Deborah Peart, Grade 2 Lead
Dionne Aminata, Grade 2-5 Lead
Elizabeth Everett
Federico Chialvo
Kaneka Turner, Grade 4 Lead
Kathy Morris
Kristin Gray, K-5 Lead
Kristin Umland, President
Latrenda Knighten
Leigh Rupar
Marni Greenstein, Grade K-3 Lead
Mike Henderson
Mike Nakamaye, K-5 Mathematician in Residence
Ni'Kia Graham
Noelle Conforti Preszler, Grades 3-5
Raven Redmond
Sahar Khatri
Sara Baranakus, Grade 1 Lead
Sarah Caban, Grade 5 Lead
Sarah Ryan
William McCallum, CEO
Yenche Tioanda, Grade 4
Zack Hill, Grade 3 Lead

Supports for Students with Disabilities and English Learners

Ann Kim
Erin Smith, EL Lead
Janelle Franco
Jeannette Rivera
Kelly Carnevale
Lily Jones
Liz Ramirez, Supports Lead
Myuriel von Aspen
Patti Drawdy
Penny King
Rachael Burke
Raven Redmond
Rosie Leyva
Sadie Estrella
Samantha Ellerbeck
Tammy Foster

Copy Editing

Emily Flanagan
Robert Puchalik, Lead
Toni Brokaw

Project Management

Olivia Russell, VP
Program Management
Todd Tate, Lead

Engineering

Brendan Shean
Dylan Dechant
Eric Connally
Karl Nelson, CIO
Nik Doran, Lead

Teacher Professional Learning

Kevin Liner, Lead
Lauren Giordano
Moniquea Willingham

Image Development

Justin Wisby
Rodney Cooke
Siavash Tehrani

Materials Production

Andrew Giang
Becca Phillips
Brigitte Lahme
Celeana Garrett
Christina Jackyra, Co-lead
Jenessa Peterson
Kathryn Phillips
Kristen Taylor
Mike Nakamaye
Patti Drawdy, Lead
Shelley Mourtgos
Susan Lovell, InDesign

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