Polynomial Patterns Learning Task

Part 1 – What is a Polynomial?

Let's break down the word: poly- and –nomial. What does "poly" mean?

1. A *monomial* is a numeral, variable, or the product of a numeral and one or more variables. For example: -1, ½, 3x, 2xy. Give a few examples of other monomials:
2. What is a *constant*? Give a few examples:
3. A *coefficient* is the numerical factor of a monomial or the _____ in front of the variable in a monomial.
4. Give some examples of monomials and their coefficients.
5. What is the degree of the monomial $4x^2$? $-7x$? 3?
6. Explain in your own words what a polynomial is.

Part 2 – Polynomial Functions and Degree

A ***polynomial function*** is defined as a function, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \Lambda + a_{n-2}x^2 + a_{n-1}x^1 + a_n$, where the coefficients are real numbers.

The ***degree of a polynomial*** (n) is equal to the greatest exponent of its variable.

The ***coefficient*** of the variable with the greatest exponent (a_0) is called the leading coefficient. For example, f(x)= $4x^3 - 5x^2 + x - 8$ is a third degree polynomial with a leading coefficient of 4.

Previously, you have learned about linear functions, which are first degree polynomial functions, y= $a_0x^1 + a_1$, where a_0 is the slope of the line and a_1 is the intercept (Recall: y=mx+b; here m is replaced by a_0 and b is replaced by a_1.)

Also, you have learned about quadratic functions, which are 2nd degree polynomial functions, which can be expressed as y = $a_0x^2 + a_1x^1 + a_2$.

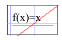
A cubic function is a third degree polynomial function. There are also fourth, fifth, sixth, etc. degree polynomial functions.

1. To get an idea of how these functions look, we can graph the first through fifth degree polynomials with leading coefficients of 1. For each polynomial function, make a table of 7 points and then plot them so that you can determine the shape of the graph. Choose points that are both positive and negative so that you can get a good idea of the shape of the graph. Also, include the x intercept as one of your points.

   For example, for the first order polynomial function: $y = x^1$. You may have the following table and graph:

x	$y = x^1$	$y = x^3$	$y = x^5$
-3			
-2			
-1			
0			
1			
2			
3			

x	$y=x^2$	$y=x^4$
-3		
-2		
-1		
0		
1		
2		
3		



2. Compare your five graphs. By looking at the graphs, describe in your own words how $y=x^2$ is different from $y=x^4$. Also, how is $y=x^3$ different from $y=x^5$?

3. In this unit, we are going to discover different characteristics of polynomial functions by looking at patterns in their behavior. Polynomials can be classified by the **number** of monomials, or *terms*, as well as by the **degree of the polynomial**. The degree of the polynomial is the same as the term with the highest degree. Complete the following chart. Make up your own expression for the last row.

Example	Degree	Name	No. of terms	Name
2		Constant		Monomial
$2x^2+3$		Quadratic		Binomial
$-x^3$		Cubic		
x^4+3x^2		Quartic		
$3x^5-4x+2$		Quintic		Trinomial
$(x+6)(x-3)$				
$x^2(x-5)(x+2)(x-1)$				
$(x+4)(x-5)(x+5)$				

Part 3 – Roots of Polynomials

This task will have you explore different characteristics of polynomial functions. In order to examine their characteristics in detail so that we can find the patterns that arise in the behavior of polynomial functions, we can study some examples of polynomial functions and their graphs. Attached to this task are 8 polynomial functions and their accompanying graphs that we will refer back to throughout the task.

Each of the 8 equations can be re-expressed as a product of linear factors by factoring the equations. Both expressions for each function are shown.

1. **1.** List the x-intercepts of each function with its graph. How are these intercepts related to the linear factors?

2. **2.** Why might it be useful to know the linear factors of a polynomial function?

3. **3.** Although we will not factor higher order polynomial functions in this unit, you have factored quadratic functions in Math II. For review, factor the following second degree polynomials, or quadratics.

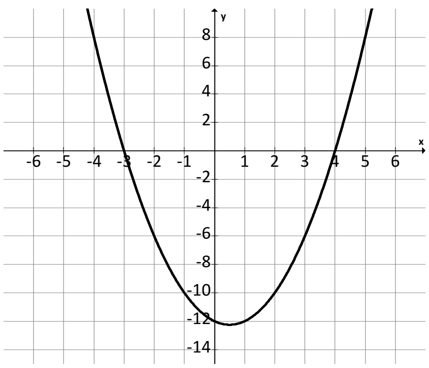
   $y = x^2 - x - 12$

   $y = x^2 + 5x - 6$

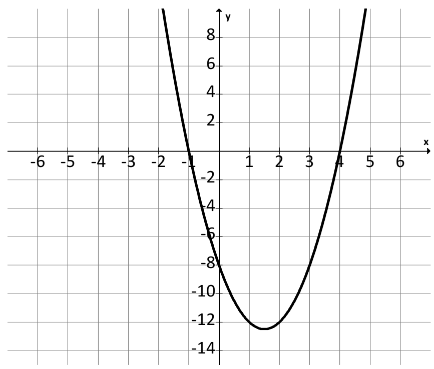
   $y = 2x^2 - 6x - 8$

4. **4.** Using these factors, find the roots of these three equations.

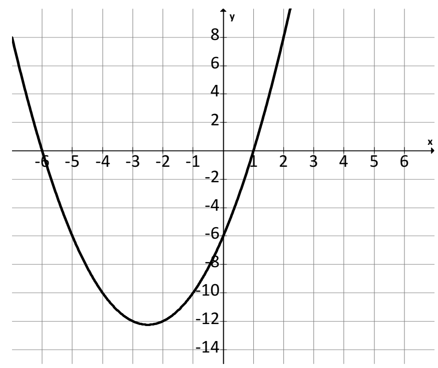
5. **5.** Now knowing the roots (x-intercepts) of these equations, match each equation to its graph.



y = ________________



y = _________________



y = _________________

6. **6.** Although you will not need to be able to find all of the roots of higher order polynomials until a later unit, using what you already know, you can factor some polynomial equations and find their roots in a similar way.
   Try this one: $y = x^5 + x^4 - 2x^3$.

   What are the roots of this fifth order polynomial function?

   How many roots are there?

   Why are there not five roots since this is a fifth degree polynomial?

   Check the roots by generating a graph of this equation using your calculator.

7. **7.** For other polynomial functions, we will not be able to draw upon our knowledge of factoring quadratic functions to find roots. For example, you may not be able to factor $y = x^3 + 8x^2 + 5x - 14$, but can you still find its zeros by graphing it in your calculator? How?

   What are the roots of this polynomial function?

Part 4 – Symmetry

Another characteristic of polynomial functions that we will consider is symmetry.

1. Sketch a function you have seen before that has symmetry about the y-axis.

   Describe in your own words what it means to have symmetry about the y-axis.

   What is do we call a function that has symmetry about the y-axis?

2. Sketch a function you have seen before that has symmetry about the origin (point symmetry).

   Describe in your own words what it means to have symmetry about the origin.

   What do we call a function that has symmetry about the origin?

3. Using the table below and your handout of the following eight polynomial functions, classify the functions by their symmetry.

Function	Symmetry about the y axis?	Symmetry about the origin?	Even, Odd, o Neither
$f(x)=x^2-4x$			
$g(x)=x^3-25x$			
$h(x)=-x^3-4x^2+x+4$			
$i(x)=x^4-5x^2+4$			
$j(x)=-\frac{1}{3}x^2+12$			
$k(x)=-\frac{1}{2}x^5-3x^4+\frac{1}{2}x^3+18x^2-5x-24$			
$l(x)=-\frac{1}{4}x^5+4x^3$			
$m(x)=x^4-5x^3-19x^2+29x+42$			

4. Now, sketch your own higher order polynomial function (an equation is not needed) with symmetry about the y-axis (no parabolas).

5. Now, sketch your own higher order polynomial function with symmetry about the origin.

6. Using the table in part c, what do you notice about the exponents of functions that are symmetrical about the y axis? What do you notice about the exponents of functions that are symmetrical about the origin?

7. Why don't we talk about functions that have symmetry about the x-axis? Sketch a graph that has symmetry about the x-axis. What do you notice?

Part 5 – Domain and Range

Another characteristic of functions that you have studied is domain and range. For each polynomial function, determine the domain and range.

Function	Domain	Range

$f(x) = x^2 - 4x$		
$g(x) = x^3 - 25x$		
$h(x) = -x^3 - 4x^2 + x + 4$		
$i(x) = x^4 - 5x^2 + 4$		
$j(x) = -\frac{1}{3}x^2 + 12$		
$k(x) = -\frac{1}{2}x^5 - 3x^4 + \frac{1}{2}x^3 + 18x^2 - 5x - 24$		
$l(x) = -\frac{1}{4}x^5 + 4x^3$		
$m(x) = x^4 - 5x^3 - 19x^2 + 29x + 42$		

1. What do you notice about the domain of even functions compared to that of odd functions?
2. What comparisons can you make about the range of even and odd functions?

Part 6 – Zeroes

1. We can also describe the functions by determining some points on the functions. We can find the x-intercepts (roots) for each function as we discussed before. Under the column labeled "x-intercepts" write the ordered pairs (x,y) of each intercept and record the number of intercepts in the next column. Also record the degree of the polynomial (you will fill in the final two columns later).

Function	Degree	X-intercepts	Zeros	Number of Zeros
$f(x) = x^2 - 4x$	2	(0,0) (4,0)	0 and 4	2
$g(x) = x^3 - 25x$				
$h(x) = -x^3 - 4x^2 + x + 4$				
$i(x) = x^4 - 5x^2 + 4$				
$j(x) = -\frac{1}{3}x^2 + 12$				
$k(x) = -\frac{1}{2}x^5 - 3x^4 + \frac{1}{2}x^3 + 18x^2 - 5x - 24$				
$l(x) = -\frac{1}{4}x^5 + 4x^3$				
$m(x) = x^4 - 5x^3 - 19x^2 + 29x + 42$				

2. These x-intercepts are called the *zeros* of the polynomial functions. Why do you think they have this name?

3. Fill in the column labeled "Zeroes" by writing the zeroes that correspond to the x-intercepts of each polynomial function, and also record the number of zeroes each function has.

4. Make a conjecture about the relationship of degree of the polynomial and number of zeroes.

5. Test your conjecture by graphing the following polynomial functions using your calculator: $y = x^2, y = x^2(x-1)(x+4), y = x(x-1)^2$.

Function	Degree	X-Intercepts	Zeroes	Number of Zeroes
$y = x^2$		(0,0)		
$y = x^2(x-1)(x+4)$		(0,0); (0,-1);(0-4)		
$y = x(x-1)^2$				

6. How are these three functions different from the first eight functions?

7. Now amend your conjecture about the relationship of the degree of the polynomial and the number of x-intercepts.

8. Make a conjecture for the maximum number of x-intercepts the following polynomial function will have: $p(x) = 2x^{11} + 4x^6 - 3x^2$

Part 7 – End Behavior

In determining the range of the polynomial functions, you had to consider the *end behavior* of the functions, that is the value of $f(x)$ as x approaches infinity and negative infinity.

Polynomials exhibit patterns of end behavior that are helpful in sketching polynomial functions.

1. Graph each function in the table below on your calculator. Using the graph, decide if the function:

- is Even or Odd
- has a positive or negative leading coefficient
- rises or falls to the left
- rises or falls to the right

Complete the table by filling in the appropriate characteristic for each function.

Function	**Degree?**	**Leading Coefficient?**	**Left End Behavior?**	**Right End Behavior?**
$y = x$				
$y = x^2$				
$y = -3x$				
$y = 5x^4$				
$y = x^3$				
$y = 2x^5$				
$y = -x^2$				
$y = -3x^4$				
$y = -x^3$				
$y = -2x^5$				
$y = -3x^6$				
$y = 7x^3$				

2. Write a conjecture about the **end behavior**, whether it rises or falls at the ends, of a function of the form $f(x) = ax^n$ for each pair of conditions below. Then test your conjectures on some of the 8 polynomial functions graphed on your handout.

   Condition a: When *n* is even and *a* > 0,

   Condition b: When *n* is even and *a* < 0,

   Condition c: When *n* is odd and *a* > 0,
   Condition d: When *n* is odd and *a* < 0,

3. Based on your conjectures in part (b), sketch a fourth degree polynomial function with a negative leading coefficient.

4. Now sketch a fifth degree polynomial with a positive leading coefficient.

Now we can sketch the graph with the end behavior even though we cannot determine where and how the graph behaves otherwise without an equation or without the zeros.

If we are given the real zeros of a polynomial function, we can combine what we know about end behavior to make a rough sketch of the function.

5. Sketch the graph of the following functions using what you know about end behavior and zeros:
   a. $f(x) = (x - 2)(x - 3)$

   b. $f(x) = -x(x - 1)(x + 5)(x - 7)$

Part 8 – Critical Points

Other points of interest in sketching the graph of a polynomial function are the points where the graph begins or ends increasing or decreasing.

1. What does it mean for a point of a function to have an *absolute minimum* or an *absolute maximum*?

2. Which of the twelve graphs from **Part 6** have an absolute maximum?

   Which have an absolute minimum?

   What do you notice about the degree of these functions?

3. Can you ever have an absolute maximum AND an absolute minimum in the same function? If so, sketch a graph with both. If not, why not?

4. For each of the eight reference graphs, locate the turning points and the related intervals of increase and decrease, as you have determined previously for linear and quadratic polynomial functions. Then record which turning points are *relative minimum* (the lowest point on a given portion of the graph) and *relative maximum* (the highest point on a given portion of the graph) values.

Function	Degree	Turning Points	Intervals of Increase	Intervals of Decrease	Relative Minimum	Relative Maximum
f(x)						

g(x)						
h(x)						
i(x)						
j(x)						
k(x)						
l(x)						
m(x)						

5. Make a conjecture about the relationship of the degree of the polynomial and the number of turning points that the polynomial has. Recall that this is the maximum number of turning points a polynomial of this degree can have because these graphs are examples in which all zeros have a multiplicity of one.

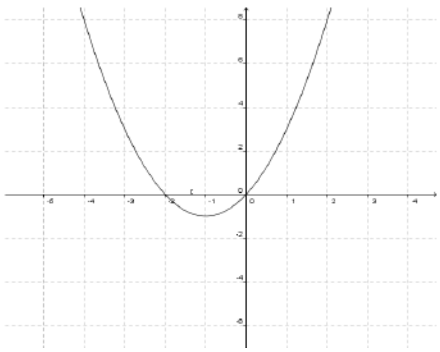
6. Sometimes points that are relative minimums or maximums are also absolute minimums or absolute maximum. Are any of the relative extrema in your table also absolute extrema?

Now that you have explored various characteristics of polynomial functions, you will be able to describe and sketch graphs of polynomial functions when you are given their equations.

If you are given the function: $f(x) = (x-3)(x+1)^2$, then what can you tell me about the graph of this function? Make a sketch of the graph of this function, describe its end behavior, and locate its critical point and zeroes.

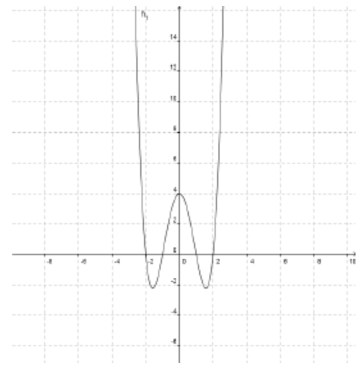
$f(x) = x^2 + 2x$

$f(x) = x(x + 2)$



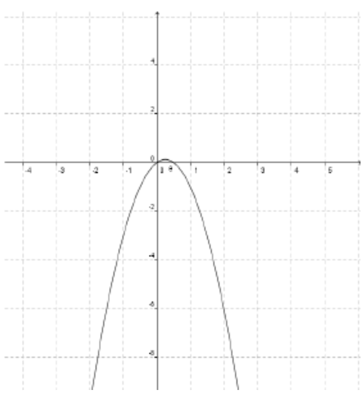
$k(x) = x^4 - 5x^2 + 4$

$k(x) = (x-1)(x+1)(x-2)(x+2)$



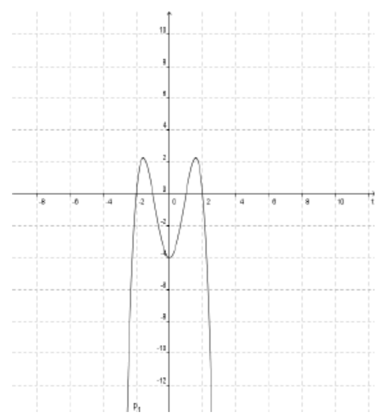
$g(x) = -2x^2 + x$

$g(x) = -x(2x-1)$



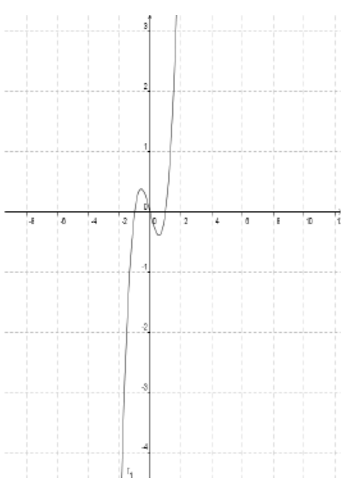
$l(x) = -(x^4 - 5x^2 + 4)$

$l(x) = -(x-1)(x+1)(x-2)(x+2)$



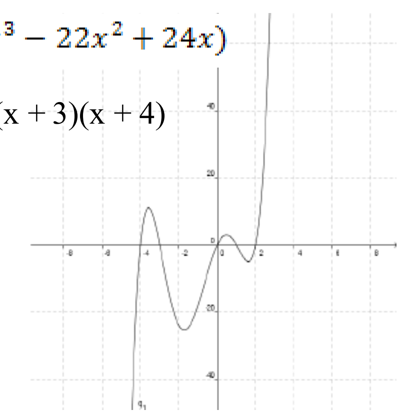
$h(x) = x^3 - x$

$h(x) = x(x-1)(x+1)$

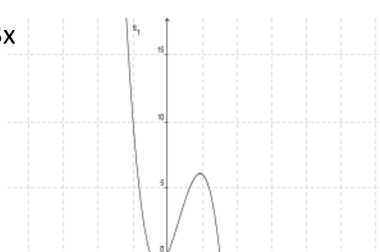


$m(x) = \frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$

$m(x) = \frac{1}{2}x(x-1)(x-2)(x+3)(x+4)$



$j(x) = -x^3 + 2x^2 + 3x$



$n(x) = -\frac{1}{2}(x^5 + 4x^4 - 7x^3 - 22x^2 + 24x)$;

$j(x) = -x(x-3)(x+1)$

$n(x) = \frac{-1}{2}x(x-1)(x-2)(x+3)(x+4)$