Transformations

Exploring Rigid Motion in a Plane

What You Should Learn
Why You Should Learn It

- Goal 1: How to identify the three basic rigid transformations in a plane
- Goal 2: How to use transformations to identify patterns and their properties in real life
- You can use transformations to create visual patterns, such as stencil patterns for the border of a wall

Identifying Transformations (flips, slides, turns)

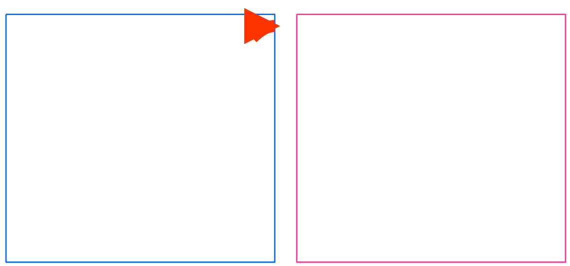
- Figures in a plane can be **reflected**, **rotated**, or **slid** to produce new figures.
- The new figure is the **image**, and the original figure is the **preimage**
- The operation that maps (or moves) the preimage onto the image is called a **transformation**

Blue: preimage

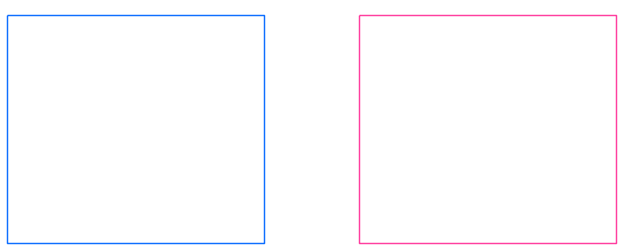
Pink: image

3 Basic Transformations

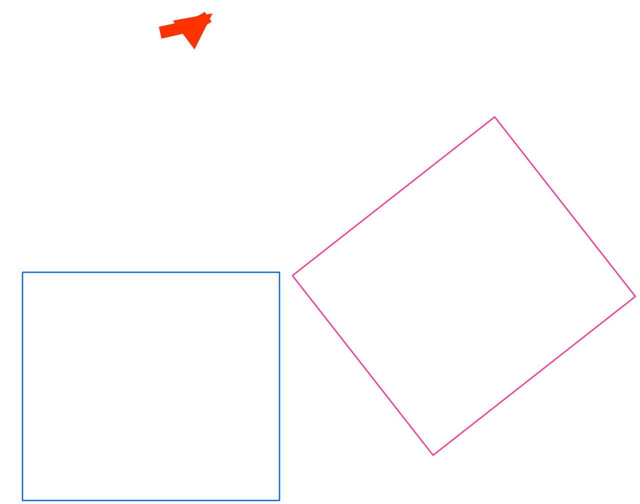
- Reflection (flip)



- Translation (slide)

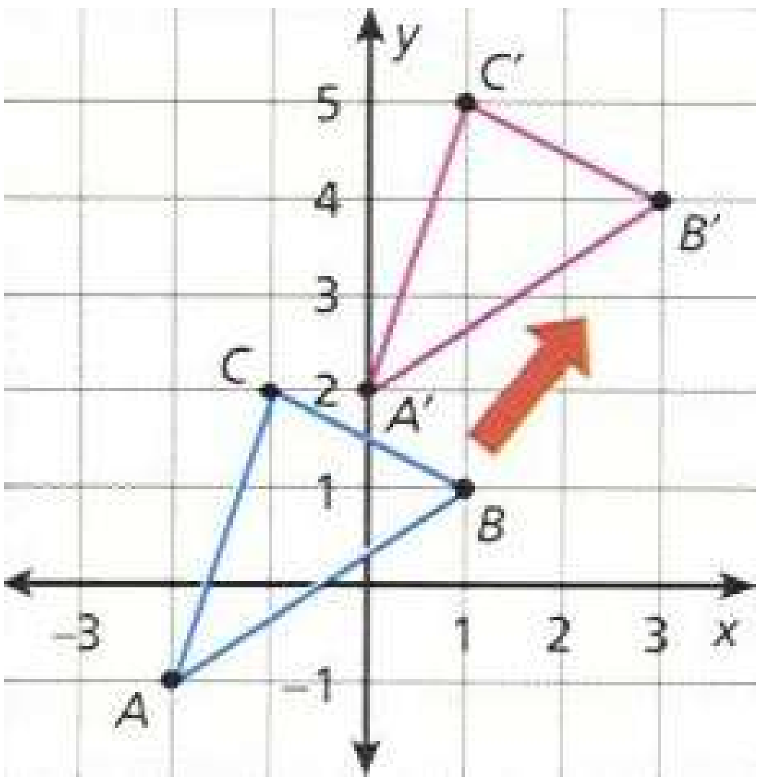


Rotation (turn)



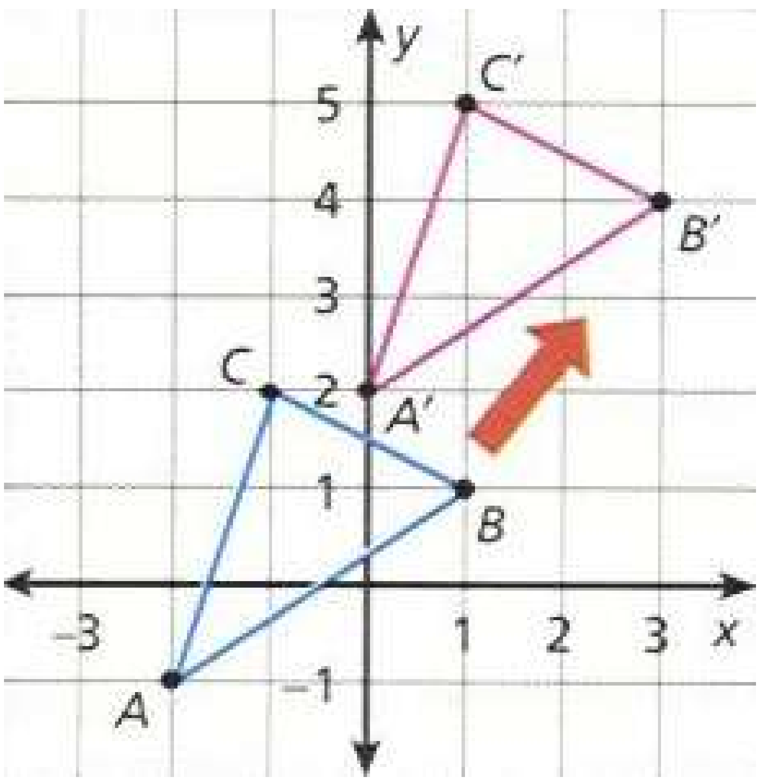
http://standards.nctm.org/document/eexamples/chap6/6.4/index.htm

Example 1
Identifying Transformations



- Identify the transformation shown at the left.

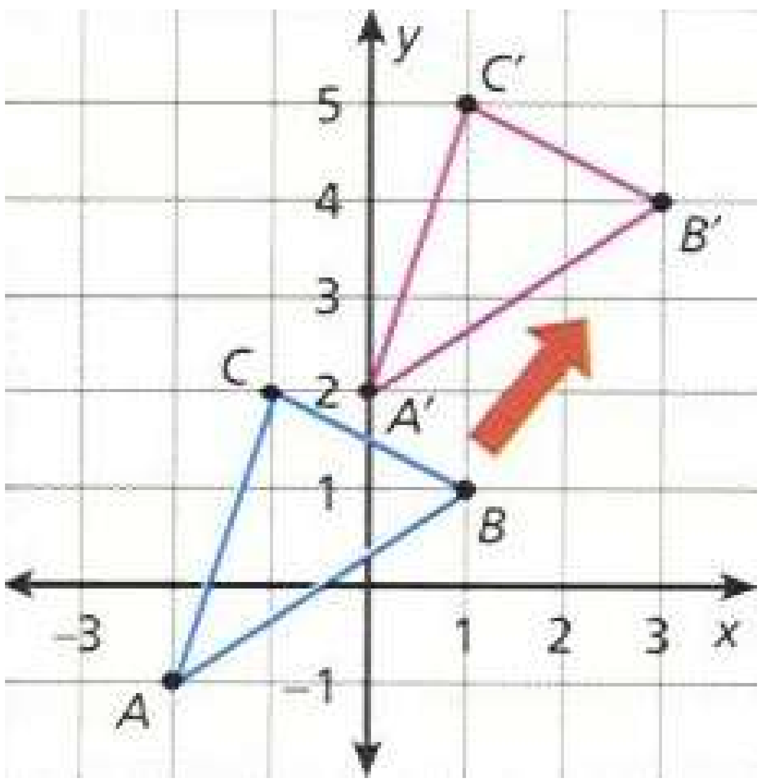
Example 1
Identifying Transformations



▪Translation

●To obtain ΔA'B'C', each point of ΔABC was slid 2 units to the right and 3 units up.

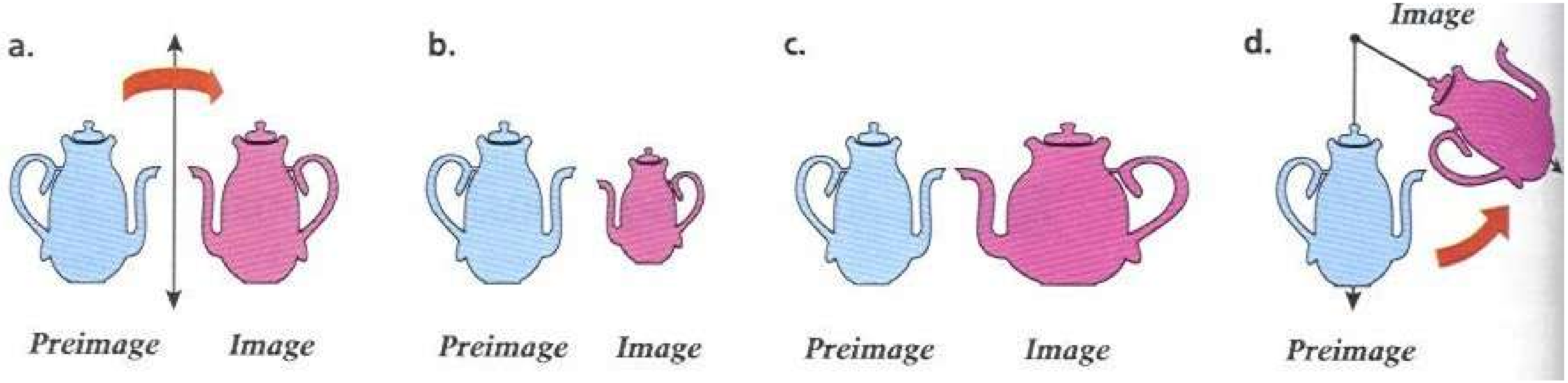
Rigid Transformations



- A transformation is rigid if every image is congruent to its preimage
- This is an example of a rigid transformation b/c the pink and blue triangles are congruent

Example 2
Identifying Rigid Transformations

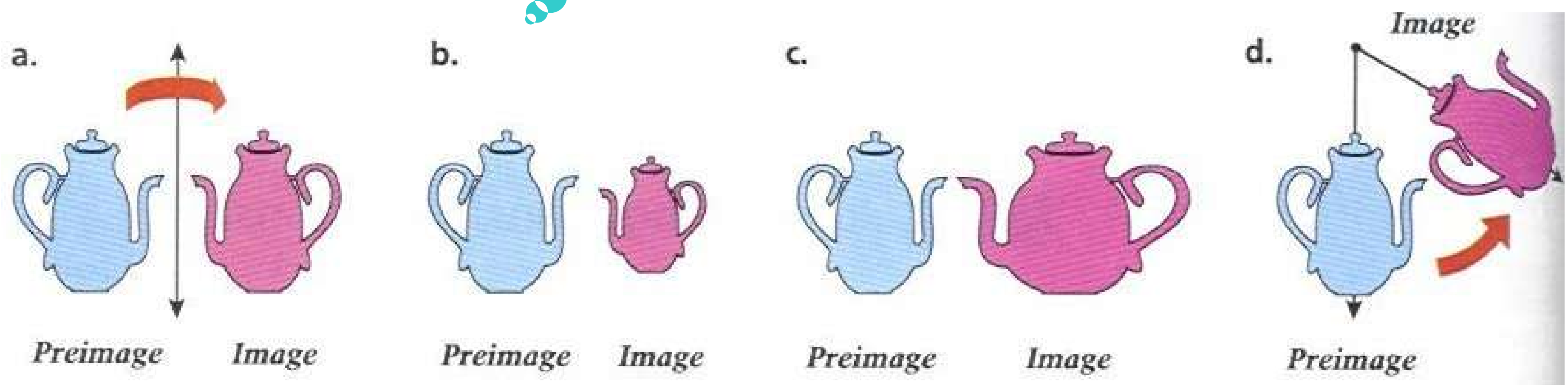
- Which of the following transformations appear to be rigid?



Example 2
Identifying Rigid Transformations

- Which of the following transformations appear to be rigid?

The image is not congruent to the preimage, it is smaller

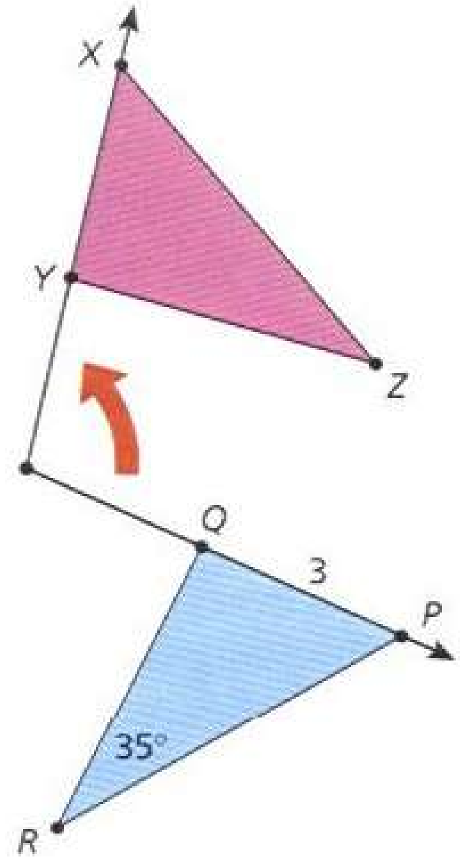


The image is not congruent to the preimage, it is fatter

Definition of Isometry

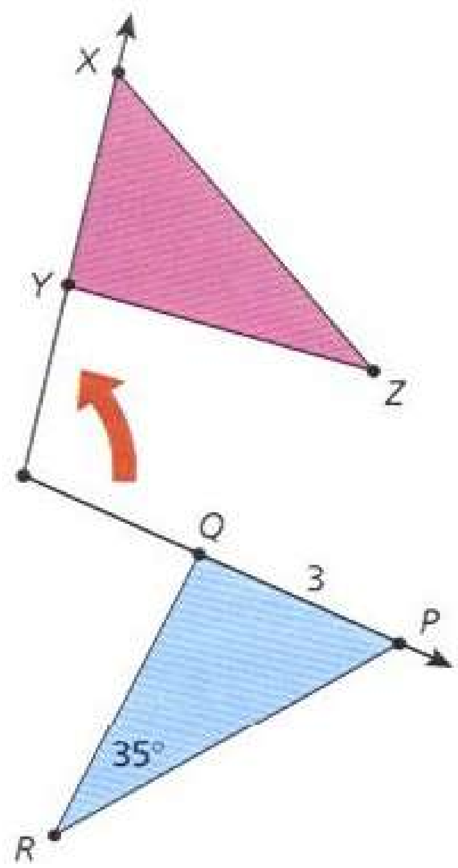
- A rigid transformation is called an **isometry**
- A transformation in the plane is an **isometry** if it preserves lengths. (That is, every segment is congruent to its image)
  - It can be proved that isometries not only preserve lengths, they also preserves angle measures, parallel lines, and betweenness of points

Example 3
Preserving Distance and Angle Measure



- In the figure at the left, ΔPQR is mapped onto ΔXYZ. The mapping is a rotation. Find the length of XY and the measure of ∠Z

Example 3
Preserving Distance and Angle Measure

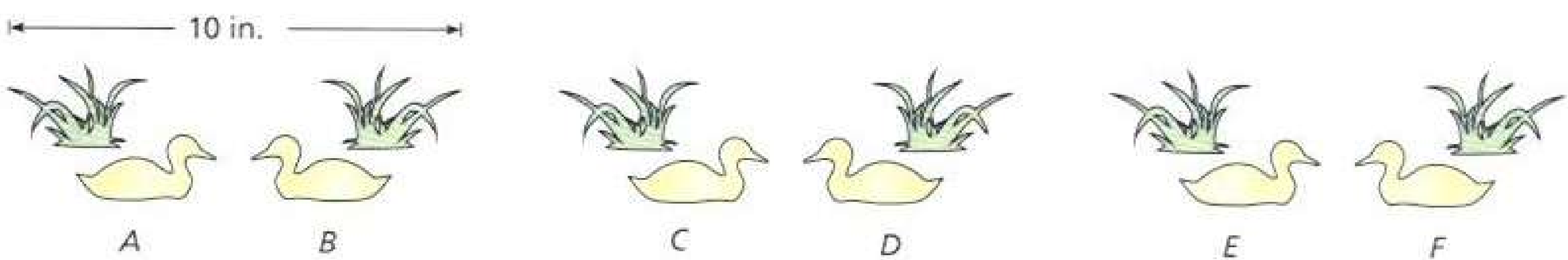


- In the figure at the left, ΔPQR is mapped onto ΔXYZ. The mapping is a rotation. Find the length of XY and the measure of ∠Z
- B/C a rotation is an isometry, the two triangles are congruent, so XY=PQ=3 and m∠Z= m∠R =35°

Note that the statement "ΔPQR is mapped onto ΔXYZ" implies the correspondence P→X, Q→Y, and R→Z

Example 4
Using Transformations in Real-Life
Stenciling a Room

- You are using the stencil pattern shown below to create a border in a room. How are the ducks labeled, B, C, D, E, and F related to Duck A? How many times would you use the stencil on a wall that is 11 feet, 2 inches long?



Example 4
Using Transformations in Real-Life
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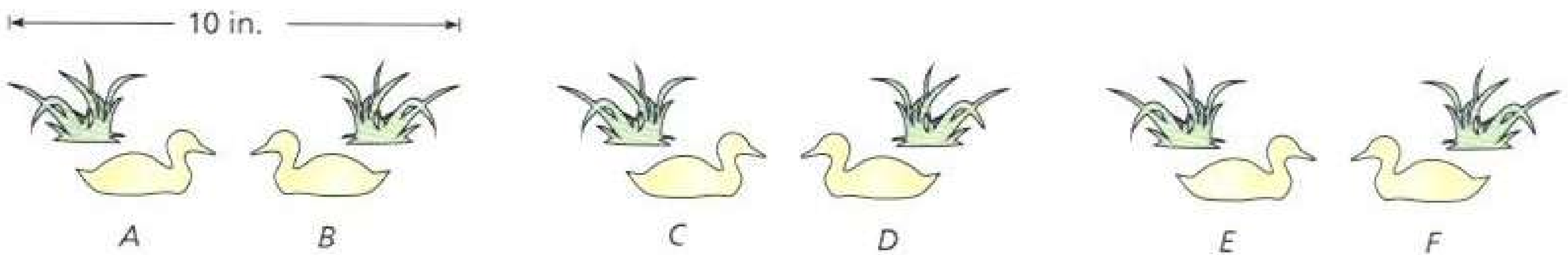
Duck C and E are translations of Duck A



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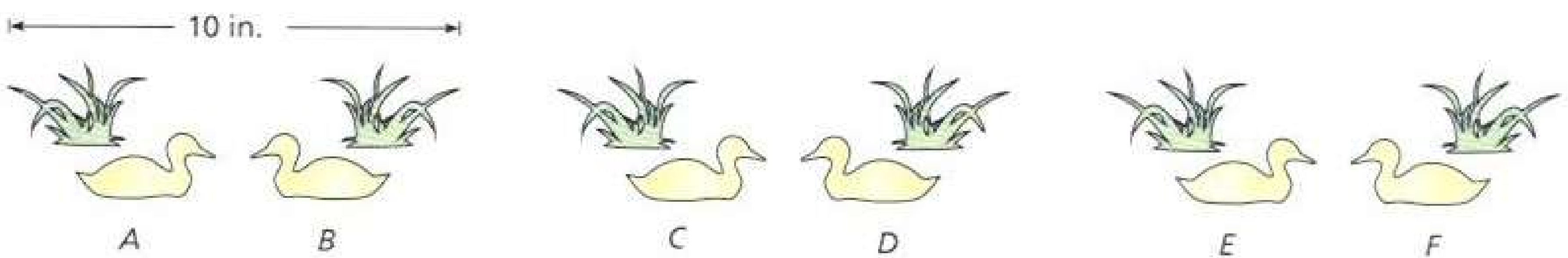
Ducks B,D and F are reflections of Duck A

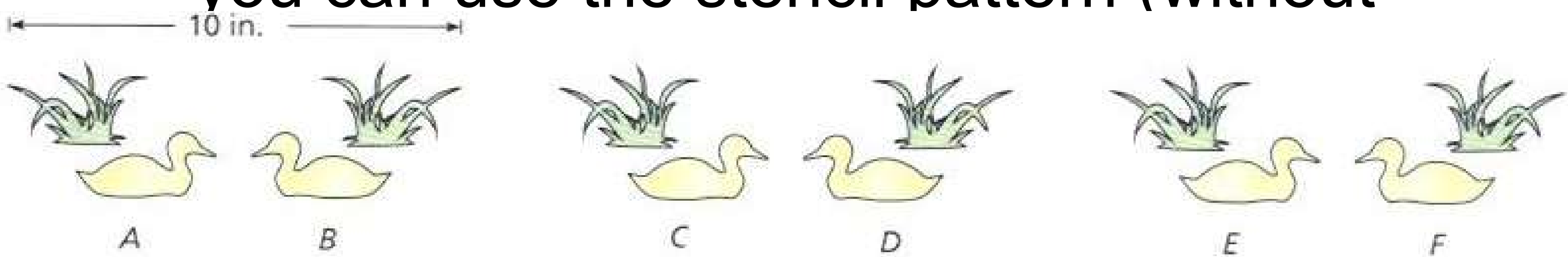


Example 4
Using Transformations in Real-Life
Stenciling a Room

- You are using the stencil pattern shown below to create a border in a room. How are the ducks labeled, B, C, D, E, and F related to Duck A? How many times would you use the stencil on a wall that is 11 feet, 2 inches long?

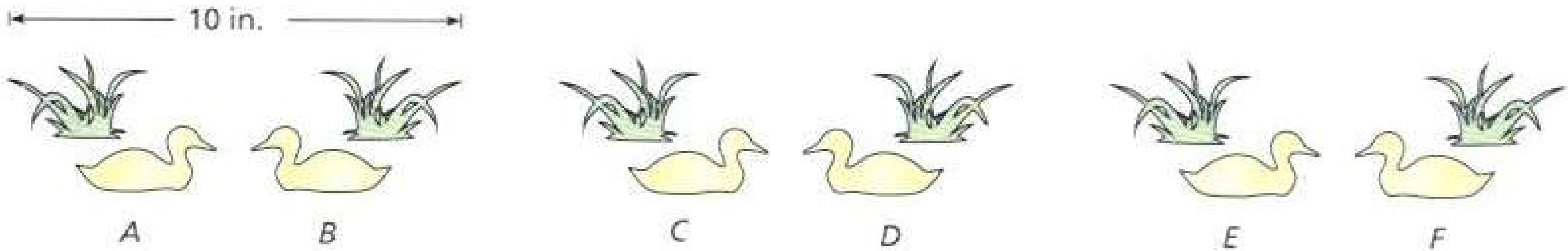
11'2" = 11 x 12 + 2 = 134 inches

134 ÷ 10 = 13.4, the maximum # of times you can use the stencil pattern (without



Example 4
Using Transformations in Real-Life
Stenciling a Room

- You are using the stencil pattern shown below to create a border in a room. How are the ducks labeled, B, C, D, E, and F related to Duck A? How many times would you use the stencil on a wall that is 1 feet, 2 inches long?
- If you want to spread the patterns out more, you can use the stencil only 11 times. The patterns then use 110 inches of space. The remaining 24 inches allow the patterns to be 2 inches part, with 2 inches on each end



Translations

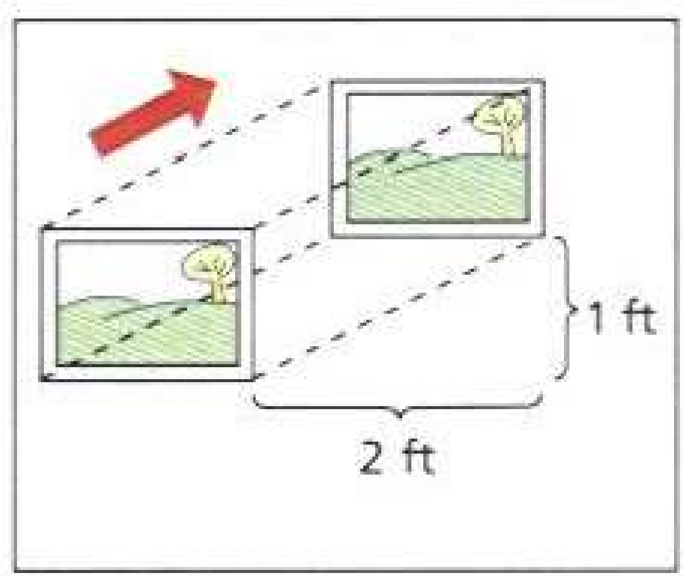
(slides)

What You Should Learn
Why You Should Learn It

- How to use properties of translations
- How to use translations to solve real-life problems
- You can use translations to solve real-life problems, such as determining patterns in music

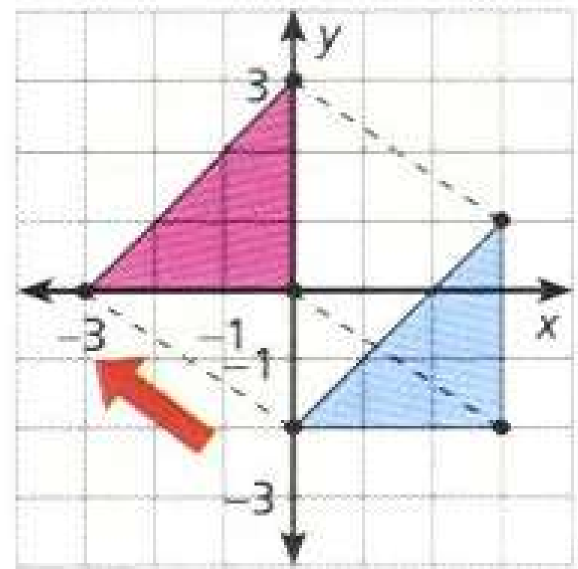
A translation (slide) is an isometry

Figure 1



Sliding a picture on a wall

Figure 2



Sliding a triangle in a plane

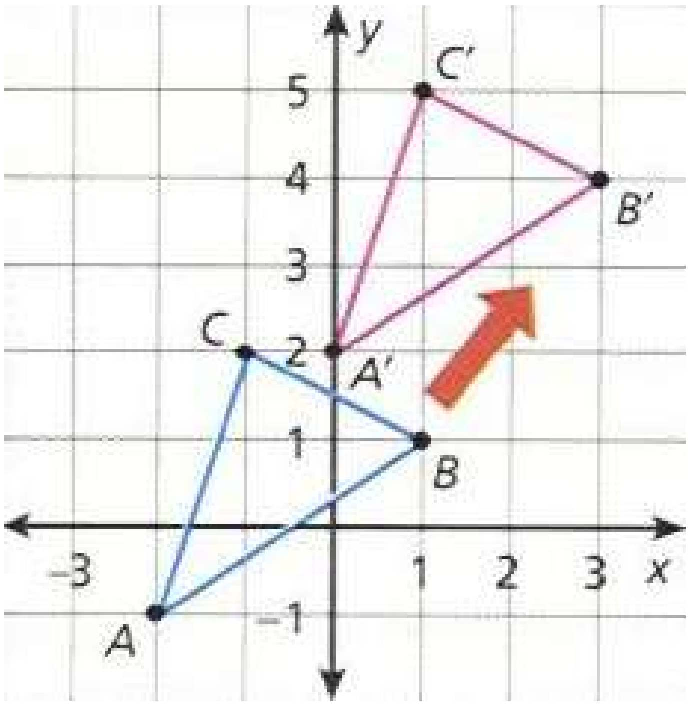
The picture is moved 2 feet to the right and 1 foot up

The points are moved 3 units to the left and 2 units up

Examples

- http://www.shodor.org/interactivate/activities/transform/index.html

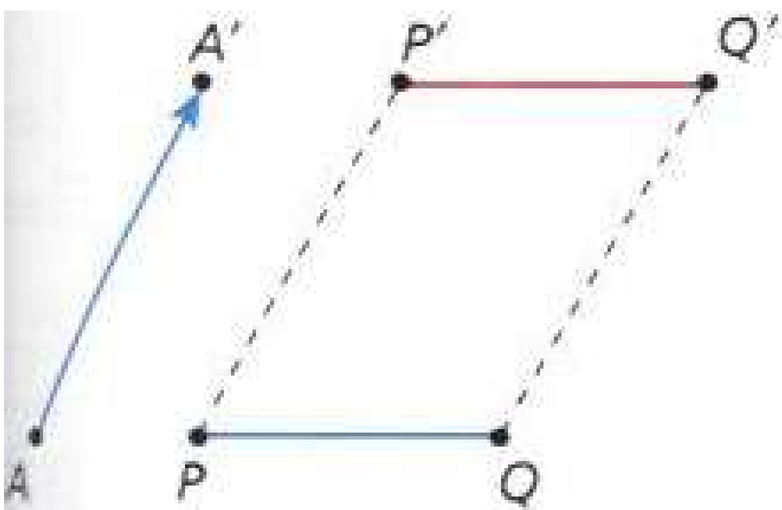
Prime Notation

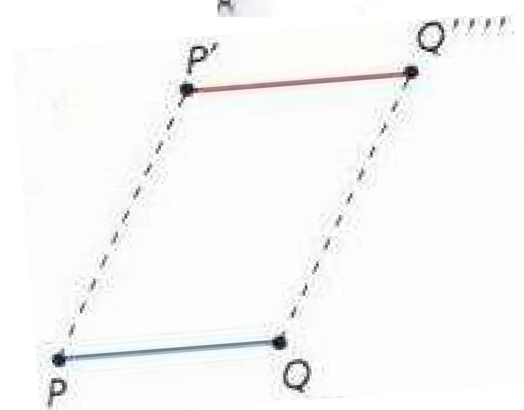


- Prime notation is just a ‘ added to a number
- It shows how to show that a figure has moved
- The preimage is the blue ΔABC and the image (after the movement) is ΔA’B’C’

Using Translations

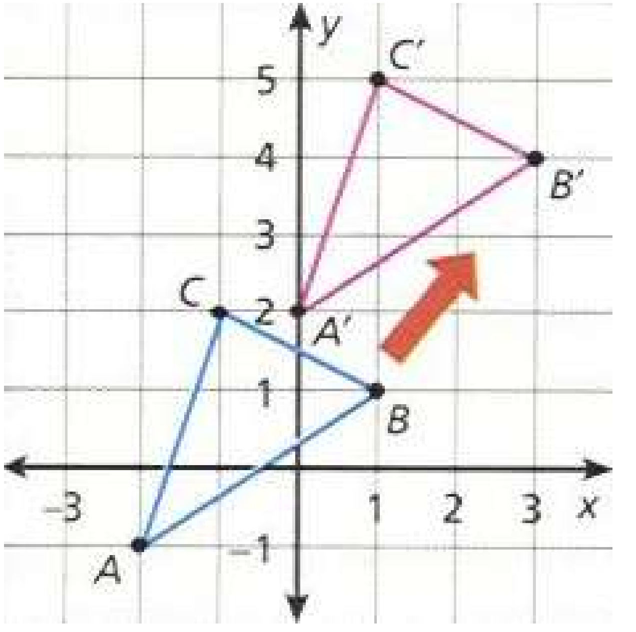
- A **translation** by a vector AA' is a transformation that maps every point P in the plane to a point P', so that the following properties are true.
  - 1. PP' = AA'
  - 2. PP' || AA' or PP' is collinear with AA'





Coordinate Notation

- Coordinate notation is when you write things in terms of x and y coordinates.
- You will be asked to describe the translation using coordinate notation.
- When you moved from A to A', how far did your x travel (and the direction) and how far did your y travel (and the direction).
- Start at point A and describe how you would get to A':
  - Over two and up three…
  - Or (x + 2, y + 3)

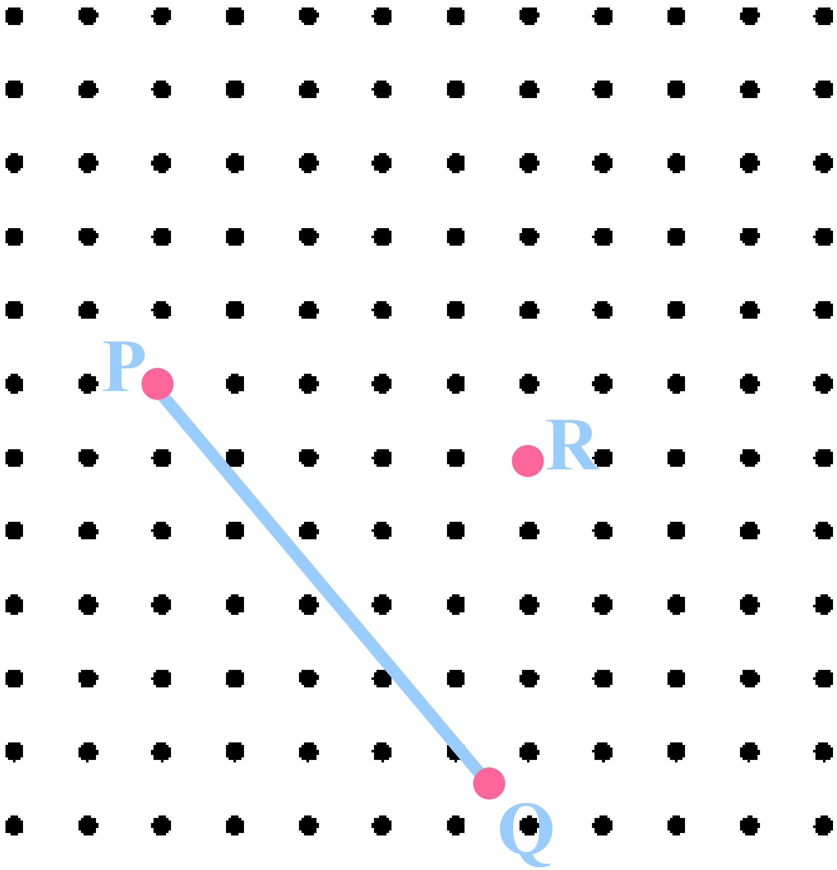


Vector Notation

Example 1
Constructing a Translation

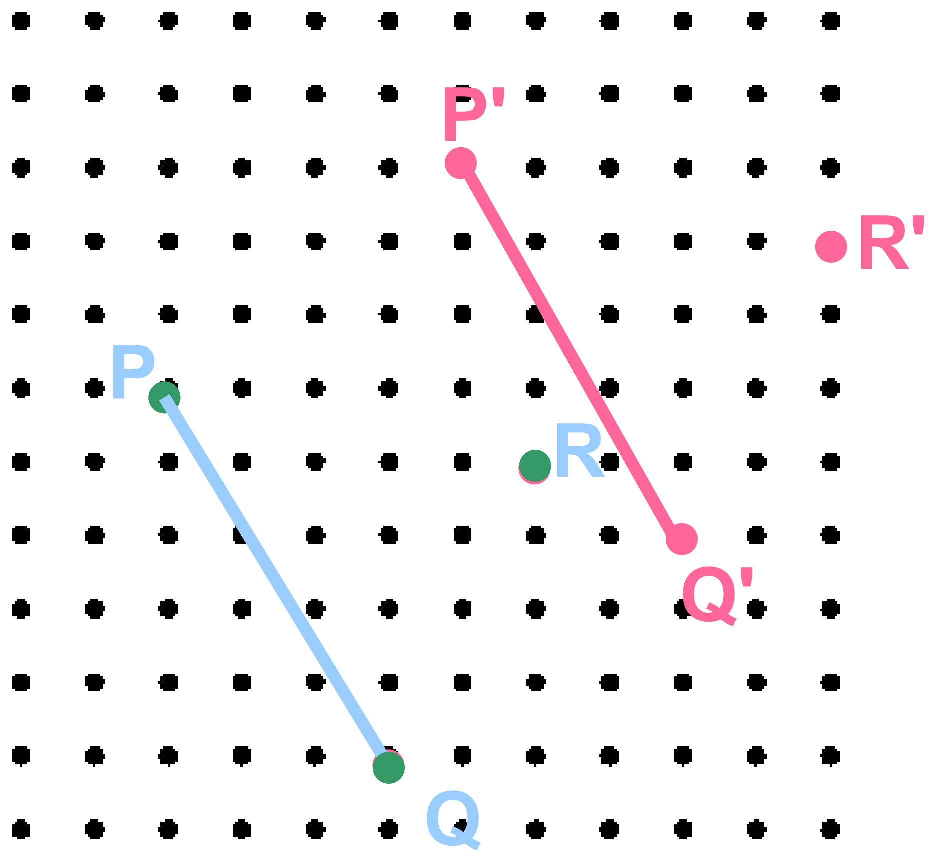
- Use a straightedge and dot paper to translate ΔPQR by the vector

- Hint: In a vector the 1st value represents horizontal distance, the 2 value represents vertical distance



Example 1
Constructing a Translation

- Use a straightedge and dot paper to translate ΔPQR by the vector



- What would this be in coordinate notation?
  - (x + 4, y + 3)

Using Translations in Real Life

- Example 2 (Translations and Rotations in Music)

a. The two measures shown below are the beginning of a musical piece titled *Barcarolle* by Jacques Offenbach. The second measure is a translation of the first measure.

Formula Summary

- Coordinate Notation for a translation by (a, b):

$$(x + a, y + b)$$

- Vector Notation for a translation by (a, b)

$$<a, b>$$

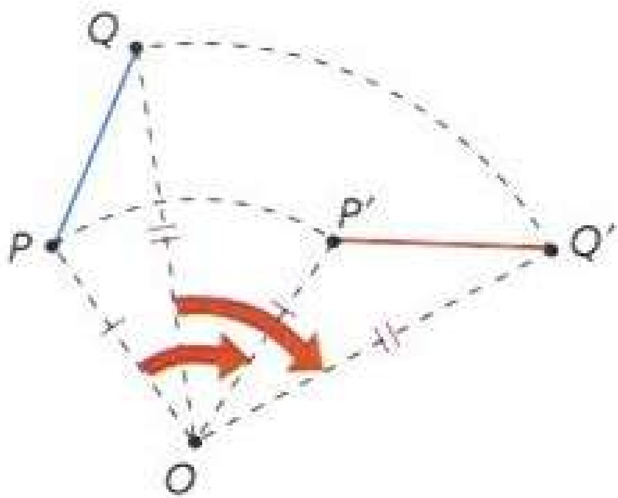
Rotations

What You Should Learn
Why You Should Learn It

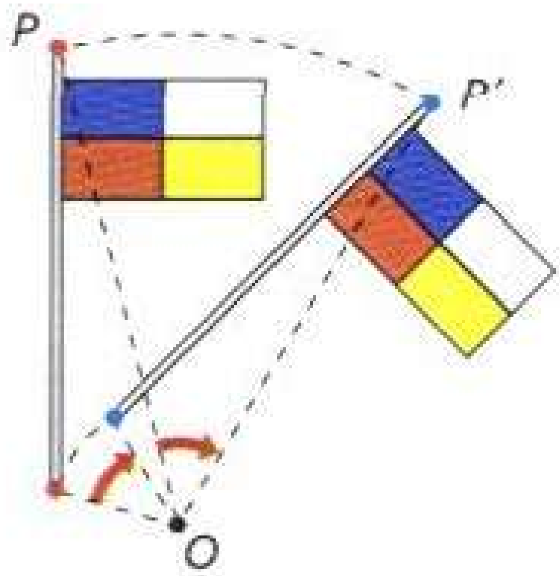
- How to use properties of rotations
- How to relate rotations and rotational symmetry
- You can use rotations to solve real-life problems, such as determining the symmetry of a clock face

Using Rotations

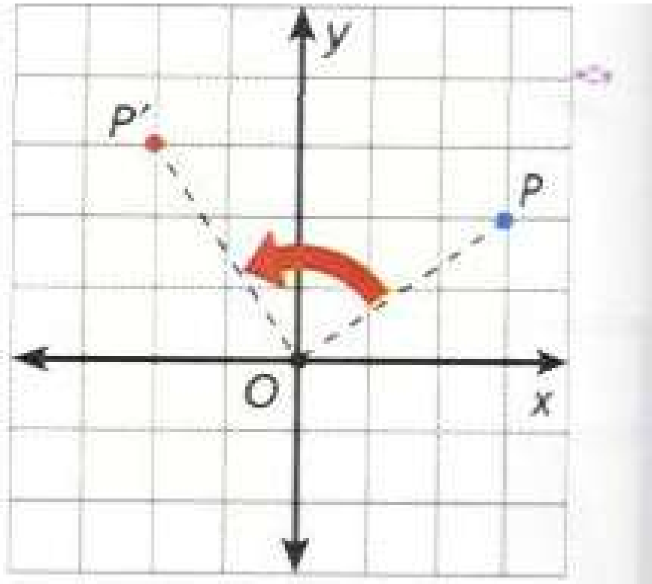
- A rotation about a point *O* through x degrees (x°) is a transformation that maps every point P in the plane to a point P', so that the following properties are true
- 1. If *P* is not Point *O*, then *PO* = *P′O* and m *POP′* = x°
- 2. If P is point *O*, then *P* = *P′*



Examples of Rotation



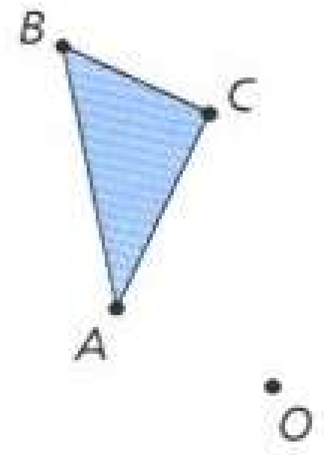
Clockwise rotation of 45°



Counterclockwise rotation of 90°

Example 1
Constructing a Rotation

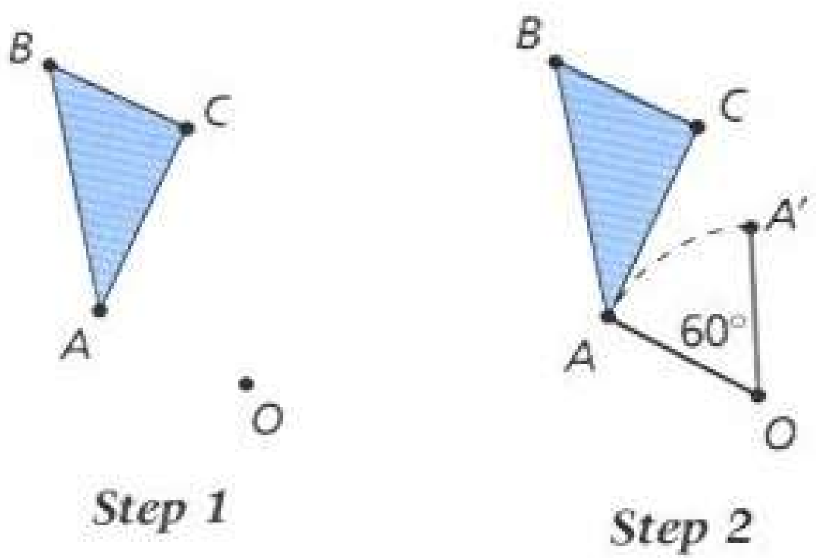
- Use a straightedge, compass, and protractor to rotate ΔABC 60° clockwise about point *O*



Step 1

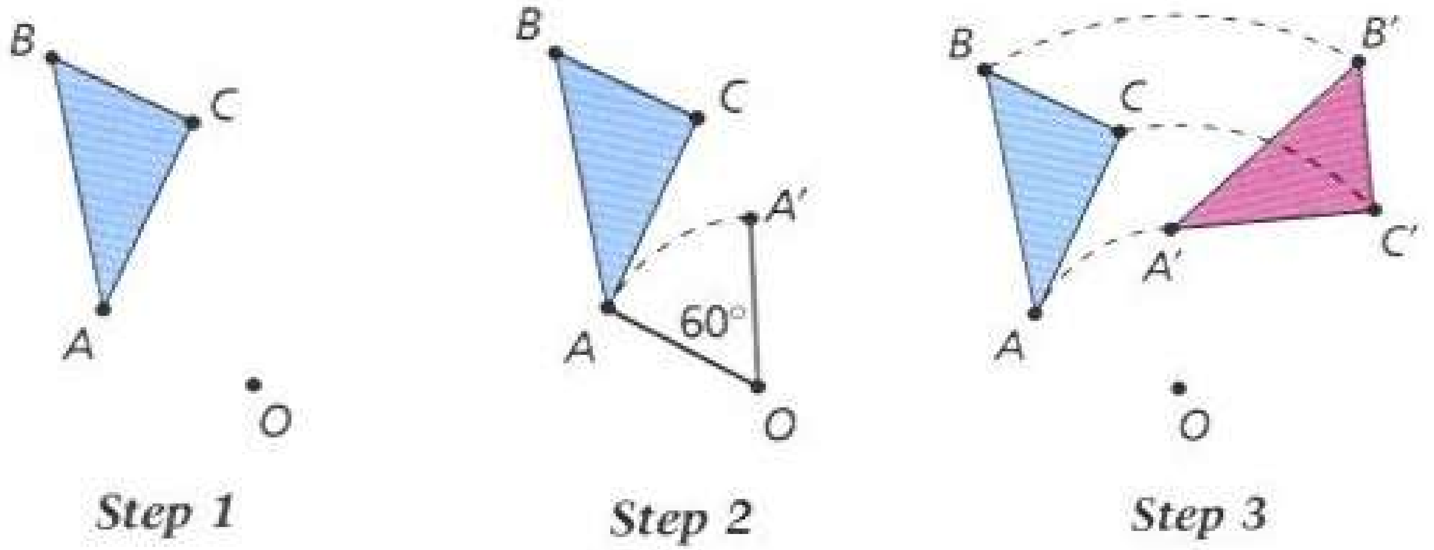
Example 1
Constructing a Rotation Solution

- Place the point of the compass at *O* and draw an arc clockwise from point *A*
- Use the protractor to measure a 60° angle, $\angle AOA'$
- Label the point A'



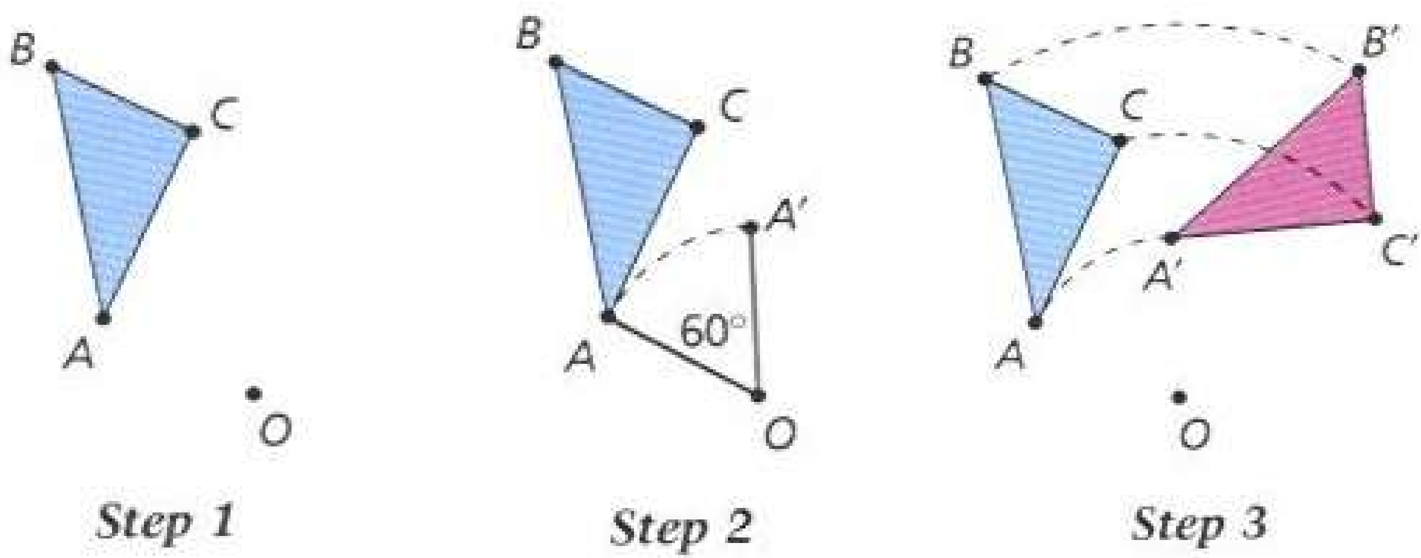
Example 1
Constructing a Rotation Solution

- Place the point of the compass at *O* and draw an arc clockwise from point *B*
- Use the protractor to measure a 60° angle, ∠*BOB'*
- Label the point B'



Example 1
Constructing a Rotation Solution

- Place the point of the compass at *O* and draw an arc clockwise from point *C*
- Use the protractor to measure a 60° angle, $\angle COC'$
- Label the point C'



Formula Summary

- Translations
  - Coordinate Notation for a translation by (a, b): $(x + a, y + b)$
  - Vector Notation for a translation by (a, b): $\langle a, b \rangle$
- Rotations
  - Clockwise (CW):

    90 $(x, y) \rightarrow (y, -x)$

    180 $(x, y) \rightarrow (-x, -y)$

    270 $(x, y) \rightarrow (-y, x)$
  - Counter-clockwise (CCW):

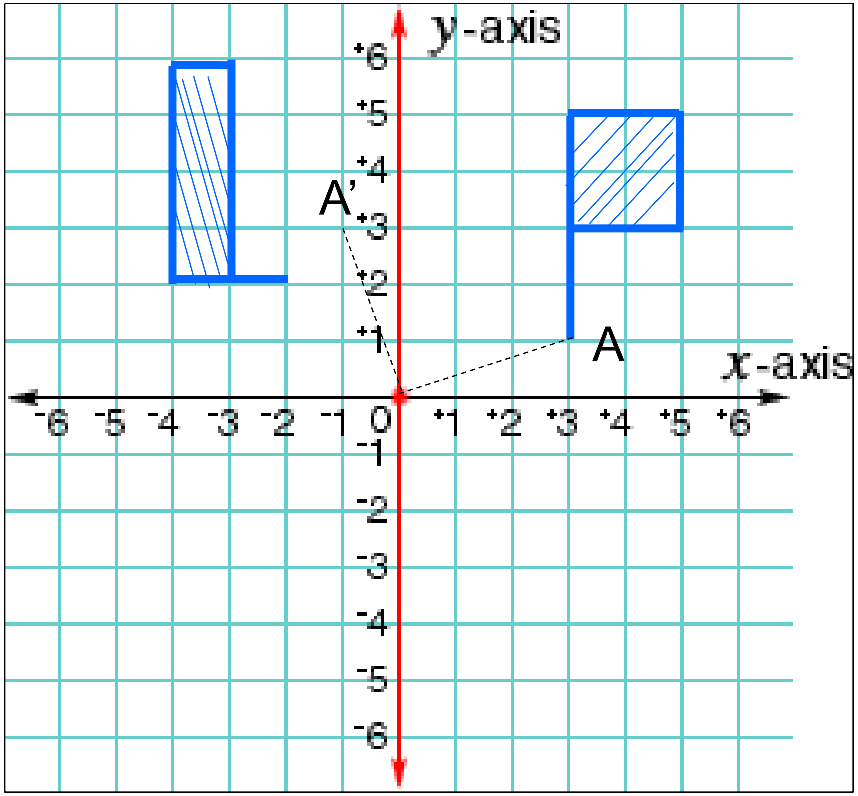
    90 $(x, y) \rightarrow (-y, x)$

    180 $(x, y) \rightarrow (-x, -y)$

    270 $(x, y) \rightarrow (y, -x)$

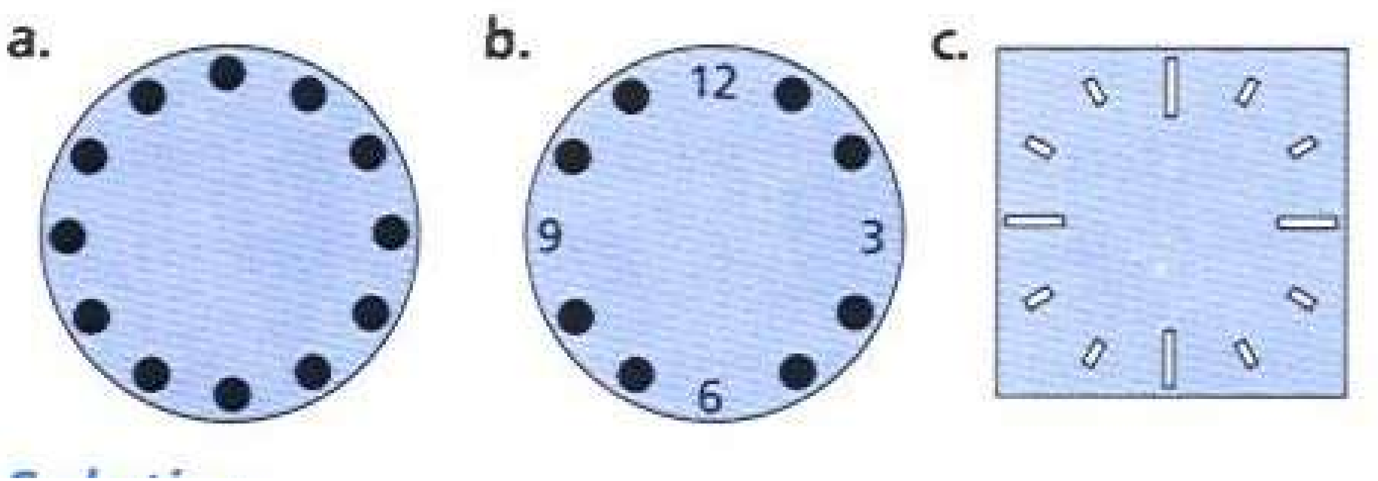
Rotations

- What are the coordinates for A?
- A(3, 1)
- What are the coordinates for A'?
- A'(-1, 3)



Example 2
Rotations and Rotational Symmetry

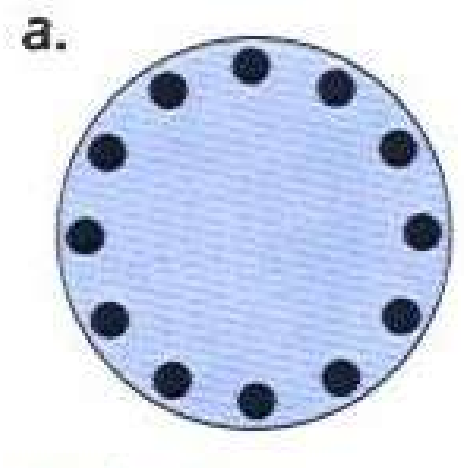
- Which clock faces have rotational symmetry? For those that do, describe the rotations that map the clock face onto itself.



Example 2
Rotations and Rotational Symmetry

- Which clock faces have rotational symmetry? For those that do, describe the rotations that map the clock face onto itself.
- Rotational symmetry about the center, clockwise or counterclockwise
- 30°,60°,90°,120°,150°,180°
-

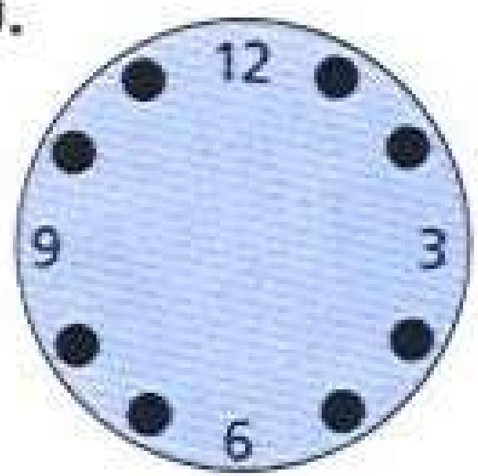
Moving from one dot to the next is (1/12) of a complete turn or (1/12) of 360°



Example 2
Rotations and Rotational Symmetry

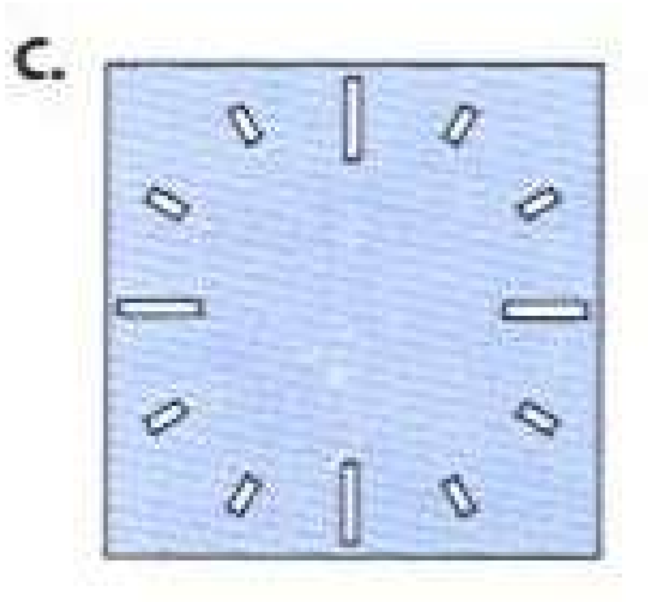
- Which clock faces have rotational symmetry? For those that do, describe the rotations that map the clock face onto itself.
- Does not have rotational symmetry
-

b.



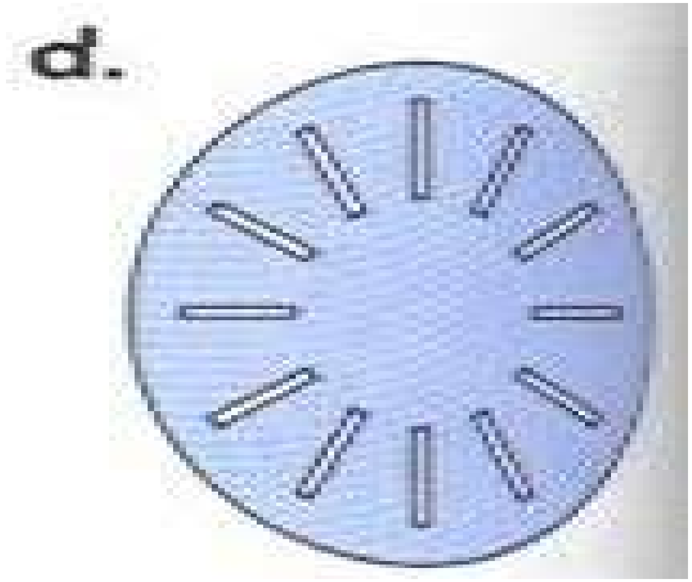
Example 2
Rotations and Rotational Symmetry

- Which clock faces have rotational symmetry? For those that do, describe the rotations that map the clock face onto itself.
- Rotational symmetry about the center
- Clockwise or Counterclockwise 90° or 180°



Example 2
Rotations and Rotational Symmetry

- Which clock faces have rotational symmetry? For those that do, describe the rotations that map the clock face onto itself.
- Rotational symmetry about its center
- 180°
-
-



Reflections

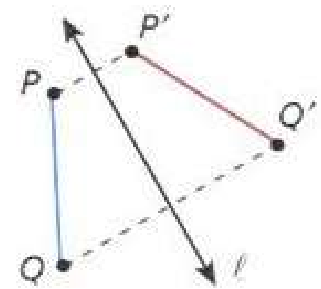
What You Should Learn
Why You Should Learn It

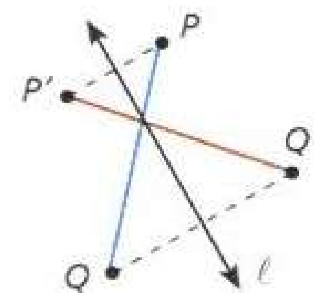
Goal 1: How to use properties of reflections
Goal 2: How to relate reflections and line symmetry
You can use reflections to solve real-life problems, such as building a kaleidoscope

Using Reflections

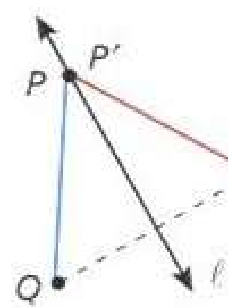
A reflection in a line *L* is a transformation that maps every point *P* in the plane to a point *P'*, so that the following properties are true

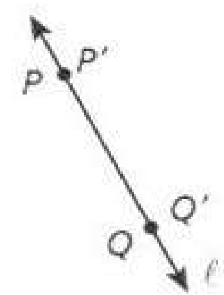
1. If *P* is not on *L*, then *L* is the perpendicular bisector of *PP'*





2. If *P* is on *L*, then *P* = *P'*



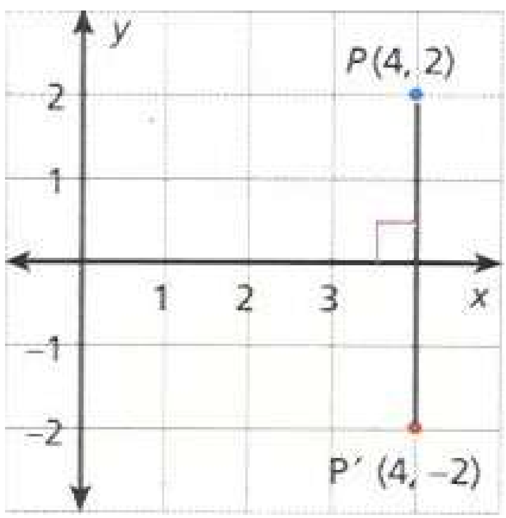


Reflection in the Coordinate Plane

Suppose the points in a coordinate plane are reflected in the x-axis.

So then every point (x,y) is mapped onto the point (x,-y)

P (4,2) is mapped onto *P'* (4,-2)



What do you notice about the x-axis?

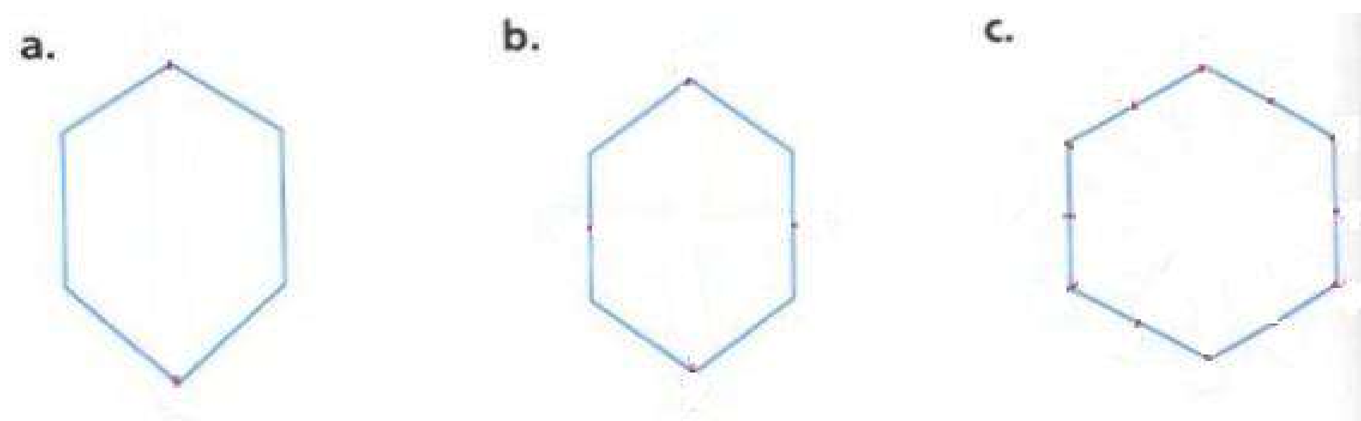
It is the line of reflection

It is the perpendicular bisector of *PP'*

Reflections & Line Symmetry

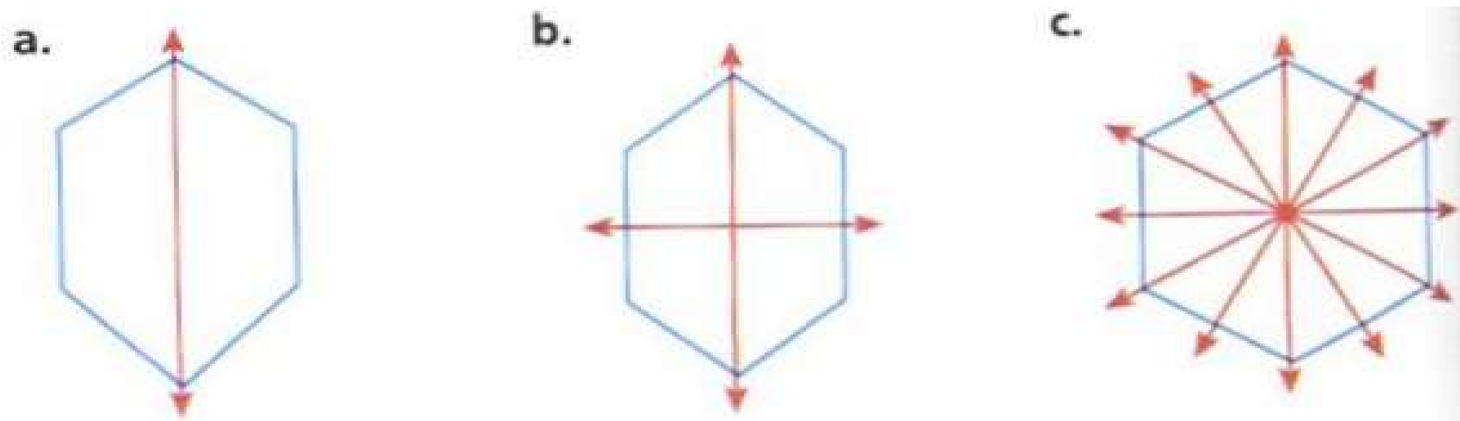
A figure in the plane has a **line of symmetry** if the figure can be mapped onto itself by a reflection

How many lines of symmetry does each hexagon have?



Reflections & Line Symmetry

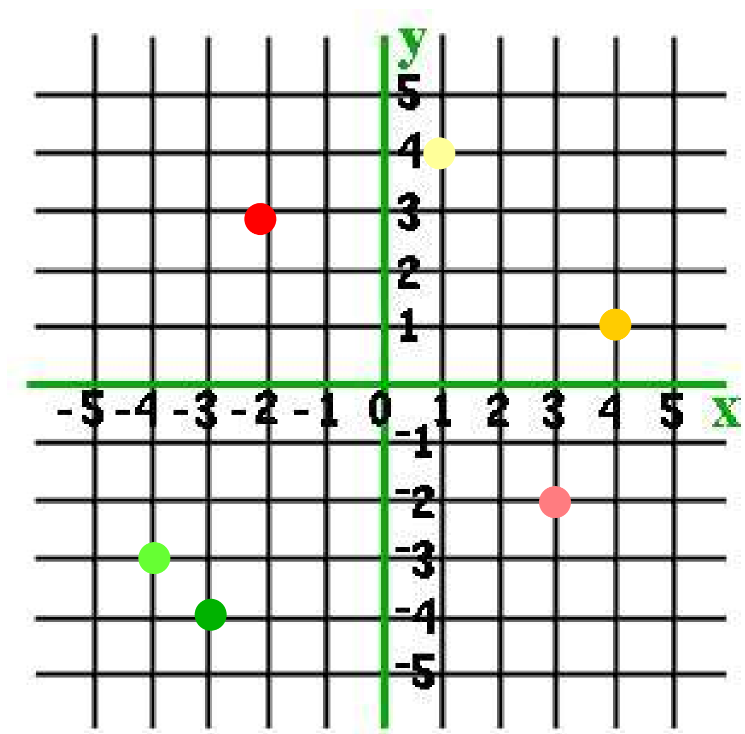
How many lines of symmetry does each hexagon have?



1 2 6

Reflection in the line y = x

Generalize the results when a point is reflected about the line y = x



y = x

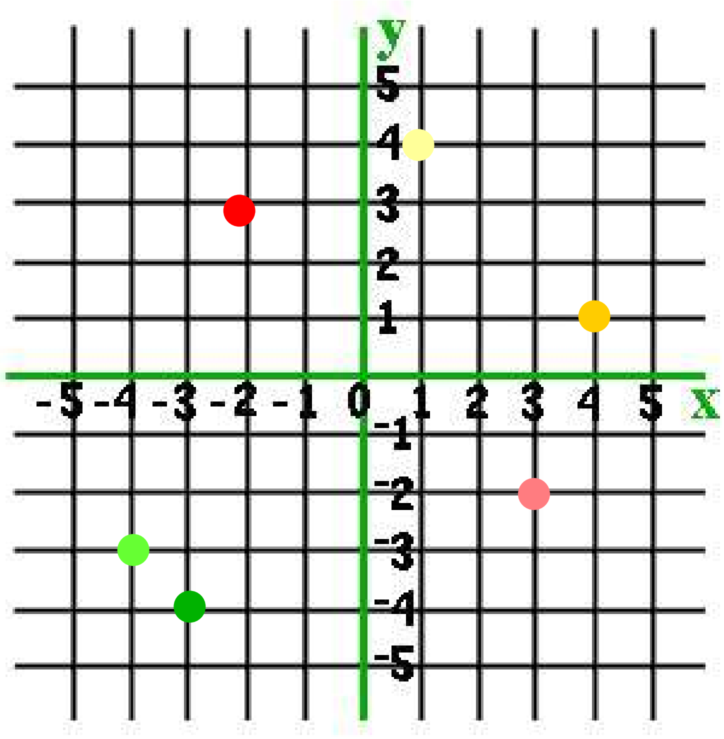
(1,4)→ (4,1)

(-2,3)→ (3,-2)

(-4,-3)→ (-3,-4)

Reflection in the line y = x

Generalize the results when a point is reflected about the line y = x



y = x

(x,y) maps to (y,x)

Formulas

Translations

- Coordinate Notation for a translation by (a, b):

  $(x + a, y + b)$

- Vector Notation for a translation by (a, b): <a, b>

Rotations

- Clockwise (CW):

90 $(x, y) \rightarrow (y, -x)$

180 $(x, y) \rightarrow (-x, -y)$

270 $(x, y) \rightarrow (-y, x)$

- Counter-clockwise (CCW):

90 $(x, y) \rightarrow (-y, x)$

180 $(x, y) \rightarrow (-x, -y)$

270 $(x, y) \rightarrow (y, -x)$

Reflections

x-axis (y = 0) $(x, y) \rightarrow (x, -y)$

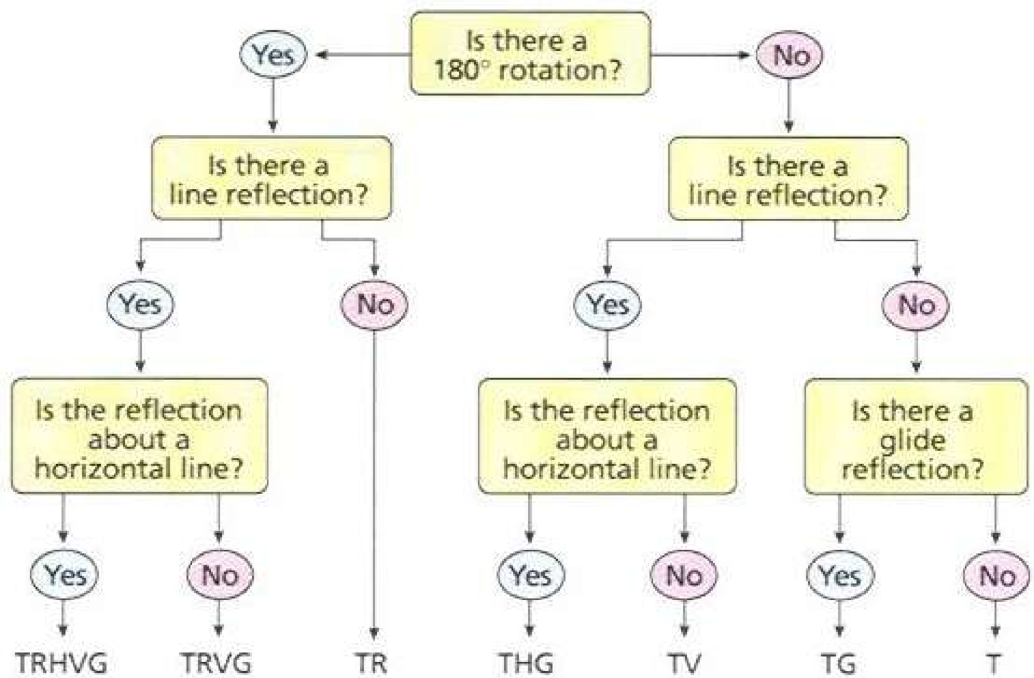
y-axis (x = 0) $(x, y) \rightarrow (-x, y)$

Line y = x $(x, y) \rightarrow (y, x)$

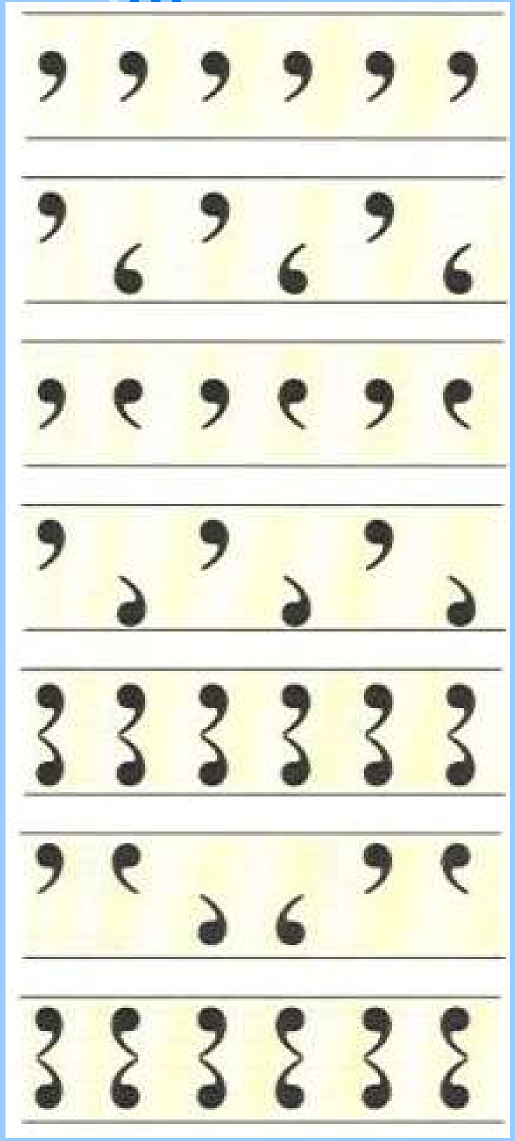
Line y = -x $(x, y) \rightarrow (-y, -x)$

Any horizontal line (y = n): $(x, y) \rightarrow (x, 2n - y)$

Any vertical line (x = n): $(x, y) \rightarrow (2n - x, y)$



7 Categories of Frieze Patterns



T	Translation only
TR	Translation and 180° rotation
TV	Translation and vertical line reflection
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THG	Translation, horizontal line reflection, and glide reflection
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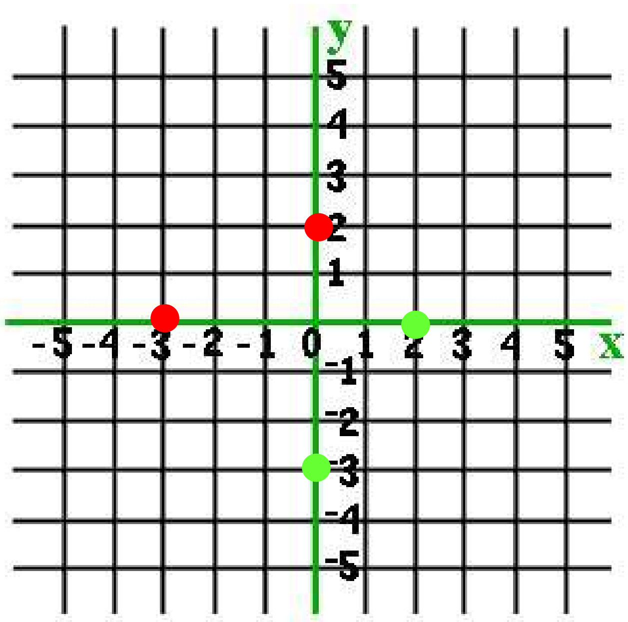
TRHVG	Translation, 180° rotation, horizontal line reflection, vertical line reflection, and glide reflection

Reflection in the line y = x

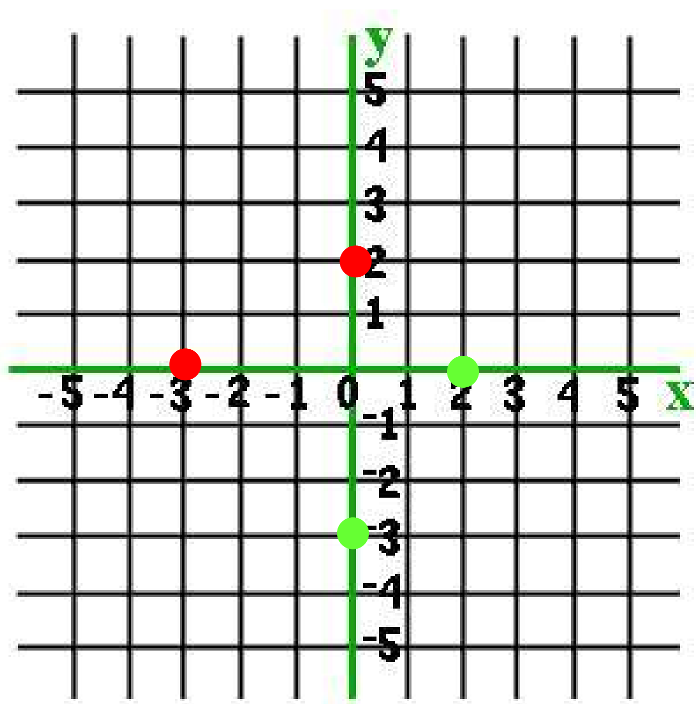
Generalize what happens to the slope, m, of a line that is reflected in the line y = x

y = x



Reflection in the line y = x

Generalize what happens to the slope, m, of a line that is reflected in the line y = x



The new slope is 1
m

The slopes are reciprocals of each other

Find the Equation of the Line

Find the equation of the line if
$y = 4x - 1$ is reflected over $y = x$

Find the Equation of the Line

Find the equation of the line if

$y = 4x - 1$ is reflected over $y = x$

$Y = 4x - 1$; $m = 4$ and a point on the line is $(0,-1)$

So then, $m = \frac{1}{4}$ and a point on the line is $(-1,0)$

$Y = mx + b$

$0 = \frac{1}{4}(-1) + b$

$\frac{1}{4} = b$

$y = \frac{1}{4}x + \frac{1}{4}$

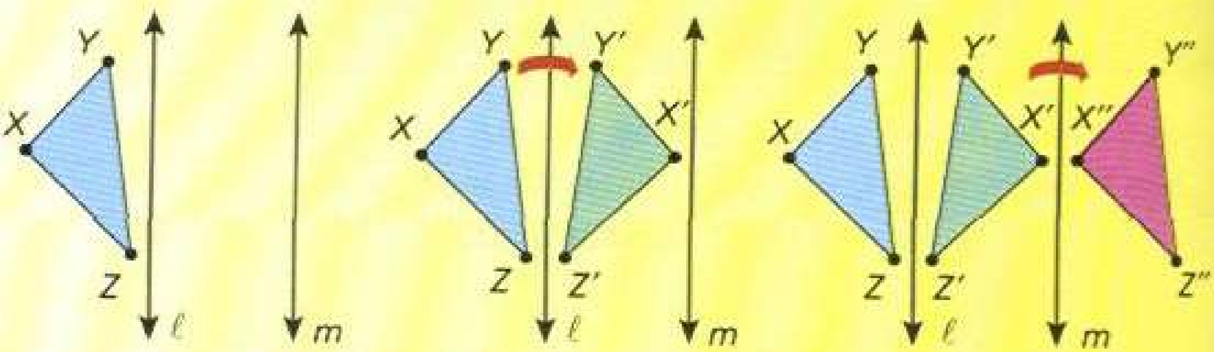
Lesson Investigation

Investigating Reflections and Translations

Use a computer drawing program or construction tools to perform the following steps.

1. Draw a triangle $\triangle XYZ$ and parallel lines ℓ and m.
2. Reflect $\triangle XYZ$ in line ℓ to obtain $\triangle X'Y'Z'$.
3. Reflect $\triangle X'Y'Z'$ in line m to obtain $\triangle X''Y''Z''$.

How is $\triangle X''Y''Z''$ related to $\triangle XYZ$?



Draw $\triangle XYZ$ and lines ℓ and m.

Reflect $\triangle XYZ$ about line ℓ.

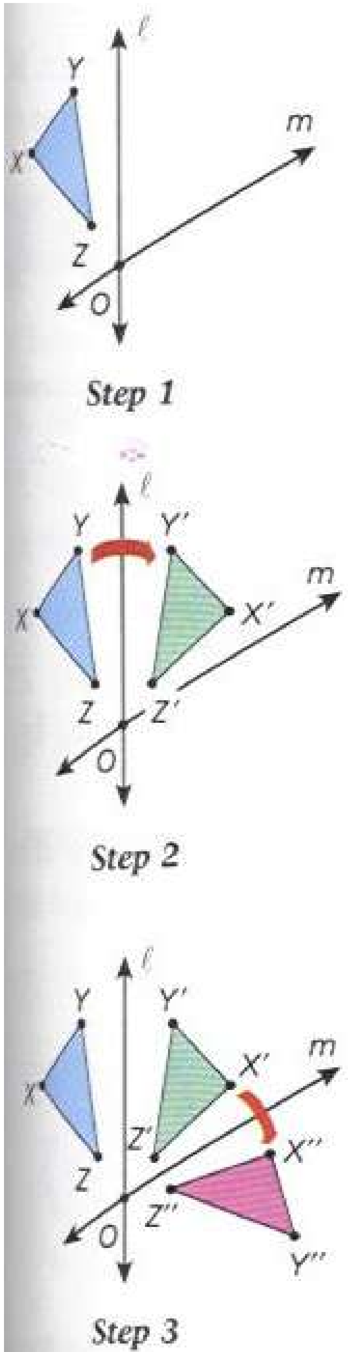
Reflect $\triangle X'Y'Z'$ about line m.

It is a translation and YY'' is twice LM

Theorem

- If lines L and M are parallel, then a reflection in line L followed by a reflection in line M is a translation. If P'' is the image of P after the two reflections, then PP'' is perpendicular to L and PP'' = 2d, where d is the distance between L and M.

Lesson Investigation



▪ *Investigating Reflections and Rotations*

Use a computer drawing program or construction tools to perform the following steps. The steps are illustrated at the left.

1. Draw a triangle, $\triangle XYZ$. Draw lines ℓ and m that intersect at point O.
2. Reflect $\triangle XYZ$ in line ℓ to obtain $\triangle X'Y'Z'$.
3. Reflect $\triangle X'Y'Z'$ in line m to obtain $\triangle X''Y''Z''$.

How is $\triangle X''Y''Z''$ *related to* $\triangle XYZ$?

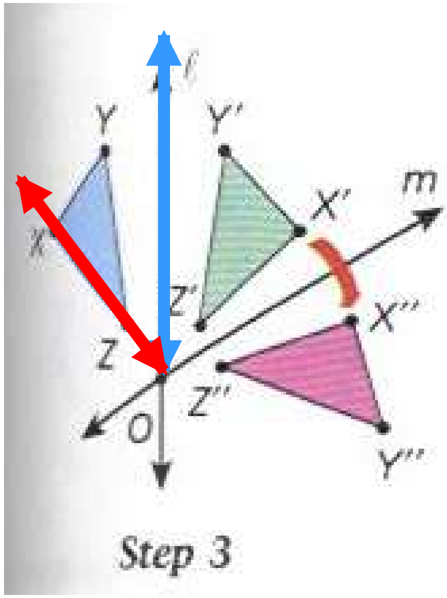
Compare the measure of ∠*XOX″* to the acute angle formed by L and m

It's a rotation

Angle XOX' is twice the size of the angle formed by L and m

Theorem

- If two lines, *L* and *m*, intersect at point *O*, then a reflection in *L* followed by a reflection in *m* is a rotation about point *O*. The angle of rotation is 2x°, where x° is the measure of the acute or right angle between *L* and *m*



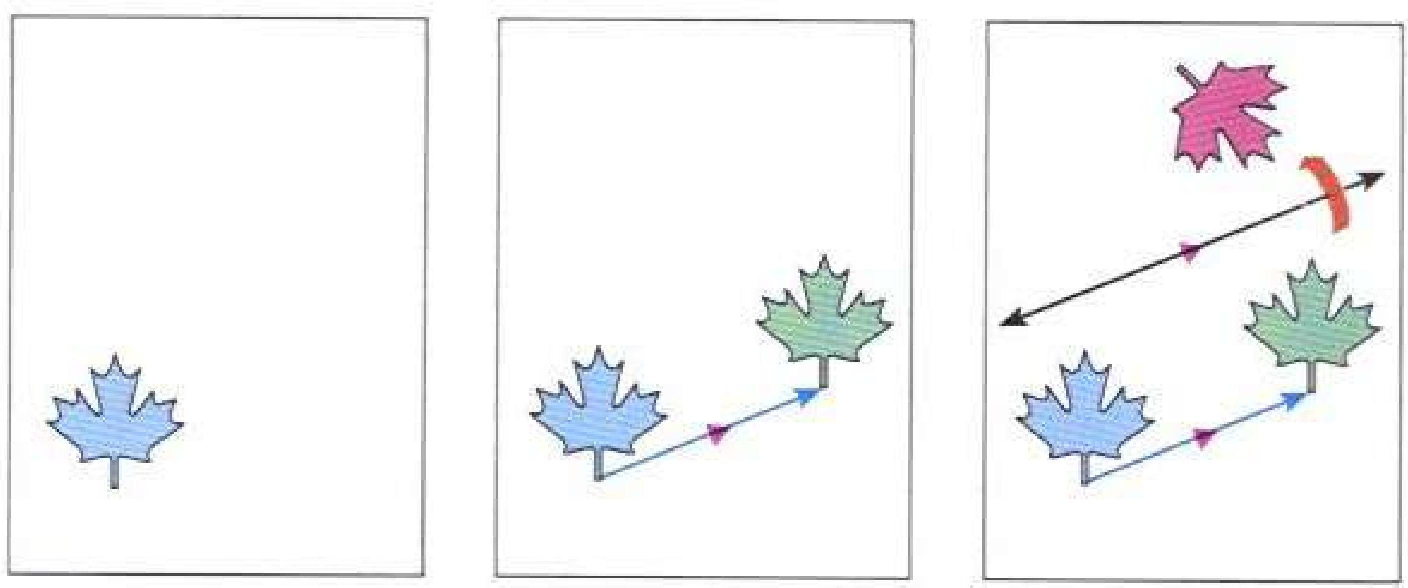
Glide Reflections & Compositions

What You Should Learn
Why You Should Learn It

- How to use properties of glide reflections
- How to use compositions of transformations
- You can use transformations to solve real-life problems, such as creating computer graphics

Using Glide Reflections

- A glide reflection is a transformation that consists of a translation by a vector, followed by a reflection in a line that is parallel to the vector



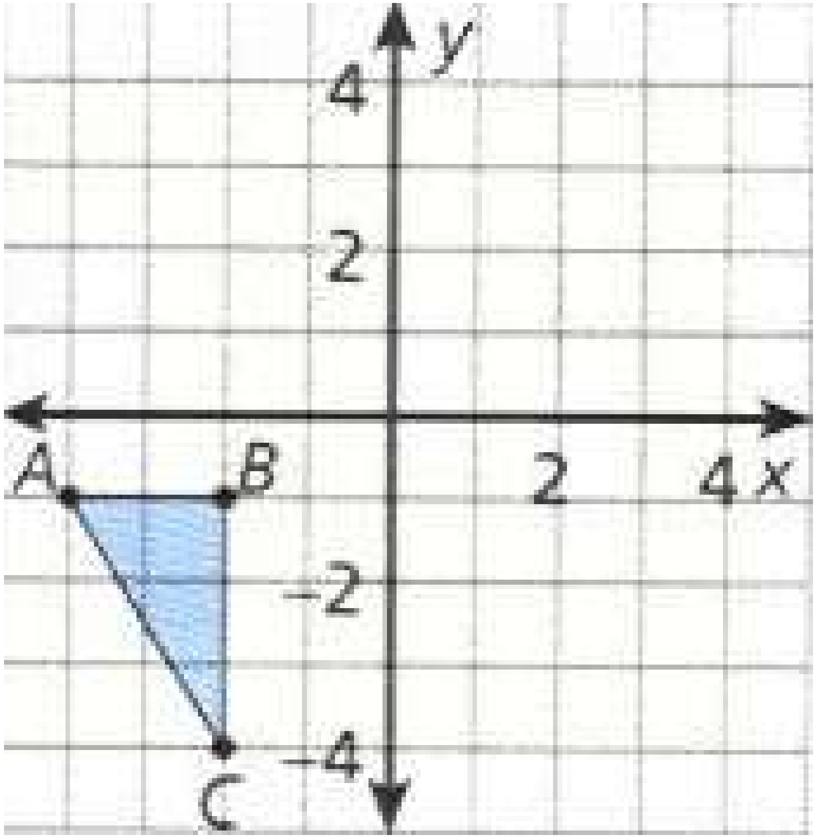
Preimage *Translate* *Reflect*

Composition

- When two or more transformations are combined to produce a single transformation, the result is called a composition of the transformations
- For instance, a translation can be thought of as composition of two reflections

Example 1
Finding the Image of a Glide Reflection

- Consider the glide reflection composed of the translation by the vector , followed by a reflection in the x-axis. Describe the image of ΔABC



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- Consider the glide reflection composed of the translation by the vector ,
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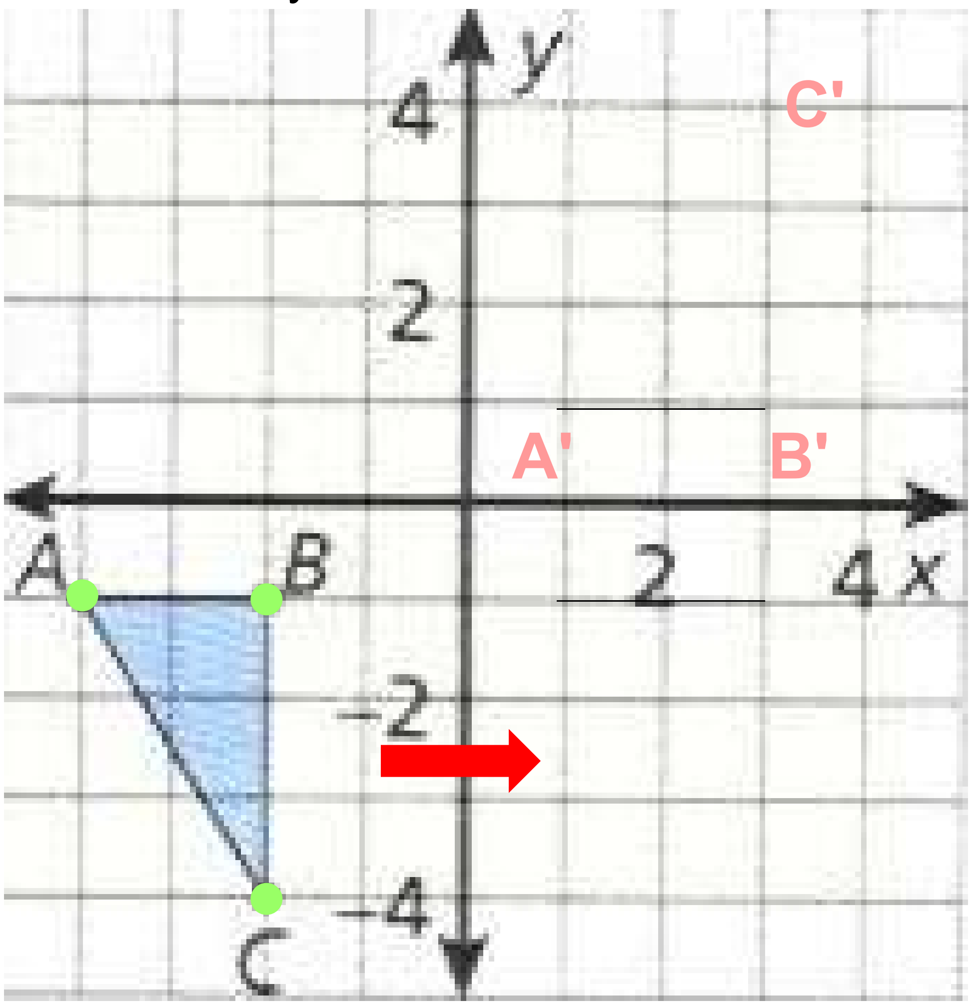


The image of ΔABC is ΔA'B'C' with these vertices:

A'(1,1)

B' (3,1)

C' (3,4)

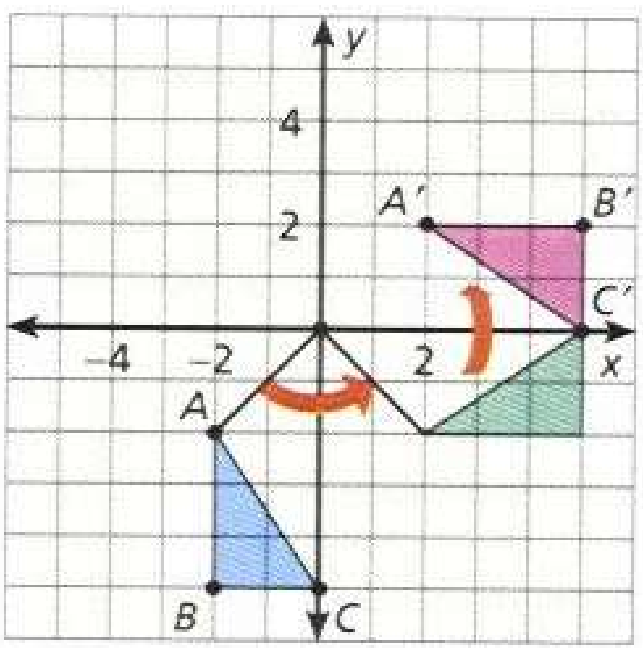
Theorem

- The composition of two (or more) isometries is an isometry
- Because glide reflections are compositions of isometries, this theorem implies that glide reflections are isometries

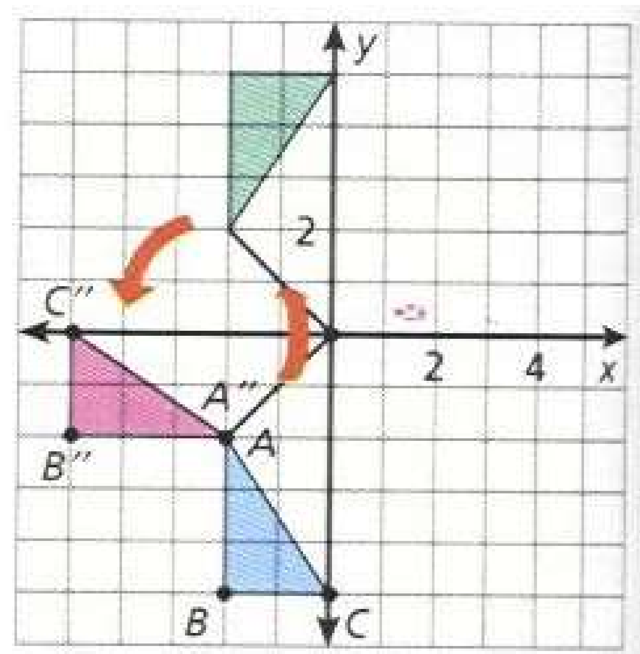
Example 2
Comparing Compositions

- Compare the following transformations of ΔABC. Do they produce congruent images? Concurrent images?

Hint: Concurrent means meeting at the same point



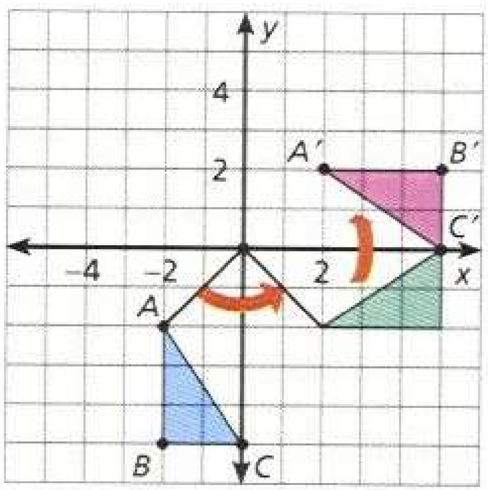
Rotate 90° counterclockwise about origin, then reflect in x-axis.



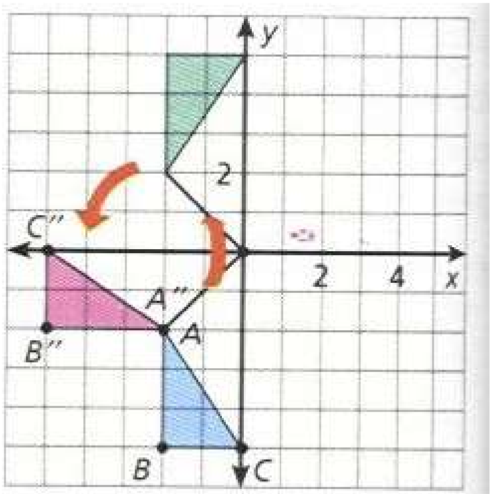
Reflect in x-axis, then rotate 90° counterclockwise about origin.

Example 2
Comparing Compositions

- Compare the following transformations of ΔABC. Do they produce congruent images? Concurrent images?
- From Theorem 7.6, you know that both compositions are isometries. Thus the triangles are congruent.
- The two triangles are not concurrent, the final transformations (pink triangles) do not share the same vertices



Rotate 90° counterclockwise about origin, then reflect in x-axis.



Reflect in x-axis, then rotate 90° counterclockwise about origin.

- Does the order in which you perform two transformations affect the resulting composition?
- Describe the two transformations in each figure

Figure 1

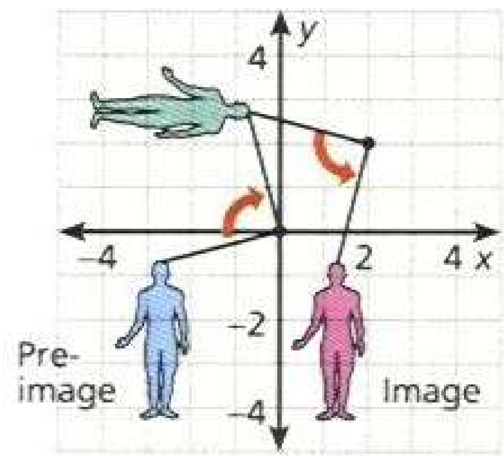
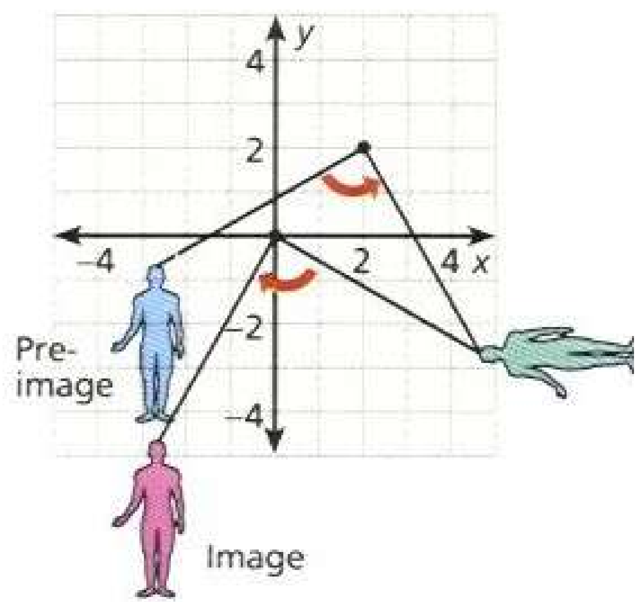


Figure 2



- Does the order in which you perform two transformations affect the resulting composition?
- Describe the two transformations in each figure

Figure 1

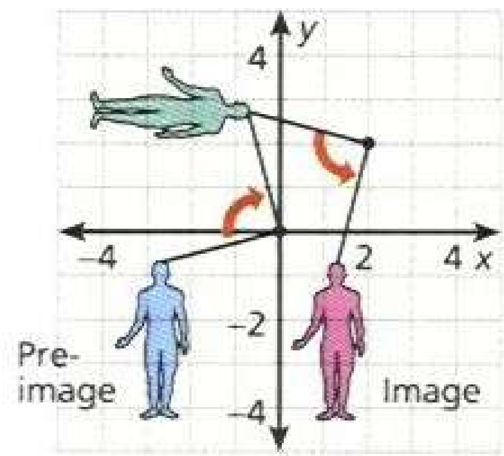
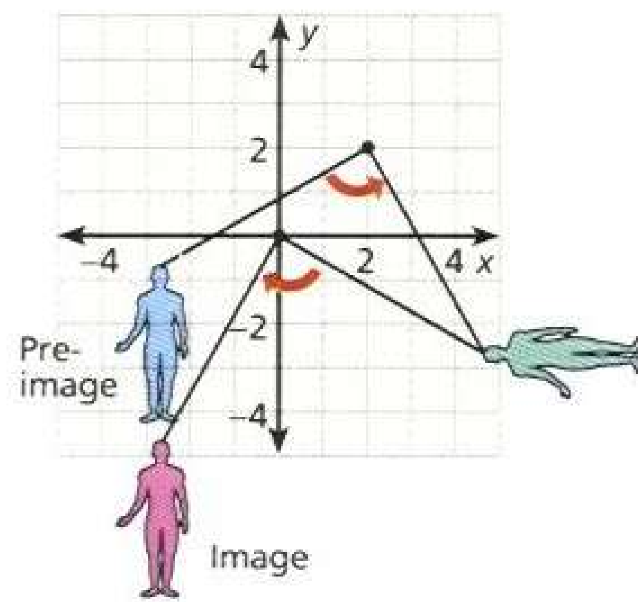


Figure 2



- Does the order in which you perform two transformations affect the resulting composition? YES
- Describe the two transformations in each figure
- Figure 1: Clockwise rotation of 90° about the origin, followed by a counterclockwise rotation of 90° about the point (2,2)
- Figure 2: a clockwise rotation of 90° about the point (2,2) , followed by a counterclockwise rotation of 90° about the origin

Figure 1

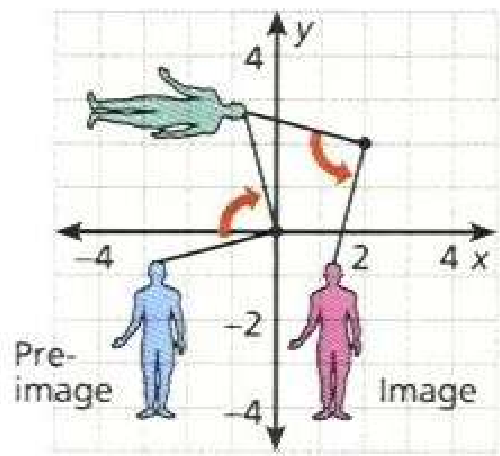
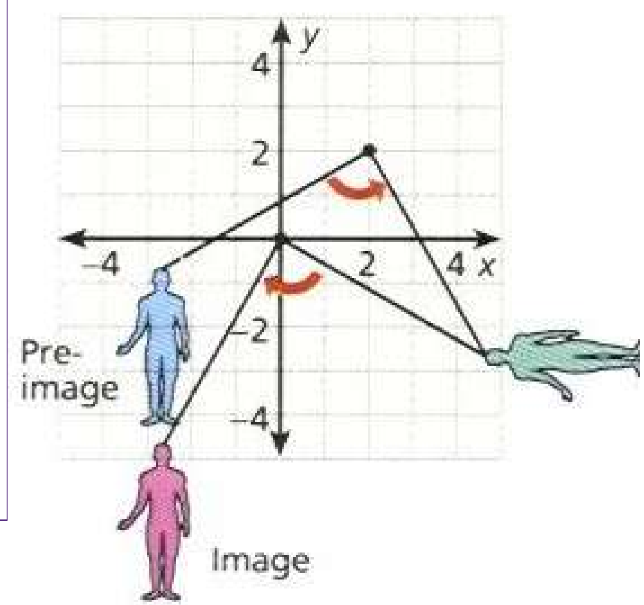
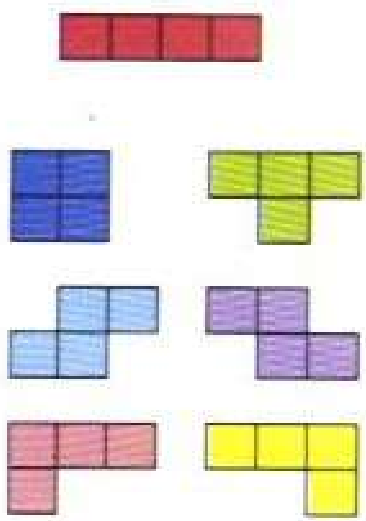


Figure 2

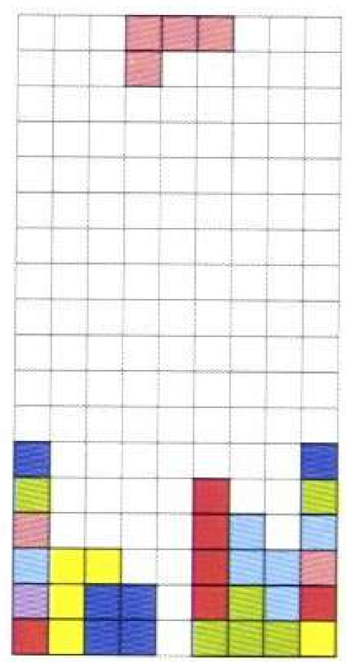


Example 3
Using Translations and Rotations in Tetris

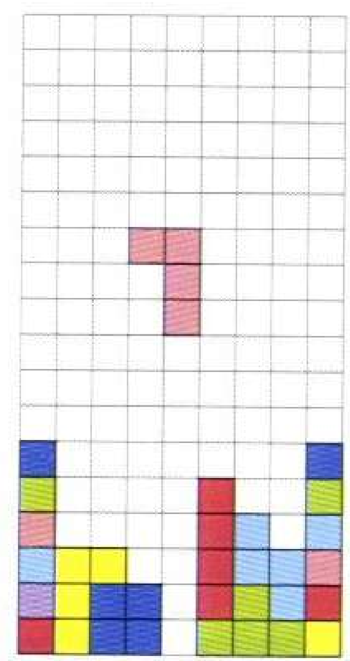
The computer game *Tetris* is a tiling game that uses seven tiles, each composed of four squares. The playing screen is a 9 × 20 grid of squares. During the game, tiles fall from the top of the screen. Your goal is to translate or rotate each falling tile to fill complete rows on the screen. Each time a row is filled, it will disappear from the screen. For instance, by rotating and translating the L-shaped tile shown below, you can fill three complete rows. Once you have done this, all three rows will disappear.



Tetris has 7 tiles. Each is composed of 4 squares.



L-Shaped tile begins to fall.



Rotate and translate to fill bottom 3 rows. ■

Online Tetris

Frieze Patterns

What You Should Learn
Why You Should Learn It

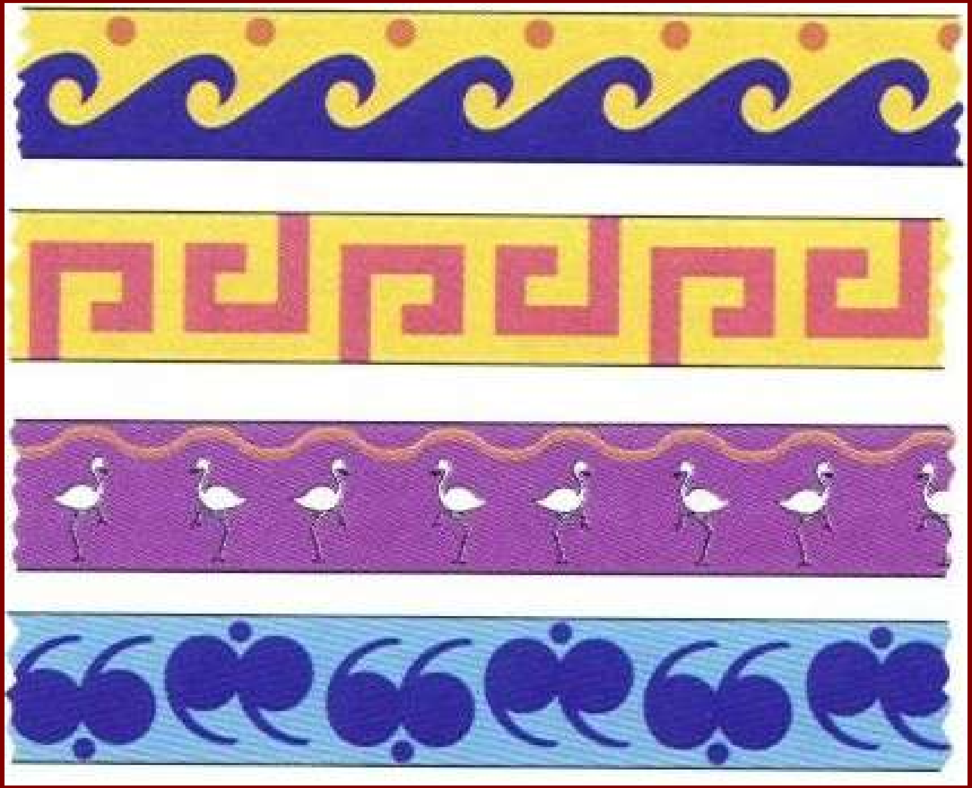
- How to use transformations to classify frieze patterns
- How to use frieze patterns in real life
- You can use frieze patterns to create decorative borders for real-life objects such as fabric, pottery, and buildings

Classifying Frieze Patterns

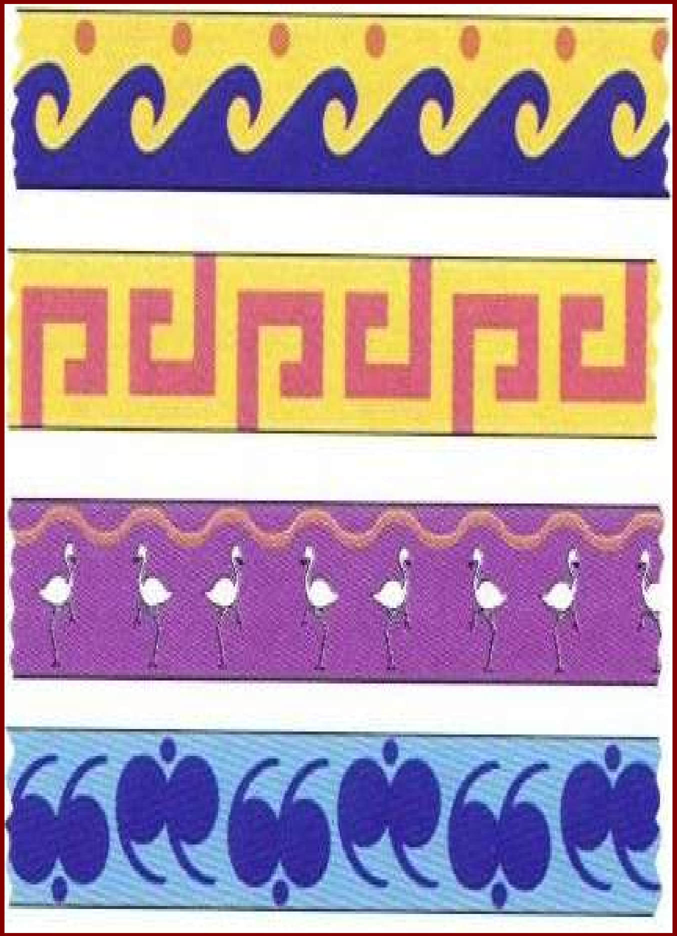
- A **frieze pattern** or **strip pattern** is a pattern that extends infinitely to the left and right in such a way that the pattern can be mapped onto itself by a horizontal translation
  - Some frieze patterns can be mapped onto themselves by other transformations:
    - A 180° rotation
    - A reflection about a horizontal line
    - A reflection about a vertical line
    - A horizontal glide reflection

Example 1
Examples of Frieze Patterns

- Name the transformation that results in the frieze pattern



- Name the transformation that results in the frieze pattern



Horizontal Translation

Horizontal Translation

Or

180° Rotation

Horizontal Translation

Or

Reflection about a vertical line

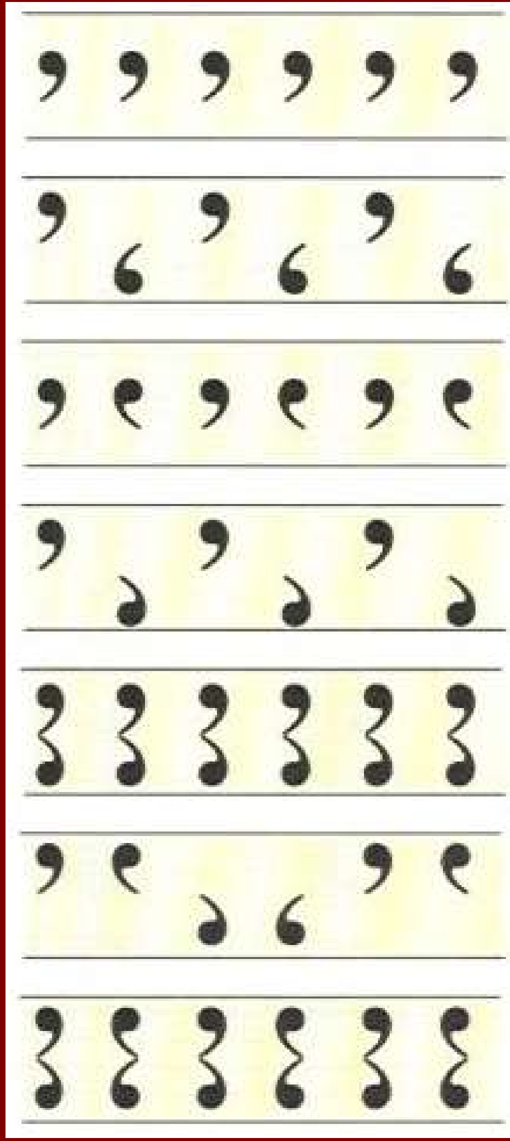
Horizontal Translation

Or

Horizontal glide reflection

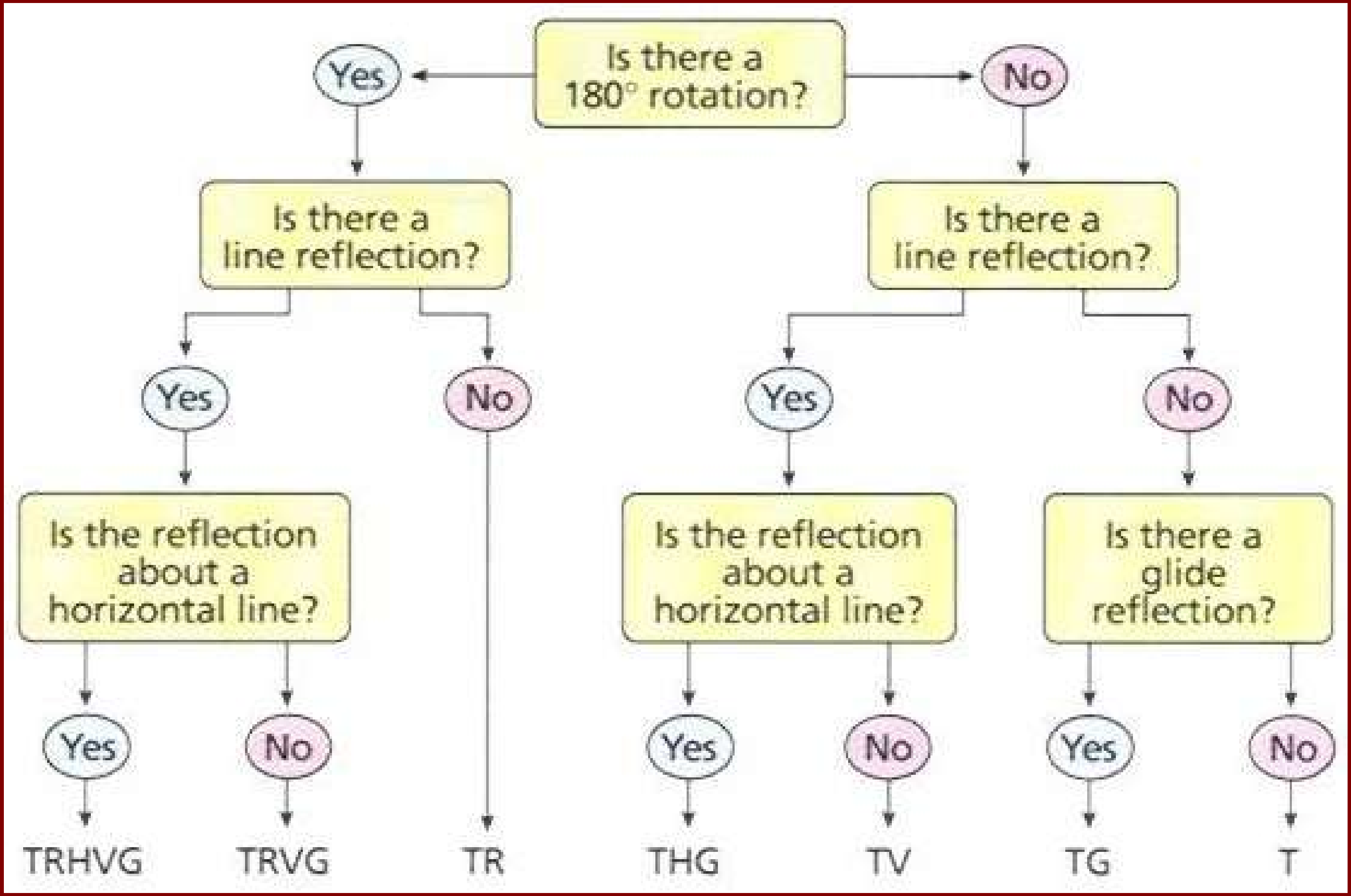
Frieze Patterns in Real-Life

7 Categories of Frieze Patterns



T	Translation only
TR	Translation and 180° rotation
TV	Translation and vertical line reflection
TG	Translation and glide reflection
THG	Translation, horizontal line reflection, and glide reflection
TRVG	Translation, 180° rotation, vertical line reflection, and glide reflection
TRHVG	Translation, 180° rotation, horizontal line reflection, vertical line reflection, and glide reflection

Classifying Frieze Patterns Using a Tree Diagram

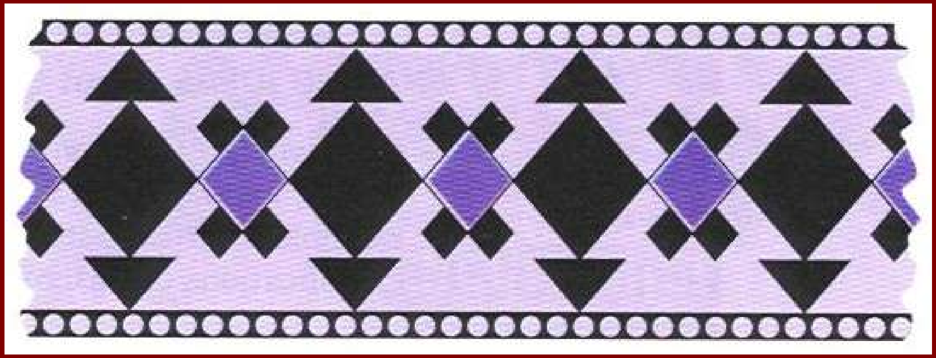


Example 2
Classifying Frieze Patterns

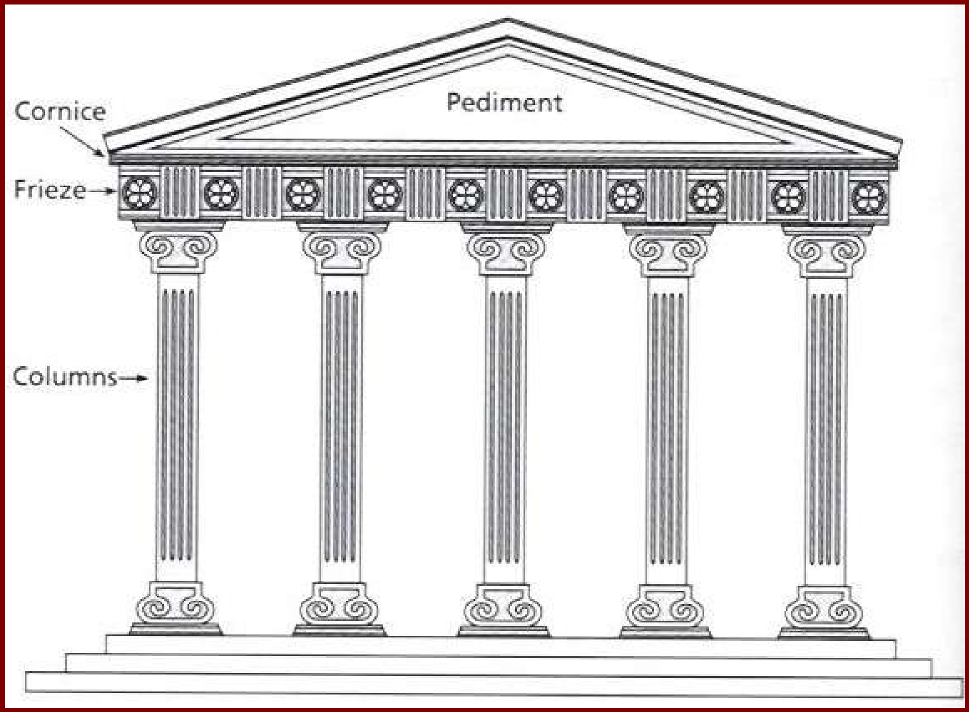
- What kind of frieze pattern is represented?

Example 2
Classifying Frieze Patterns

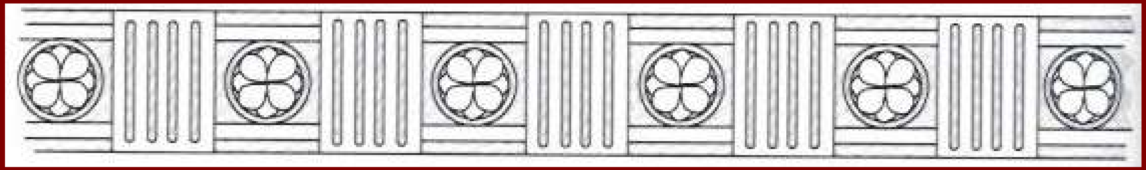
- What kind of frieze pattern is represented?
- TRHVG
- It can be mapped onto itself by a translation, a 180° rotation, a reflection about a horizontal or vertical line, or a glide reflection



Example 3 Classifying a Frieze Pattern



In architecture, the term frieze refers to the horizontal band between the cornice and the columns, as shown at the right. Many Greek and Roman buildings used friezes that were decorated with repeating patterns.



A portion of the frieze pattern on the above building is shown. Classify the frieze pattern.

TRHVG