MOT Charter School Scope and Sequence
Discipline: Math
Subject: Integrated Math 1
Text Support: Interactive Mathematics Program, 2nd Edition

	Week	UNIT	BIG IDEAS	ESSENTIAL QUESTIONS/LEARNING TARGETS	Standards Alignment	Assessments
Aug/September	1 2 3 4 5	Patterns	Math as the science of patterns. Introduction on how to work on and think about mathematical problems. Using concrete mathematical models.	Finding, analyzing, and generalizing geometric and numeric patterns. Analyzing and creating in-out tables. Using variables to express generalizations. Developing and using general principles for working with variables, including the distributive property. Applying algebraic ideas in geometric settings Developing proofs concerning consecutive sums and other topics.	CC.A-SSE.1 CC.F-IF.1 CC.F-IF.3 CC.F-BF.1 CC.F-BF.1 CC.F-BF.1A CC.F-BF.2 CC.G-C0.1	Problem of the Week Written Assignments: • Calculator Exploration • Pulling Out Rules • You're the Chef • Consecutive Sums • Angular Summary • Border Varieties End of Unit In Class End of Unit Take Home Portfolio
October/November	6 7 8 9 10 11 12 13 14	Game of Pig	Probability. How to reason mathematically. How to communicate mathematical reasoning.	Calculating probabilities as fractions, decimals and percents by emphasizing equally likely outcomes. Constructing mathematical models, including area models and tree diagrams. Calculating and interpreting expected values. Using simulations to estimate probabilities and compare strategies. Comparing the theoretical analysis of a situation with experimental results. Examining how the number of trials in a simulation affects the results.	CC.S-ID.1 CC.S-ID.9 CC.S-IC.2 CC.S-CP.1 CC.S-CP.2 CC.S-CP.3 CC.S-CP.6 CC.S-CP.7 CC.S-CP.8 CC.S-MD.1 CC.S-MD.2 CC.S-MD.3 CC.S-MD.5 CC.S-MD.5a CC.S-MD.5b CC.S-MD.6 CC.S-MD.7	

MOT Charter School Scope and Sequence
Discipline: Math
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December/January	15 16 17 18 19 20 21	Overland Trail	Linear functions and their representations. How to write equations to represent specific contexts. Using tables, graphs, and symbols to solve linear equations and systems of linear equations. How to use graphs and symbols representing lines to solve problems.	Developing numeric algorithms and expressing algorithms in words and symbols. Developing and interpreting algebraic expressions. Using graphs to represent two-variable equations and data sets. Using graphs, tables, and algebraic relationships to describe situations. Solving linear equations for one variable in terms of another. Solving problems involving two linear conditions. Solving linear equations in one variable. Making and interpreting graphs on a graphing calculator.	CC.N-Q.1 CC.N-Q.2 CC.A-SSE.1a CC.A-SSE.2 CC.A-SSE.3 CC.A-CED.1 CC.A-REI.1 CC.A-REI.3 CC.A-REI.10 CC.A-REI.11 CC.F-IF.2 CC.F-IF.4 CC.F-IF.5 CC.F-IF.6 CC.F-IF.7 and CC.F-IF.7b CC.F-LE.1a and 1b CC.F-LE.5 CC.S-ID.6, 6, 6b & 6c CC.S-ID.7 CC.S-ID.8	

MOT Charter School Scope and Sequence
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February/March	22 23 24 25 26 27 28 29	Pit & Pendulum	Recognizing the normal distribution as a model for certain kinds of data. Applying standard deviation and the normal distribution in problem contexts. Using standard deviation to decide whether a variation in experimental results is significant.	Collecting and analyzing data. Expressing experimental results and other data using frequency bar graphs. Making area estimates to understand the normal distribution Developing concepts of data spread, especially standard deviation . Working with symmetry and concavity in connection with the normal distribution and standard deviation Distinguishing between population standard deviation and sample standard deviation. Calculating the mean and standard deviation of data sets, both by hand and with calculators. Using function notation. Fitting a function to data using a graphing calculator. Making predictions based on curve-fitting.	CC.N-Q.3 CC.A-SSE.1b CC.F-IF.7 & 7b CC.F-BF.3 CC.F-LE.2 CC.S-ID.2 CC.S-ID.3 CC.S-ID.4 CC.S-IC.1 CC.S-MD.4	

MOT Charter School Scope and Sequence
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April	30	Shadows	Similarity and Congruence		CC.A-CED.4 CC.G-CO.1	
	31		Algebra of proportions		CC.G-CO.6 CC.G-CO.7	
	32		Logical Reasoning and proof		CC.G-CO.8 CC.G-CO.9	
May/June	33				CC.G-CO.10 CC.G-CO.11	
	34				CC.G-SRT.1a & 1b CC.G-SRT.2	
	35				CC.G-SRT.3 CC.G-SRT.4	
	36				CC.G-SRT.5 CC.G-SRT.6	
	37				CC.G-SRT.7 CC.G-SRT.8 CC.G-GPE.6	

MOT Charter School Scope and Sequence
Discipline: Math
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Text Support: Interactive Mathematics Program, 2nd Edition

	Week	UNIT	BIG IDEAS	ESSENTIAL QUESTIONS/LEARNING TARGETS	Standards Alignment	Assessments
Aug/September	1 2 3 4 5	Patterns	Math as the science of patterns. Introduction on how to work on and think about mathematical problems. Using concrete mathematical models.	Finding, analyzing, and generalizing geometric and numeric patterns. Analyzing and creating in-out tables. Using variables to express generalizations. Developing and using general principles for working with variables, including the distributive property. Applying algebraic ideas in geometric settings Developing proofs concerning consecutive sums and other topics.	CC.A-SSE.1 CC.F-IF.1 CC.F-IF.3 CC.F-BF.1 CC.F-BF.1 CC.F-BF.1A CC.F-BF.2 CC.G-CO.1	Problem of the Week Written Assignments: • Calculator Exploration • Pulling Out Rules • You're the Chef • Consecutive Sums • Angular Summary • Border Varieties End of Unit In Class End of Unit Take Home Portfolio
October/November	6 7 8 9 10 11 12 13 14	Game of Pig	Probability. How to reason mathematically. How to communicate mathematical reasoning.	Calculating probabilities as fractions, decimals and percents by emphasizing equally likely outcomes. Constructing mathematical models, including area models and tree diagrams. Calculating and interpreting expected values. Using simulations to estimate probabilities and compare strategies. Comparing the theoretical analysis of a situation with experimental results. Examining how the number of trials in a simulation affects the results.	CC.S-ID.1 CC.S-ID.9 CC.S-IC.2 CC.S-CP.1 CC.S-CP.2 CC.S-CP.3 CC.S-CP.6 CC.S-CP.7 CC.S-CP.8 CC.S-MD.1 CC.S-MD.2 CC.S-MD.3 CC.S-MD.5 CC.S-MD.5a CC.S-MD.5b CC.S-MD.6 CC.S-MD.7	Problem of the Week Written Assignments: • Pig Strategies • 0 to 1, or Never to Always • Two-Dice Sums and Products • Spinner Give and Take • Spins and Draws • A Fair Deal for the Carrier? • Little Pig Strategies • The Best Little Pig End of Unit In Class End of Unit Take Home Portfolio

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December/January	15 16 17 18 19 20 21	Overland Trail	Linear functions and their representations. How to write equations to represent specific contexts. Using tables, graphs, and symbols to solve linear equations and systems of linear equations. How to use graphs and symbols representing lines to solve problems.	Developing numeric algorithms and expressing algorithms in words and symbols. Developing and interpreting algebraic expressions. Using graphs to represent two-variable equations and data sets. Using graphs, tables, and algebraic relationships to describe situations. Solving linear equations for one variable in terms of another. Solving problems involving two linear conditions. Solving linear equations in one variable. Making and interpreting graphs on a graphing calculator.	CC.N-Q.1 CC.N-Q.2 CC.A-SSE.1a CC.A-SSE.2 CC.A-SSE.3 CC.A-CED.1 CC.A-REI.1 CC.A-REI.3 CC.A-REI.10 CC.A-REI.11 CC.F-IF.2 CC.F-IF.4 CC.F-IF.5 CC.F-IF.6 CC.F-IF.7 and CC.F-IF.7b CC.F-LE.1a and 1b CC.F-LE.5 CC.S-ID.6, 6, 6b & 6c CC.S-ID.7 CC.S-ID.8	Problem of the Week Written Assignments: • Creating Families • Laced Travelers • Ox Expressions at Home • Graph Sketches • Who Will Make It? • All Four, One--Linear Functions • Straight Line Reflections • More Fair Share for Hired Hands • Family Comparisons by Algebra End of Unit In Class End of Unit Take Home Portfolio First Semester Assessment

MOT Charter School Scope and Sequence
Discipline: Math
Subject: Integrated Math 1
Text Support: Interactive Mathematics Program, 2nd Edition

February/March	22 23 24 25 26 27 28 29	Pit & Pendulum	Recognizing the normal distribution as a model for certain kinds of data. Applying standard deviation and the normal distribution in problem contexts. Using standard deviation to decide whether a variation in experimental results is significant.	Collecting and analyzing data. Expressing experimental results and other data using frequency bar graphs. Making area estimates to understand the normal distribution Developing concepts of data spread, especially standard deviation . Working with symmetry and concavity in connection with the normal distribution and standard deviation Distinguishing between population standard deviation and sample standard deviation. Calculating the mean and standard deviation of data sets, both by hand and with calculators. Using function notation. Fitting a function to data using a graphing calculator. Making predictions based on curve-fitting.	CC.N-Q.3 CC.A-SSE.1b CC.F-IF.7 & 7b CC.F-BF.3 CC.F-LE.2 CC.S-ID.2 CC.S-ID.3 CC.S-ID.4 CC.S-IC.1 CC.S-MD.4	Problem of the Week Written Assignments: • Initial Experiments • Pulse Analysis • Kai and Mai Spread Data • Penny Weight Revisited • Pendulum Conclusions • Graphing Summary • Mathematics and Science End of Unit In Class End of Unit Take Home Portfolio

MOT Charter School Scope and Sequence
Discipline: Math
Subject: Integrated Math 1
Text Support: Interactive Mathematics Program, 2nd Edition

April	30 31 32	Shadows	Similarity and Congruence Algebra of proportions Logical Reasoning and proof		CC.A-CED.4 CC.G-CO.1 CC.G-CO.6 CC.G-CO.7 CC.G-CO.8 CC.G-CO.9 CC.G-CO.10 CC.G-CO.11 CC.G-SRT.1a & 1b CC.G-SRT.2 CC.G-SRT.3 CC.G-SRT.4 CC.G-SRT.5 CC.G-SRT.6 CC.G-SRT.7 CC.G-SRT.8 CC.G-GPE.6	Problem of the Week Written Assignments: • Shadow Data Gathering and Working with Shadow Data • Similar Problems • Angles and Counterexamples • Angles, Angles, Angles • Mirror Madness • A Shadow of a Doubt • The Tree and the Pendulum • A Bright, Sunny Day End of Unit In Class End of Unit Take Home Portfolio End of 2nd Semester Assessment
May/June	33 34 35 36 37					

MOT Charter School Scope and Sequence
Discipline: Math
Subject: Integrated Math 2
Text Support: Interactive Mathematics Program, 2nd Edition

	Week	UNIT	BIG IDEAS	ESSENTIAL QUESTIONS/LEARNING TARGETS	Standards Alignment	Assessments
Aug/September	1 2 3 4 5	Do Bees Build it Best?	Why do bees build their honeycombs as a collection of right hexagonal prisms? Area The Pythagorean Theorem Surface Area and Volume Applying right triangle trigonometry	***Area*** • Understand the role of units in measuring area • Establish standard units for area, especially those based on units of length • Recognize that a figure's perimeter alone does not determine its area • Discover formulas for the areas of rectangles, triangles, parallelograms, and trapezoids • Establish that a square has the greatest area of all rectangles with a fixed perimeter • Develop a formula for the area of a regular polygon with a given perimeter in terms of the number of sides • Discover that for a fixed perimeter, the more sides a regular polygon has, the greater its area • Discover that the ratio of the areas of similar figures is equal to the square of the ratio of their corresponding linear dimensions ***The Pythagorean Theorem*** • Discover the Pythagorean theorem by comparing the areas of the squares constructed on the sides of a right triangle • Prove the Pythagorean theorem using an area argument • Apply the Pythagorean theorem in a variety of situations	CC.N-Q.3 CC.A-REI.2 CC.REI.4b CC.F-IF.5 CC.G-SRT.4 CC.G-SRT.8 CC.G-SRT.10 CC.G-SRT.11 CC.G-GMD.1 CC.G-CMD.2 CC.G-CMD.3 CC.G-GMD.4 CC.G-MG.3	Problem of the Week Written Assignments: • How Many Can You Find? • That's All There Is! • More Gallery Measurements • Any Two Sides Work, Make the Lines Count, and The Power of Pythagoras • Leslie's Fertile Flowers • More Fencing, Bigger Corrals • Not a Sound End of Unit In Class End of Unit Take Home Portfolio
October	6 7 8					

MOT Charter School Scope and Sequence
Discipline: Math
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				Surface Area and Volume • Understand the role of units in measuring surface area and volume • Establish standard units for surface area and volume, especially those based on a unit of length • Recognize that a solid figure's surface area alone does not determine its volume • Develop principles relating the volume and surface area of a prism to the area and perimeter of its base • Discover that the ratio of the surface areas of similar solids is equal to the square of the ratio of their corresponding linear dimensions, and that the ratio of the volumes of similar solids is equal to the cube of the ratio of their corresponding linear dimensions ***Trigonometry*** • Understand right-triangle trigonometry • Find the ranges of the basic trigonometric functions (for acute angles) • Know the terminology and notation of inverse trigonometric functions • Examine the concept of tessellation and discovering which regular polygons tessellate • Develop some properties of square-root radicals • Develop the general concept of an inverse function		

MOT Charter School Scope and Sequence
Discipline: Math
Subject: Integrated Math 2
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	UNIT	BIG IDEAS	ESSENTIAL QUESTIONS/LEARNING TARGETS	Standards Alignment	Assessments
9 10 11 12 13 14 15 16	Cookies	Using variables to represent problems Variables, equations and inequalities Graphing linear inequalities and systems of linear inequalities	Express real-world situations in terms of equations and inequalities. Solve linear equations for one variable in terms of another. Use several methods for solving systems of linear equations in two variables. Use graphing calculator to solve non-routine equations. Find the equation of a straight line and the inequality for a half plane. Write and graph linear inequalities in two variables. Develop methods for solving linear programming problems with two variables, and creating problems that can be solved with these methods. Understand dependent, inconsistent, and independent pairs of linear equations. Understand that setting a linear expression equal to a series of constraints produces a family of parallel lines. Creating problems that can be solved using two equations in two unknowns.	CC.A-CED.1 CC.A-CED.2 CC.A.CED.3 CC.A.REI.3 CC.A.REI.5 CC.A-REI.6 CC.A.REI.7 CC.A.REI.12 CC.F-IF.4 CC.F-IF.7a	Problem of the Week Written Assignments: • Inequality Stories, Part I • Profitable Pictures • Changing What You Eat • Get the Point: This investigation will give you insight into students' abilities to think about systems of linear equations in flexible ways. • A Reflection on Money • "How Many of Each Kind?" Revisited • Producing Programming Problems End of Unit In Class End of Unit Take Home Portfolio

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January	17 18 19 20	Is There Really a Difference?	Setting up Statistical Investigations. Understanding the role of the null hypothesis in statistical reasoning. Interpreting Data. Using the chi-square statistic to make decisions.	Design, conduct and interpret a statistical experiment. Make and test hypotheses by analyzing sample data. Construct and draw inferences from charts, tables and graphs, including frequency bar graphs and double-bar graphs. Formulate a null hypothesis. Determine whether to accept or reject a null hypothesis and understand the consequence of rejecting a null hypothesis. Understand the role of the null hypothesis in statistical reasoning. Use the chi-square statistic to make decisions. Understand some limitations in applying the chi-square statistic. Solve problems that involve conditional probability.	CC.S-ID.5 CC.S-ID.9 CC.S-IC.1 CC.S-IC.2 CC.S-IC.3 CC.S-IC.4 CC.S-IC.5 CC.S-IC.6 CC.S-CP.4 CC.S-CP.5 CC.S-MD.4	Problem of the Week Written Assignments: • Changing the Difference, Part I • Loaded or Not? • Decisions with Deviation • Measuring Weirdness with • Late in the Day • "Two Different Differences" Revisited End of Unit In Class End of Unit Take Home Portfolio
February	21 22 23 24					
March	25					

MOT Charter School Scope and Sequence
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April	26 27 28 29 30	Fireworks	How to calculate how long a fireworks rocket will take to reach the top of its trajectory, how high it will be when in reaches the top, and how long it will take to fall back to the ground. • Quadratic Functions and their representations • Graphs of quadratic functions • Solving quadratic equations • Understand the connection between algebra and geometry.	***Mathematical Modeling*** • Express real-world situations in terms of functions and equations • Interpret mathematical results in terms of real-world situations ***Graphs of Quadratic Functions*** • Understand the roles of the vertex and x-intercept in the graphs of quadratic functions • Recognize the significance of the sign of the x^2 term in determining the orientation of the graph of a quadratic function • Use graphs to understand and solve problems involving quadratic functions ***Working with Algebraic Expressions*** • Use an area model to understand multiplication of binomials, factoring of quadratic expressions, and completing the square of quadratic expressions • Transform quadratic expressions into vertex form • Simplify expressions involving parentheses • Identify certain quadratic expressions as perfect squares ***Solving Quadratic Equations*** • Interpret quadratic equations in terms of graphs and vice versa • Estimate x-intercepts using a graph • Find roots of an equation using the vertex form of the corresponding function • Use the zero product rule of multiplication to solve equations by factoring	CC.A-SSE.2 CC.A-SSE.3a & 3b CC.A-APR.1 CC.A-APR.2 CC.A-APR.3 CC.A-REI.4, 4a, 4b CC.F-IF.7a, 7c CC.F-IF.8, 8a CC.F-IF.9 CC.F-BF.3	Problem of the Week Written Assignments: • Using Vertex Form • Squares and Expansions • How Much Can they Drink? • Another Rocket • A Firework's Summary • A Quadratic Summary End of Unit In Class End of Unit Take Home Portfolio

May/June	31 32 33 34 35 36 37	All About Alice	Exponents, Scientific Notation & Logarithms	***Extending the Operation of Exponentiation*** Define the operation for an exponent of zero Define the operation for negative integer exponents Define the operation for fractional exponents ***Laws of Exponents*** Develop the additive law of exponents Develop the law of repeated exponentiation ***Graphing*** Describe the graphs of exponential functions Compare graphs of exponential functions for different bases Describe the graphs of logarithmic functions Compare graphs of logarithmic functions for different bases ***Logarithms*** Understand the meaning of logarithms Make connections between exponential and logarithmic equations ***Scientific Notation*** Convert numbers from ordinary notation to scientific notation, and vice versa Develop principles for doing computations using scientific notation Use the concept of order of magnitude in estimation	CC.N-RN.1 CC.N-RN.2 CC.N-RN.3 CC.S-ID.4 CC.S-IC.1 CC.A-SSE.4 CC.F-IF.7e CC.F-IF.8b CC.F-BF.4, 4a, 4b, 4c CC.F-BF.5 CC.F-LE.1, 1a, 1c CC.F-LE.3	Problem of the Week Written Assignments: • Calculator Exploration • Pulling Out Rules • You're the Chef • Consecutive Sums • Angular Summary • Border Varieties End of Unit In Class End of Unit Take Home Portfolio

MOT Charter School Scope and Sequence
Discipline: Math
Subject: Integrated Math 3
Text Support: Interactive Mathematics Program, 2nd Edition

	Week	UNIT	BIG IDEAS	ESSENTIAL QUESTIONS/LEARNING TARGETS	Standards Alignment	Assessments
Aug/September	1 2 3 4 5	Orchard Hideout	*How soon after planting an orchard will the center of the lot become a true "orchard hideout"?* ✓ *Coordinate Geometry* ✓ *Circles* ✓ *Writing proofs*	Develop and apply the distance formula. Develop the standard form for the equation of a circle with a given center and radius. Find the distance from appoint to a line in a coordinate setting. Develop and apply the midpoint formula. Explain the relationship of the area and circumference of a circle to its radius. Using circumscribed polygons to see that the "circumference coefficient" for the circle is twice the "area coefficient" for the circle. Develop and apply the formulas for the circumference and area of a circle. Find the formula for the perimeter and area of regular polygons circumscribed about a circle. Identify and describe a set of points satisfying a geometric condition. Prove that the set of points equidistant from two given points is the perpendicular bisector of the segment connecting the given points. Define the distance from a point to a line and	CC.G-CO.1 CC.G-CO.9 CC.G-CO.12 CC.G-CO.13 CC.G-SRT.9 CC.G-C.1 CC.G-C.2 CC.G-C.3 CC.G-C.4 CC.G-GPE.1 CC.G-GPE.2 CC.G-GPE.3 CC.G-GPE.4 CC.G-GPE.5 CC.G-GPE.7 CC.G-GMD.3 CC.G-MG.1	Problem of the Week Written Assignments: • Sprinkler in the Orchard • Proving with Distance • Polygoning the Circle • Orchard Growth Revisited • Cable Ready • Hiding in the Orchard End of Unit In Class End of Unit Take Home Portfolio
October	6 7					

				prove that the perpendicular distance is the shortest. Prove that any line through the midpoint of a segment is equidistant from the endpoints of the segment. Prove that the set of points equidistant from two intersecting lines consists of the bisectors of the angles formed by the lines. Use the technique of completing the square t transform equations of circles into standard form. Use "if and only if" in describing sets of points fitting given criteria. Use the concept of the converse of a statement.		

MOT Charter School Scope and Sequence
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November	8	Meadows or Malls?	How to solve a linear programming problem in six variables, using matrices to solve systems of linear equations.	Generalize the method of finding corner points to more than two variables. Use substitution, graphing, and guess-and-check to solve systems of linear equations in two variables. Use the elimination method to solve systems of linear equations in two or more variables. Use the concepts of inconsistent, dependent, and independent systems of equations. Extend the concepts of coordinates to three variables by introducing a third axis perpendicular to the first two. Graph linear equations in three variables and understand that these graphs are planes in 3-space. Understand that 2 distinct points always determine a unique line and that two distinct lines in the plane determine a unique point unless the lines are parallel. Understand the possible intersections of planes in 3-space. Relate the possible intersections of lines and planes to the algebra of solving linear systems in two or three variables. Use matrices to represent information.	CC.N-VM.6 CC.N-VM.7 CC.N-VM.8 CC.N-VM.9 CC.N-VM.10 CC.N-VM.11 CC.N-VM.12 CC.A-CED.3 CC.A.REL.8 CC.A-REL.9	Problem of the Week Written/Oral Assignments: • Programming Puzzles • Just the Plane Facts • Three Variables, Continued • Matrices in the Oven • Inverses and Equations • Meadows or Malls, Revisited End of Unit In Class End of Unit Take Home Portfolio
	9					
	10					
	11					
	12					
	13					
	14					
	15					
	16					
January	17					

				Use the operations of matrix addition and multiplication to solve problems. Understand that systems of linear equations are equivalent to certain types of matrix equations. Understand the role of identity and inverse elements in solving certain types of matrix equations. Find matrix inverses by hand by solving systems of linear equations. Understand the relationship between a system of linear equations having a unique solution and the coefficient matrix being invertible. Use a graphing calculator to multiply and invert matrices and to solve systems of linear equations. Apply the concepts of linear programming to problems with several variables. Use equations of planes in 3-space coordinate geometry. Define polar coordinates.		

February	18, 19, 20, 21, 22, 23, 24	Small World, Isn't It?	*If population growth continues according to its current pattern, how long will it be until people are squashed up against one another?* Using population growth to understand how to fit an exponential function to a set of data.	Evaluate average rate of change in terms of the coordinates of points on a graph. Understand the relationship between the rate of change of a function and the appearance of its graph. Develop an algebraic definition of slope. Using similarity, prove that a line has a constant slope. Understand the significance of negative slope for a graph and an applied context. Understand that the slope of a line is equal to the coefficient of x in the y=a+bx representation of the line. Use slope to develop equations for lines. Develop the concept of the derivative of a function at a point. Understand that the derivative of a function at a point is the slope of the tangent line at that point. Find numerical estimates for the derivatives of functions at specific points.	CC.A-SSE.3c CC.A-SSE.4 CC.F-IF.5 CC.F-IF.8b CC.F-BF.2 CC.F-LE.1c CC.F-LE.2 CC.F-LE.4 CC.G-MG.2 CC.S-ID.6a	Problem of the Week Written/Oral Assignments: • How Many More People? • Points, Slopes, and Equations • Photo Finish • What's It All About? • Slippery Slopes • Return to "A Crowded Place" End of Unit In Class End of Unit Take Home Portfolio

				Understand that for functions of the form y=b ^x the derivative at each point of the graph is proportional to the y-value at that point. Use exponential functions to model real-life situations. Apply the principles that a^b*a^c=a^b+c. Understand that a^logab = b. Understand that any exponential function can be expressed using any positive number other than 1 as a base. Understand the meaning of the terms natural logarithm and common logarithm. Use an exponential function to fit a curve to numerical data. Estimate the value of b for which the function y=b^x has a derivative at each point on its graph equal to the y-value at that point.		

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<table>
<tr><td rowspan="4">March</td><td>25</td><td rowspan="6">Pennant Fever</td><td rowspan="6">Probability

Combinatorial Coefficients

Binomial Theorem</td><td rowspan="6">Probability and statistics
• Develop a mathematical model for a complex probability situation
• Use area diagrams and tree diagrams to find and explain probabilities
• Use a simulation to understand a situation, to help analyze probabilities, and to support a theoretical analysis
• Find expected value
• Find and use probabilities for sequences of events
• Use specific problem contexts to develop the binomial distribution, and find a formula for the associated probabilities
• Use probability to evaluate null hypotheses
Counting principles
• Develop systematic lists for complex situations
• Use the multiplication principle for choosing one element from each of several sets
• Define and using the concepts of permutation and combination
• Understand and use standard notation for counting permutations and combinations
• Develop formulas for the permutation and combinatorial coefficients
Pascal's triangle and combinatorial coefficients
• Find patterns and properties within Pascal's triangle
• Recognize that Pascal's triangle consists of combinatorial coefficients</td><td rowspan="6">CC.A-APR.4
CC.A-APR.5
CC.S-CP.9</td><td rowspan="6">Problem of the Week
Written/Oral Assignments:
• Baseball Probabilities
• How Likely Is All Wins?
• Monthly Matches
• Cones from Bowls, Bowls from Cones
• Who's on First?
• About Bias
• Race for the Pennant Revisited
End of Unit In Class
End of Unit Take Home
Portfolio</td></tr>
<tr><td>26</td></tr>
<tr><td>27</td></tr>
<tr><td>28</td></tr>
<tr><td rowspan="2">April</td><td>29</td></tr>
<tr><td>30</td></tr>
</table>

				• Explain the defining pattern and other properties of Pascal's triangle using the meaning of combinatorial coefficients • Develop and explain the binomial theorem		
May/June	31 32 33 34 35 36 37	High Dive	*How can I use trigonometry and physics to determine when a diver should start his fall off of a stationary ferris wheel in order to land in a moving pool of water on the ground?*	***Trigonometry*** • Apply the trigonometric functions to all angles • Understand the importance of similarity in the definitions of the trigonometric functions • Graph the trigonometric functions and variations on those functions • Define the inverse trigonometric functions and principal values • Understand the Pythagorean identity $\sin^2 \vartheta + \cos^2 \vartheta = 1$, and other trigonometric identities • Define polar coordinates and finding rectangular coordinates from polar coordinates and vice versa ***Physics*** • Develop quadratic expressions for the height of free-falling objects, based on the principle of constant acceleration • Recognize that a person falling from a moving object will follow a different path than someone falling from a stationary object ***Quadratic Equations*** • Develop simple quadratic equations to describe the behavior of falling objects	CC.A-SSE.1b CC.A-APR.6 CC.A-APR.7 CC.F-IF.4 CC.F-IF.7e CC.F-TF.2 CC.F-TF.5 CC.F-TF.7 CC.F-TF.8	Problem of the Week Written/Oral Assignments • As the Ferris Wheel Turns • Testing the Definition • More Beach Adventures • A Practice Jump • Moving Cart, Turning Ferris Wheel End of Unit In Class End of Unit Take Home Portfolio

MOT Charter School Scope and Sequence
Discipline: Math
Subject: Integrated Math 4
Text Support: Interactive Mathematics Program, 2nd Edition

	Week	UNIT	BIG IDEAS	ESSENTIAL QUESTIONS/LEARNING TARGETS	Standards Alignment	Assessments
Aug/September	1 2 3 4 5	**THE DIVER RETURNS**	*How can I use trigonometry and physics to determine when a diver should start his fall off of a stationary ferris wheel in order to land in a moving pool of water on the ground?* A person falling from a moving object will follow a different path than someone falling from a stationary object	**Trigonometry and Geometry** • Apply extended trigonometric functions • Apply the principle that the tangent to a circle is perpendicular to the radius at the point of tangency **Physics** • Express velocity in terms of vertical and horizontal components • Represent the motion of falling objects when the vertical and horizontal components of the initial velocity are both nonzero **Quadratic Equations** • Recognize the importance of quadratic equations in the analysis of falling objects • Develop the quadratic formula • Use the quadratic formula to solve quadratic equations • Find a general solution for the falling time of objects with an initial vertical velocity **Complex Numbers** • Understand the need to extend the number system to solve certain quadratic equations • Establish basic ideas about complex number arithmetic • Represent complex numbers in the plane and seeing addition of complex numbers as a vector sum	N-CN.1 N-CN.2 N-CN.3 N-CN.4 N-CN.5 N-CN.6 N-CN.7 N-CN.8 N-CN.9 N-VM.1 N-VM.2 N-VM.3 N-VM.4, 4a, 4b, 4c N-VM.5, 5a, 5b	Problem of the Week Written/Oral Assignments: • As the Ferris Wheel Turns • Free Fall • The Simplified Dive, Revisited • Big Push • Complex Numbers and Quadratic Equations • Three O'Clock Drop • Vector Velocities • The Diver's Success End of Unit In Class End of Unit Take Home Portfolio
October	6					

	7 8 9 10 11 12 13 14 15	**THE WORLD OF FUNCTIONS**		**General Notions Regarding Functions** • Recognizing four ways of representing a function—tabular, graphical, algebraic, and situational—and moving from one representation to another • Formally defining functions as sets of ordered pairs • Reviewing some basic families of functions **Properties of Specific Families of Functions** • Finding, describing, and proving patterns in the tables of linear, quadratic, cubic, and exponential functions based on the algebraic form of the functions • Seeing the sets of linear and exponential functions as two-parameter families and comparing the two types of growth • Applying the concepts of direct and inverse proportionality and constants of proportionality • Using absolute value functions and step functions to model problem situations • Using rational functions to model problem situations **End Behavior and Asymptotes of Functions** • Finding vertical and horizontal asymptotes for specific functions and finding functions with given asymptotes • Relating asymptotic behavior to situations • Characterizing end behavior of functions and finding the behavior of particular functions **Fitting Functions to Data**	F-IF.7c F-IF.7d F-BF.1b,1c F-BF.4 F-BF.4a F-BF.4b F-BF.4c	Problem of the Week Written/Oral Assignments: • What Good Are Functions? • Exponential Tables • Families Have Many Different Members • Name That Family! • The Cost of Pollution • Better Braking End of Unit In Class End of Unit Take Home Portfolio

				• Finding the specific function in a given family to fit a given situation or set of data • Developing a measure of "quality of fit" of a function to a set of data • Applying the least-squares criterion for quality of fit • Using a calculator's regression feature to find a function that fits a given set of data **Combining and Modifying Functions** • Describe situations using arithmetic combinations of functions • Relate arithmetic operations on functions to graphs • Define arithmetic operations on functions • Define composition notation • Understand that composition is not commutative • Compose and decompose functions • Define the concept of inverse function • Find a general algebraic equation for the inverse of a linear function • Relate the concept of inverse function to graphs, tables, and situations • Understand that the graph of an inverse function is a reflection of the graph of the original function • Find the graphs and tables of transformations of functions • Use functional notation and understand its use in characterizing the transformations of functions		

MOT Charter School Scope and Sequence
Discipline: Math
Subject: Integrated Math 4
Text Support: Interactive Mathematics Program, 2nd Edition

Month	Week	Unit		Content		Assessment
January	16 17 18 19 20	**THE POLLSTER'S DILEMMA**		**General Sampling Concepts** • Establish methods of good polling, including random sampling • Using sampling from a known population to analyze the reliability of samples • Distinguish and compare sampling with replacement and sampling without replacement • Understand the terminology true proportion and sample proportion • Identify simplifying assumptions in analyzing sampling **Specific Results on Sampling with Replacement** • Make probability bar graphs for various distributions • Understand the concept of a theoretical distribution for sampling results from a given population • Use combinatorial coefficients to find the theoretical distribution of poll results for polls of various sizes • Generalize that sampling results fit a binomial distribution **The Central Limit Theorem and the Normal Distribution** • Understand that as poll size increases, the distribution of sample proportions becomes approximately normal • Understand normal distribution • Use estimates of areas to understand the normal distribution table • Apply the central limit theorem for the case of binomial distributions		Problem of the Week Written/Oral Assignments: • Graphs of the Theory • Gifts Aren't Always Free • A Normal Poll • The Search Is On! • What Does It Mean? • "The Pollster's Dilemma" Revisited End of Unit In Class End of Unit Take Home Portfolio
February	21 22 23					

				Mean and Standard Deviation • The steps for computation of standard deviation • Understand that the "large number of trials" method for computing mean and standard deviation is independent of the number of trials • Apply the concepts of mean and standard deviation to probability distributions • Define the concept of variance • Find formulas for the mean and standard deviation of the distribution of poll results in terms of the poll size and the true proportion • Decide what to use for σ if the true proportion is unknown, and finding the maximum value of σ for polling problems **Confidence Levels and Margin of Error** • Know the terms confidence level, confidence interval, and margin of error • Understand how poll size affects the standard deviation of poll results • Establish confidence intervals in terms of sample proportions and standard deviation • Estimate the size of a poll based on the reported margin of error		

March	24 25 26 27 28	**HOW MUCH? HOW FAST?**		**Accumulation** • Understand that the area under a rate curve represents an accumulation • Estimate amount of total accumulation based on linear approximations of a situation • Create and analyze graphs for accumulation as a function of time **Derivatives** • Understand a derivative as an instantaneous rate of change • Estimate derivatives from graphs • Develop formulas for derivatives of simple polynomial functions • Develop formulas for derivatives of the sine and cosine functions • Establish principles for the derivative of a sum or constant multiple **The Fundamental Theorem of Calculus** • Understand that an accumulation function is an antiderivative of the corresponding rate function • Find areas and volumes using antiderivatives **Trigonometry** • Define radian measure • Use radians in sine and cosine functions **Geometry** • Develop formulas for the volumes of pyramids and cones	F-TF.1 F-TF.3 F-TF.4	Problem of the Week Written/Oral Assignments: • Leaky Faucet • A Distance Graph • Zero to Sixty • A Pyramid of Bright Ideas • A Solar Summary End of Unit In Class End of Unit Take Home Portfolio

MOT Charter School Scope and Sequence
Discipline: Math
Subject: Integrated Math 4
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May/June	29 30 31 32 33 34 35 36 37	**AS THE CUBE TURNS**		**Coordinate Geometry** • Express geometric transformations—translations, rotations, and reflections—in terms of coordinates in two and three dimensions • Find coordinates a fractional distance along a line segment in two and three dimensions • Find the projection of a point onto a plane from the perspective of a fixed point and developing an algebraic description of the projection process • Understand the effect of change of viewpoint on projections **Matrices** • Use matrices to express geometric transformations in two and three dimensions **Programming** • Learn to use a technical manual • Use loops in programming • Understand programs from their code • Design and program animations **Synthetic Geometry and Trigonometry** • Review formulas relating the sine of an angle to the cosine of a related angle • Derive the formula for the area of a triangle in terms of the lengths of two sides and the sine of the included angle • Derive formulas for the sine and cosine of the negative of an angle • Derive formulas for the sine and cosine of the sum of two angles and related variations	F-TF.5 F-TF.9 G-CO.2 G-CO.3 G-CO.4 G-CO.5 G-GMD.4	Problem of the Week Written/Oral Assignments: • Learning the Loops • Move That Line! • Oh, Say What You Can See • Swing That Line! • And Fred Brings the Lunch • Find Those Corners! End of Unit In Class End of Unit Take Home Portfolio Second Semester Assessment

Mathematics Unit Plan

Unit Title: All About Alice

Designed by: D. Fendel, D. Resek, L. Alper, S. Fraser, 2011, *Interactive Mathematics Year 2, Second Edition* (Emeryville, CA: Key Curriculum Press).

Grade: 8/9

Time Frame (Number of Lessons): 13 days

Summary of Unit

This unit uses Lewis Carroll's story Alice's Adventures in Wonderland as the context in which students define the exponential function, derive properties of exponents, and use exponents to solve problems.

Unlike most other IMP units, All About Alice has no central problem to solve. Instead, there is a general context to the unit, as in the Year 1 unit The Overland Trail.

In particular, the Alice story provides a metaphor for understanding exponents. When Alice eats an ounce of cake, her height is multiplied by a particular whole-number amount; when she drinks an ounce of beverage, her height is multiplied by a particular fractional amount. Using this metaphor, students reason about exponential growth and decay.

Students use several approaches to extend exponentiation beyond positive integers: a contextual situation, algebraic laws, graphs, and number patterns. They then apply principles of exponents to study logarithms and scientific notation.

UNIT OVERVIEW

This unit addresses the following Common Core Standards for Math:

MATHEMATICAL PRACTICE STANDARDS

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

CONTENT STANDARDS

- CC.N-RN.1: Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define 51/3 to be the cube root of 5 because we want (51/3)3 = 5(1/3)3 to hold, so (51/3)3 must equal 5.*
- CC.N-RN.2: Rewrite expressions involving radicals and rational exponents using the properties of exponents.
- CC.N-RN.3: Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.
- CC.A-SSE.4: Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments*
- CC.F-IF.7e: Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
- CC.F-IF.8b: Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as y = (1.02)t, y = (0.97)t, y = (1.01)12t, y = (1.2)t/10, and classify them as representing exponential growth or decay.*
- CC.F-BF.4: Find inverse functions.
- CC.F-BF.4a: Solve an equation of the form *f*(*x*) = *c* for a simple function f that has an inverse and write an expression for the inverse. *For example, f(x) =2 x3 for x > 0 or f(x) = (x+1)/(x–1) for x ≠ 1.*
- CC.F-BF.4b: (+) Verify by composition that one function is the inverse of another.
- CC.F- BF.4c: (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
- CC.F-BF.5: (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.
- CC.F-LE.1: Distinguish between situations that can be modeled with linear functions and with exponential functions.
- CC.F-LE.1a: Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

- CC.F-LE. 1c: Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another
- CC.F-LE.3: Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

Big Idea// Unit Essential Question:

What can we learn about the laws of exponents from looking at Alice in Wonderland?

Unit Enduring Understanding(s):

✓ Using a contextual situation, algebraic laws, graphs, and number patterns to extend exponentiation beyond positive integers.

✓ Applying principles of exponents to study logarithms and scientific notation.

Knowledge and Skills:

✓ Define the operation for an exponent of zero, negative integer exponents, and fractional exponents.
✓ Develop the additive law of exponents
✓ Develop the law of repeated exponentiation
✓ Describe the graphs of exponential functions
✓ Compare graphs of exponential functions for different bases
✓ Describe the graphs of logarithmic functions
✓ Compare graphs of logarithmic functions for different bases
✓ Understand the meaning of logarithms
✓ Understand the connections between exponential and logarithmic equations
✓ Convert numbers from ordinary notation to scientific notation, and vice versa
✓ Develop principles for doing computations using scientific notation
✓ Use the concept of order of magnitude in estimation

Assessments:

Teachers will have a variety of opportunities to formatively assess student understanding, in addition to end of unit summative assessments. Each of these assessment tools are included within the lesson activities or at the end of this unit.

✓ Problem of the Week
✓ Written/Oral Assignments:
  - Graphing Alice: This assignment will provide information about how well students understand the basic Alice metaphor and about their comfort with nonlinear graphs.
  - Having Your Cake and Drinking Too: This activity will reveal students' ability to work with the Alice metaphor in a complex situation.
  - Negative Reflections: This assignment will demonstrate how well students understand the extension of exponentiation to negative exponents.
  - All Roads Lead to Rome: This activity will reveal students' ability to synthesize a variety of approaches to understanding a mathematical concept.

- Alice on a Log: This assignment will provide information on students' understanding of the basics about logarithms.

✓ End of Unit In Class
✓ End of Unit Take Home

✓ Student Portfolio

Accommodations/Differentiation

IMP offers both extension and reinforcement activities to address the varying needs of students in the classroom. All of these activities are noted and including in the lesson activity pages.

All About Alice

Exponents, Scientific Notation, and Logarithms

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Contents

Introduction

All About Alice Unit Overview

Intent

This unit uses Lewis Carroll's story *Alice's Adventures in Wonderland* as the context in which students define the exponential function, derive properties of exponents, and use exponents to solve problems.

Mathematics

Unlike most other IMP units, *All About Alice* has no central problem to solve. Instead, there is a general context to the unit, as in the Year 1 unit *The Overland Trail.*

In particular, the Alice story provides a metaphor for understanding **exponents.** When Alice eats an ounce of cake, her height is multiplied by a particular whole-number amount; when she drinks an ounce of beverage, her height is multiplied by a particular fractional amount. Using this metaphor, students reason about exponential growth and decay.

Students use several approaches to extend exponentiation beyond positive integers: a contextual situation, algebraic laws, graphs, and number patterns. They then apply principles of exponents to study **logarithms** and **scientific notation.**

The main concepts and skills students will encounter and practice during the course of this unit are summarized by category here.

Extending the Operation of Exponentiation

Defining the operation for an exponent of zero

Defining the operation for negative integer exponents

Defining the operation for fractional exponents

Laws of Exponents

Developing the additive law of exponents

Developing the law of repeated exponentiation

Graphing

Describing the graphs of exponential functions

Comparing graphs of exponential functions for different bases

Describing the graphs of logarithmic functions

Comparing graphs of logarithmic functions for different bases

Logarithms

Understanding the meaning of logarithms

Making connections between exponential and logarithmic equations

Scientific Notation

Converting numbers from ordinary notation to scientific notation, and vice versa

Developing principles for doing computations using scientific notation

Using the concept of order of magnitude in estimation

Progression

The unit begins with a brief introduction to the Alice metaphor. Next, students develop a set of rules for computing with exponents and generalize these rules to include zero, negative, and fractional exponents. Finally, students "undo" the exponential function to define the logarithmic function and learn about scientific notation. There are two POWs in this unit.

Who's Alice?

Extending Exponentiation

Curiouser and Curiouser!

Turning Exponents Around

Supplemental Activities

Unit Assessments

Pacing Guides

50-Minute Pacing Guide (20 days)

Day	Activity	In-Class Time Estimate
1	Who's Alice?	
	Alice in Wonderland	30
	POW 12: Logic from Lewis Carroll	15
	Homework: *Graphing Alice*	5
2	*Alice in Wonderland* (continued)	25
	Discussion: *Graphing Alice*	20
	Homework: *A Wonderland Lost*	5
3	Discussion: *A Wonderland Lost*	10
	Extending Exponentiation	
	Here Goes Nothing	30
	POW 12: Logic from Lewis Carroll (progress check)	10
	Homework: *A New Kind of Cake*	0
4	Discussion: *A New Kind of Cake*	10
	Piece After Piece	40
	Homework: *When Is Nothing Something?*	0
5	Discussion: *When Is Nothing Something?*	15
	Many Meals for Alice	35
	Homework: *In Search of the Law*	0
6	Discussion: *In Search of the Law*	15

	Having Your Cake and Drinking Too	30
	Homework: *Rallods in Rednow Land*	5
7	Discussion: *Rallods in Rednow Land*	15
	Having Your Cake and Drinking Too (continued)	30
	Homework: *Continuing the Pattern*	5
8	Discussion: *Continuing the Pattern*	45
	Homework: *Negative Reflections*	5
9	Presentations: *POW 12: Logic from Lewis Carroll*	15
	Discussion: *Negative Reflections*	5
	Curiouser and Curiouser!	
	A Half Ounce of Cake	30
	Homework: *It's in the Graph*	0
10	*A Half Ounce of Cake* (continued)	15
	Discussion: *It's in the Graph*	15
	POW 13: A Digital Proof	15
	Homework: *Stranger Pieces of Cake*	5
11	Discussion: *Stranger Pieces of Cake*	35
	POW 13: A Digital Proof (continued)	15
	Homework: *Confusion Reigns*	0
12	Discussion: *Confusion Reigns*	20
	All Roads Lead to Rome	25
	Homework: *Measuring Meals for Alice*	5
13	Discussion: *Measuring Meals for Alice*	15
	All Roads Lead to Rome (continued)	25

	POW 13: A Digital Proof (continued) Turning Exponents Around Homework: *Sending Alice to the Moon*	10 0
14	Discussion: *Sending Alice to the Moon* Homework: *Alice on a Log*	40 10
15	Discussion: *Alice on a Log* *Taking Logs to the Axes* Homework: *Base 10 Alice*	15 30 5
16	Discussion: *Base 10 Alice* *POW 13: A Digital Proof* (work time) Homework: *Warming Up to Scientific Notation*	30 10 10
17	Discussion: *Warming Up to Scientific Notation* *Big Numbers* Homework: *An Exponential Portfolio*	10 40 0
18	Presentations: *POW 13: A Digital Proof* Discussion: *An Exponential Portfolio* Homework: *"All About Alice" Portfolio*	20 10 20
19	*In-Class Assessment* Homework: *Take-Home Assessment*	40 10
20	Exam Discussion Unit Reflection	30 20

90-minute Pacing Guide (12 days)

Day	Activity	In-Class Time Estimate
1	Who's Alice?	
	Alice in Wonderland	55
	POW 12: Logic from Lewis Carroll	25
	Homework: *Graphing Alice*	5
	Homework: *A Wonderland Lost*	5
2	Discussion: *Graphing Alice*	20
	Discussion: *A Wonderland Lost*	10
	Extending Exponentiation	
	Here Goes Nothing	30
	Piece After Piece	30
	Homework: *A New Kind of Cake*	0
	Homework: *When Is Nothing Something?*	0
3	Discussion: *A New Kind of Cake*	10
	Piece After Piece (continued)	10
	Discussion: *When Is Nothing Something?*	15
	Many Meals for Alice	35
	POW 12: Logic from Lewis Carroll (progress check)	10
	Homework: *In Search of the Law*	5
	Homework: *Rallods in Rednow Land*	5
4	Discussion: *In Search of the Law*	15

	Discussion: *Rallods in Rednow Land*	15
	Having Your Cake and Drinking Too	60
	Homework: *Continuing the Pattern*	0
5	Discussion: *Continuing the Pattern*	40
	Curiouser and Curiouser!	
	A Half Ounce of Cake	50
	Homework: *Negative Reflections*	0
6	Presentations: *POW 12: Logic from Lewis Carroll*	15
	Discussion: *Negative Reflections*	5
	Stranger Pieces of Cake	60
	POW 13: A Digital Proof	10
	Homework: *It's in the Graph*	0
7	*POW 13: A Digital Proof* (continued)	20
	Discussion: *It's in the Graph*	15
	All Roads Lead to Rome	50
	Homework: *Confusion Reigns*	5
8	Discussion: *Confusion Reigns*	20
	Measuring Meals for Alice	45
	POW 13: A Digital Proof (work time)	20
	Turning Exponents Around	
	Homework: *Sending Alice to the Moon*	5
9	Discussion: *Sending Alice to the Moon*	40
	Alice on a Log	45
	Homework: *Taking Logs to the Axes*	0

	Homework: *Base 10 Alice*	5
10	Discussion: *Taking Logs to the Axes*	10
	Discussion: *Base 10 Alice*	30
	Warming Up to Scientific Notation	40
	POW 13: A Digital Proof (work time)	10
	Homework: *An Exponential Portfolio*	0
11	Presentations: *POW 13: A Digital Proof*	20
	Big Numbers	40
	Discussion: *An Exponential Portfolio*	10
	Homework: *"All About Alice" Portfolio*	20
	Homework: *Take-Home Assessment*	0
12	*In-Class Assessment*	40
	Exam Discussion	30

Materials and Supplies

All IMP classrooms should have a set of standard supplies and equipment, and students are expected to have materials available for working at home on assignments and at school for classroom work. Lists of these standard supplies are included in the section "Materials and Supplies for the IMP Classroom" in *A Guide to IMP.* There is also a comprehensive list of materials for all units in Year 2.

Listed below are the supplies needed for this unit. General and activity-specific blackline masters are available for presentations on the overhead projector or for student worksheets. The masters are found in the *All About Alice* Unit Resources under *Blackline Masters*.

All About Alice

- Adding machine tape (3 or 4 rolls for the entire class)

More About Supplies

- Graph paper is a standard supply for IMP classrooms. Blackline masters of 1-Centimeter Graph Paper, ¼-Inch Graph Paper, and 1-Inch Graph Paper are provided so you can make copies and transparencies. (You'll find links to these masters in "Materials and Supplies for Year 2" [link] of the Year 2 guide and in the Unit Resources of each unit.)

Assessing Progress

All About Alice concludes with two formal unit assessments. In addition, there are many opportunities for more informal, ongoing assessments throughout the unit. For more information about assessment and grading, including general information about the end-of-unit assessments and how to use them, see "Assessment and Grading" in the *Year 2: A Guide to IMP* resource.

End-of-Unit Assessments

This unit concludes with in-class and take-home assessments. The in-class assessment is intentionally short so that time pressures will not affect student performance. Students may use graphing calculators and their notes from previous work when they take the assessments.

Ongoing Assessment

Assessment is a component in providing the best possible ongoing instructional program for students. Ongoing assessment includes the daily work of determining how well students understand key ideas and what level of achievement they have attained in acquiring key skills.

Students' written and oral work provide many opportunities for teachers to gather this information. Here are some recommendations of written assignments and oral presentations to monitor especially carefully that will offer insight into student progress.

- *Graphing Alice:* This assignment will give you information about how well students understand the basic Alice metaphor and about their comfort with nonlinear graphs.
- *Having Your Cake and Drinking Too:* This activity will reveal students' ability to work with the Alice metaphor in a complex situation.
- *Negative Reflections:* This assignment will tell you how well students understand the extension of exponentiation to negative exponents.
- *All Roads Lead to Rome:* This activity will give you information on students' ability to synthesize a variety of approaches to understanding a mathematical concept.
- *Alice on a Log:* This assignment will give you information on students' understanding of the basics about logarithms.

Supplemental Activities

All About Alice contains a variety of activities at the end of the student pages that you can use to supplement the regular unit material. These activities fall roughly into two categories.

- **Reinforcements** increase students' understanding of and comfort with concepts, techniques, and methods that are discussed in class and are central to the unit.
- **Extensions** allow students to explore ideas beyond those presented in the unit, including generalizations and abstractions of ideas.

The supplemental activities are presented in the teacher's guide and the student book in the approximate sequence in which you might use them. Below are specific recommendations about how each activity might work within the unit. You may wish to use some of these activities, especially the later ones, after the unit is completed.

***Inflation, Depreciation, and Alice* (reinforcement)** This activity provides other contexts for exponential growth and "shrinkage," or decay. To find formulas for Questions 1 and 2, students should recognize that increasing prices by 5% is the same as multiplying them by 1.05. They should reason similarly for depreciation. This activity can be assigned as soon as the basic idea of the Alice metaphor is clear.

***A Logical Collection* (reinforcement)** This activity contains several logic problems that make a good follow-up to the discussion of *POW 12: Logic from Lewis Carroll*.

***More About Rallods* (extension)** This activity builds on the situation from *Rallods in Rednow Land*, asking students to find a general formula for the sum of the first n powers of 2 and then having them explore other geometric sequences.

***Ten Missing Digits* (extension)** This activity expands the digit puzzle in *POW 13: A Digital Proof* to ten missing digits.

***Exponential Graphing* (reinforcement)** This activity offers students more opportunities to examine the graphs of exponential functions and can be assigned after *Stranger Pieces of Cake*.

***Basic Exponential Questions* (extension)** This activity raises some challenging questions about inequalities involving exponential expressions and can be used after *Stranger Pieces of Cake*. Question 2 is intentionally trivial (as it involves base 1). Question 3 follows up with a similar but more complicated problem. Question 4 is quite difficult to solve in general. The only whole-number solutions are the cases in which X is 2 and Y is 4, and vice versa. Students may find explanations for why there are no other solutions.

***Alice's Weights and Measures* (extension)** This activity explores issues of approximation. When a measurement is only an approximation, what effect does it have on computations that make use of that measurement? This exercise makes a good follow-up to *Measuring Meals for Alice,* in which students give only approximate answers.

***A Little Shakes a Lot* (reinforcement)** In this activity, students explore an interesting real-world use of logarithms: earthquakes. You can use this activity after *Sending Alice to the Moon*.

***Who's Buried in Grant's Tomb?* (extension)** This activity offers another setting in which students can explore the relationship between exponents and logarithms. This can be used after *Sending Alice to the Moon*.

***Very Big and Very Small* (extension)** This supplement asks students to identify and investigate more situations involving very big and very small numbers in contexts they find intriguing. It can be assigned following *Big Numbers*.

Who's Alice?

Intent

These introductory activities use excerpts from Lewis Carroll's book *Alice's Adventures in Wonderland* to introduce the context of and to lay the mathematical foundation for the unit.

Mathematics

Through a retelling of the classic story, eating ounces of cake and drinking ounces of beverage will serve as metaphors for exponential, or constant percentage, growth and decay, respectively. Students discover that eating C ounces of cake multiplies Alice's height by 2^C and drinking B ounces of beverage multiplies her height by $\left(\frac{1}{2}\right)^B$. They construct In-Out tables and graphs that display these data. Finally, students examine a new context: a constant percentage decay in which the base is not $\frac{1}{2}$.

Progression

The first activity introduces the context for the unit and some important vocabulary. The other two activities introduce graphing and set the stage for more rigorous treatment of exponential functions. In addition, students begin work on the first POW of the unit, which focuses on logical reasoning.

Alice in Wonderland

POW 12: Logic from Lewis Carroll

Graphing Alice

A Wonderland Lost

Alice in Wonderland

Intent

This activity introduces the metaphor for exponential growth and decay that is used throughout the unit. Students begin thinking about how exponents work and establish basic exponential growth and decay expressions.

Mathematics

An excerpt from *Alice's Adventures in Wonderland* establishes a context for thinking about exponents, exponential expressions, and exponential functions. The metaphor will be used to motivate students and to help them extend the definition of exponentiation beyond whole-number exponents and understand laws relating to exponents. Students realize that eating *C* ounces of cake multiplies Alice's height by 2^C and drinking *B* ounces of beverage multiplies her height by $\left(\frac{1}{2}\right)^B$, eventually realizing this second multiplication process as equivalent to repeated division by 2.

Throughout the work with this metaphor, focus students' attention on the *factor* by which Alice grows rather than on the *amount* by which she grows. The cake and beverage create *multiplicative* changes, rather than *additive* changes, in Alice's height.

Progression

Working through some examples in groups, students consider *how much bigger* or *how much smaller* Alice grows and shrinks, and then they generalize the process. Class discussion then determines some agreements for interpretation of and notation for this metaphor. The activities *Graphing Alice* and *A Wonderland Lost* will strengthen the metaphor by linking the ideas of exponential growth to other representations and situations.

Approximate Time

55 minutes

Classroom Organization

Groups, followed by whole-class discussion

Materials

Cash register tape (optional)

Doing the Activity

Wait at least a day before discussing the mathematical topics to be studied in this unit, as doing so now may undermine the discovery that Alice's height is related to exponents.

Many students may be unfamiliar with Lewis Carroll's *Alice's Adventures in Wonderland,* so give a quick overview of the story to set the context, or ask a student to do so. To refresh your memory, the story begins with a young girl who dreams she spots a white rabbit who is carrying a pocket watch and muttering about how late he is. Alice follows him down a rabbit hole to a tea party with the Mad Hatter and goes on to have many other adventures. Although the cake and beverage are part of the original story, no specific numeric effect of eating or drinking is given.

Have the class read the excerpt and the paragraphs that follow it, but not the questions.

Acting Out "Doubling"

Before having students turn to the questions, it will be helpful to have them act out the concept of Alice's growth to give them a concrete sense of the effect of repeated doubling.

For example, you might use tiles on the floor to act out the doubling process. Have one student stand on the first tile, another on the second tile, another on the fourth tile, another on the eighth tile, and so on. To act out the halving process, you might give each group or pair of students a strip of cash-register tape, perhaps 20 feet long, and have them repeatedly fold it in half so they realize how quickly halving reduces the length of the strip.

After this introduction, have students work on the questions in their groups.

Nowhere in the problem are we told how tall Alice is to begin with; we know only by how much her height is multiplied. Some students may not like working with this kind of abstraction. You might suggest they make the situation more concrete by picking a particular height for Alice as a starting point, still focusing on the comparison between her starting height and her final height.

Discussing and Debriefing the Activity

Ask for volunteers to present each question as the rest of the class comments and elaborates. Given here is guidance on what needs to emerge from these presentations. As needed, remind students to focus on the *factor* by which Alice's height changes.

Eating C Ounces of Cake

Students should realize that if Alice eats 1 ounce of cake, then 1 more ounce, and so on, her height will double each time so that overall, if she eats C ounces, her height will be multiplied by 2^C. It is important that this fundamental generalization be very clear.

If necessary, have the presenter explain the specific cases in Question 1, and ask questions to bring out the use of an exponential expression in Question 2. **Is there another way to write 2 • 2? What if *C* 2s were multiplied together?**

If needed, remind students of the words **base** and **exponent** for referring to the number 2 and the *C* in the expression 2^C.

Note: When we say that eating *C* ounces of cake multiplies Alice's height by 2^C, we are assuming that eating *C* ounces all at once is the same as eating them one ounce at a time. This issue will be important in the discussion of the activity *Piece After Piece.* The generalization that eating *C* ounces of cake multiplies Alice's height by 2^C also incorporates the fact that if something is multiplied first by one factor (for example, *a*) and the resulting product is multiplied by another factor (for example, *b*), the original number has been multiplied altogether by *ab.* Thus, if the first ounce of cake doubles Alice's height and the second doubles her height again, altogether her height has been multiplied by 2 • 2, or 4. This is essentially the associative property of multiplication, as it states that $(h \bullet 2) \bullet 2 = h \bullet (2 \bullet 2)$, where *h* represents Alice's original height.

Drinking B Ounces of Beverage

Students typically find the beverage aspect of the situation somewhat more difficult. They need to realize that drinking *B* ounces of the beverage multiplies Alice's height by $\frac{1}{2^B}$, which they may initially write as $\left(\frac{1}{2}\right)^B$.

They are probably even more likely to recognize the change in Alice's height in terms of division rather than multiplication, dividing by 2^B. The unit will proceed more smoothly, however, if they focus on the question, **What has Alice's height been multiplied by?**

If students initially use division, help them make this transition, phrasing your questions in fairly explicit terms if necessary. **What is another way to express division by 2? How can you express this in terms of multiplication?** If students say 1 ounce of beverage multiplies Alice's height by 0.5, ask, **How else can you write 0.5?**

It is also important that students express this multiplication factor as a fraction rather than as a decimal, as this will make the pattern clear and help extend the operation of exponentiation, in upcoming activities, to include negative exponents.

The fact that drinking *B* ounces of beverage multiplies Alice's height by $\frac{1}{2^B}$ can be understood by students as the consequence of two basic ideas.

- Multiplying by $\frac{1}{2}$ repeatedly, *B* times, is the same as multiplying by $\left(\frac{1}{2}\right)^B$.

- Multiplying by $\left(\frac{1}{2}\right)^B$ is the same as multiplying by $\frac{1}{2^B}$. In other words,

$$\left(\frac{1}{2}\right)^B = \frac{1^B}{2^B} = \frac{1}{2^B}.$$

If students do not reason this way, you can introduce these ideas gradually, asking what Alice's height is multiplied by when she drinks a specific amount and then having students restate the result.

For example, if they say that drinking 3 ounces means Alice's height is multiplied by $\frac{1}{8}$, ask where the 8 comes from. It will be easier to pose this type of question if students write the multiplying factor as the fraction $\frac{1}{8}$ rather than the decimal 0.125.

Even after examples like this, many students will continue to interpret drinking multiple ounces of beverage as repeatedly dividing Alice's height by 2 rather than repeatedly multiplying it by $\frac{1}{2}$.

Key Questions

What has Alice's height been multiplied by?

Is there another way to write 2 • 2? What if *C* 2s were multiplied together?

What is another way to express division by 2? How can you express this in terms of multiplication?

How else can you write 0.5?

Supplemental Activity

Inflation, Depreciation, and Alice (reinforcement) provides other contexts for exponential growth and "shrinkage," or decay. To find formulas for Questions 1 and 2, students should recognize that increasing prices by 5% is the same as multiplying them by 1.05. They should reason similarly for depreciation.

POW 12: Logic from Lewis Carroll

Intent

This POW focuses on basic ideas of logic and deduction connected to the interests of Lewis Carroll, who was a mathematician as well as an author. This POW is more directed than usual, but the subject matter is likely to be new to many students, and the problem does have an open-ended component.

Mathematics

Logic and proof are at the core of the mathematician's work. This POW explores situations of logical deduction and, in particular, syllogisms. Technically, a *syllogism* is an argument in which a conclusion is drawn based on two premises. For example, the premises "Socrates is a man" and "All men are mortal" lead to the conclusion "Socrates is mortal." Students often spend considerable time studying formal logic in a traditional geometry course. This POW asks them to use this type of reasoning in a less formal setting.

Progression

Students are introduced to this POW early in the unit, with time spent to help them clarify the problem as well as to confer with one another about their deductions. Presentations follow about a week later.

Approximate Time

15 minutes for introduction
1 to 2 hours for activity (at home)
10 minutes for progress check
15 minutes for presentations

Classroom Organization

Groups, then individuals, followed by whole-class presentations

Doing the Activity

Introduce the activity and discuss the two examples. Point out that the text is not claiming that any of the individual statements is actually true. Instead, the focus is on what conclusions could be drawn *if* they were true.

If possible, provide some in-class time a few days after students have begun work on this POW as a progress check of sorts. You might give them five or ten minutes in their groups to compare ideas and then take general questions. Or have students discuss the first question or two either in groups or as a class.

You might also offer these additional examples from Lewis Carroll to help students understand the POW, or let groups spend some time trying to make up more problems of this type.

a. No quadrupeds can whistle.
b. Some cats are quadrupeds.
Conclusion: "Some cats can't whistle."

a. All clever people are popular.
b. All obliging people are popular.
No new conclusion can be drawn.

a. All puddings are nice.
b. This dish is a pudding.
c. No nice things are wholesome.
Conclusion: "This dish is not wholesome."

The day before the POW is due, identify three students to prepare presentations.

Discussing and Debriefing the Activity

One approach for presentations and discussion is to have the three students alternate presentations on the six questions. After each question, let the other presenters add any comments and then ask the rest of the class for comments. Here are some things to look for.

1. Conclusion: "Senna is not nice." As this is a straightforward example, make sure everyone understands why this conclusion is valid.

2. No new conclusion can be drawn. You might mention that shillings are coins formerly used in Britain (though this is not relevant to the problem). Make sure all students understand that they cannot conclude, "These coins are shillings."

3. Conclusions: "Some wild pigs are fat" or "Some fat pigs are wild." Students may not recognize these as legitimate conclusions. If they decided there is no possible new conclusion, that's okay; it's a matter of opinion whether these conclusions are "new." But they should realize that both of these conclusions must be true, as is the conclusion "There are some fat, wild pigs."

4. No new conclusion can be drawn. Students may suggest as a conclusion "Some unprejudiced persons are liked." According to formal logic, this doesn't follow from statement b, although everyday language usage suggests that if we say "some are" we also are saying "some are not." This is worth discussing if it arises. Point out that we often draw conclusions from things people say that are not necessarily inherent in what they say.

5. Main conclusion: "Babies cannot manage a crocodile." Other, subsidiary conclusions use only two of the three statements, such as "Babies are despised" (from a and c) and "Illogical persons cannot manage a crocodile" (from b and c). Discuss how the main conclusion can be drawn by combining one of these subsidiary statements with the remaining original statement.

6. Main conclusion: "No bird in this aviary lives on mince pies." This conclusion uses all four statements. Subsidiary conclusions use only two of the statements: "All my birds are ostriches" (from a and d), "No birds in this aviary are less than 9 feet tall" (from b and d), and "No birds that are 9 feet tall live on mince pies" (from a and c). Try not to get caught up in such technicalities as whether ostriches are *exactly* 9 feet tall or *at least* 9 feet tall. Also, help students to realize that none of the statements guarantees that there are any birds in the aviary or that "I" have any birds. In general, a statement like "All *x*'s are *y*" doesn't imply that there are any *x*'s.

Ask presenters, and perhaps volunteers, to give one of the examples they made up for Part II of the POW for the class to work on.

Supplemental Activity

A Logical Collection (reinforcement) contains several logic problems that make a good follow-up to the POW discussion.

Graphing Alice

Intent

This activity will strengthen students' understanding of exponential relationships and, as they consider larger domains, their curiosity about what more there is to know.

Mathematics

Students examine graphs of base 2 and base 3 exponential relationships and confront the dilemmas posed by scaling exponential functions. They also examine the similarities and differences among graphs of varying bases, including different natural numbers and unit fractions. The nonlinearity of the relationships is emphasized. The activity also sets the stage for examining exponential functions with domains extended beyond the natural numbers.

Progression

This activity makes a good homework assignment, with class discussion occurring after work on the activity *Alice in Wonderland* is completed. The discussion concludes by raising questions about negative and zero inputs and emphasizing nonlinear growth by introducing the concepts **absolute growth** and **percentage growth.**

Approximate Time

5 minutes for introduction
20 minutes for activity (at home or in class)
20 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Read through the introduction as a class. Discuss why students might want the scales on the two axes to be different.

Discussing and Debriefing the Activity

Have students share their findings in their groups. You might have them make chart-paper versions of their work in Questions 1 to 4 so all four graphs can be easily compared during the class discussion of Question 5.

Note: Students are expected only to plot individual points for the graphs in Questions 1 to 4, using specific positive integer values for x, rather than to graph the general functions. As the unit progresses, they will gradually move toward creating complete graphs of such functions as $y = 2^x$.

Question 1

The main focus for the discussion of Question 1 will likely be the scaling of the axes. One error many students make is scaling the *y*-axis so that the values 2^1, 2^2, 2^3, 2^4, and so on are equally spaced, thus creating a linear graph. If the presenter for Question 1 makes this mistake, ask a general question about scaling to bring this out, such as, **Is the difference between 2^2 and 2^3 the same as the difference between 2^3 and 2^4?**

When the graph is scaled correctly, students should realize that Alice's height does not grow linearly, as the points do not lie on a straight line.

Ask students to express the rule for the graph using function notation. **Using *f* to represent the function, how can you write an equation for this graph using function notation?** Using *x* for the independent variable, they will probably write $f(x) = 2^x$.

After the graph is finished, post it on chart paper, labeling it appropriately. Students can add to the graph as the definition of exponentiation is extended during the unit.

You may need to emphasize that the scales for the *x*- and *y*-axes can be different. In fact, for this question, if the scales are the same, the graph will be very tall in comparison to its width.

Question 2

The issues that arise for Question 2 will likely be similar to those for Question 1. Students will need to use a different scale for the *y*-axis than was used for Question 1.

Ask students to express this relationship using function notation and a new letter for the function name. For instance, they might write $g(x) = \left(\frac{1}{2}\right)^x$.

Questions 3 and 4

For these questions, elicit generalizations analogous to those developed in the discussion of *Alice in Wonderland.* That is, if Alice eats *C* ounces of the new cake, her height is multiplied by 3^C. If she drinks *B* ounces of the new beverage, her height is multiplied by $\frac{1}{3^B}$.

Introduce the terms *base 3 cake* and *base 3 beverage* for the new cake and beverage, and have a volunteer explain why these terms apply. Ask students what they would call the cake and beverage in the original problem. They should recognize that the appropriate terms are *base 2 cake* and *base 2 beverage.* Explain that, unless otherwise indicated, they should assume the cake and beverage in "Alice problems" are base 2 cake and base 2 beverage and that, whatever type is used, the beverage and the cake within a given problem will always use the same base.

The graphs for Questions 3 and 4 will require even more difference in the scales than those for Questions 1 and 2. Have students express these functions using function notation.

Question 5

You might let volunteers share their observations about Question 5 and have the rest of the class comment. There are several comparisons students can make.

The main observation is that in the cake problems (Questions 1 and 3), the y-value goes up rapidly as the x-value increases, while in the beverage problems (Questions 2 and 4), the y-value seems to get closer to 0 as the x-value increases. Students may also point out that the larger the base, the more extreme the change in y as x changes.

Extending the Graphs to the Left

As a lead-in to later discussion of the use of zero or negative integers as exponents, ask groups to discuss where each graph would cross the y-axis and what each graph might look like for negative inputs. **Where would each graph cross the y-axis? What will happen to the graphs as x becomes negative?** These issues should be considered only briefly at this point, in terms of what extending the graphs might suggest rather than in terms of what zero or negative exponents might mean.

After a few minutes of small-group discussion, let volunteers share ideas. Students may notice that all the graphs seem likely to cross the y-axis at the point (0, 1). They may also realize that in the graphs from Questions 1 and 3, the y-values seem likely to get closer to 0 as x becomes more negative, while in the graphs from Questions 2 and 4, the y-values seem likely to get larger and larger as x becomes more negative.

Absolute Growth Versus Percentage Growth

Now ask students to focus on the case of base 2 cake, and suggest they have Alice start from a specific height, such as 5 feet. Ask, **How does the effect on Alice's height of her third ounce of cake compare to the effect of her fifth ounce of cake?**

Using a starting height of 5 feet, they might realize that the third ounce of cake increases Alice's height from 20 feet to 40 feet while the fifth ounce increases her height from 80 feet to 160 feet. Bring out that although the latter is a greater increase (80 versus 20 feet of growth), both cases involve doubling her height. Introduce the terms **percentage growth** for the proportional rate of increase (found by dividing the final value by the initial value) and **absolute growth** for the numeric difference (found by subtracting the initial value from the final value).

Key Questions

Is the difference between 2^2 and 2^3 the same as the difference between 2^3 and 2^4?

Using *f* to represent the function, how can you write an equation for this graph using function notation?

Where would each graph cross the *y*-axis? What will happen to the graphs as *x* becomes negative?

How does the effect on Alice's height of her third ounce of cake compare to the effect of her fifth ounce of cake?

A Wonderland Lost

Intent

This activity offers students a real-world context for the phenomenon of exponential decay.

Mathematics

Students examine an exponential decay function expressed as a constant percentage decrease, beginning from a context and deriving the tabular, graphical, and symbolic representations of the associated exponential function. Translating a percentage decrease into a symbolic rule, possibly in standard exponential form, will be challenging.

Progression

Students work on this task individually and share results with the whole class. The class will return to this graph in the next activity, *Here Goes Nothing.*

Approximate Time

5 minutes for introduction
20 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Although this is an individual task, students will benefit from some initial whole-class or small-group work. After they read the task, ask them to describe the situation in their own words.

Help them to clarify that the 10 percent in the problem is always a percentage of the remaining forest, which decreases each year. You might ask, **If 10 percent of the forest is destroyed each year, how many years will it take until the forest is completely gone? Ten years, right?**

Some students will recognize that this statement is incorrect. Getting someone who understands the concept to explain why should clear up any possible confusion. If needed, introduce a simpler specific area for the rain forest, such as 100,000 square miles, for students to work with.

Discussing and Debriefing the Activity

Give students time to share ideas and ask questions in their groups before beginning the discussion.

Questions 1 and 2
Have students give their numeric results, year by year, for Questions 1 and 2. You might record their results in an In-Out table, including the initial area given in the activity as the second coordinate of a point whose first coordinate is 0.

Have someone suggest scales for a graph of this information. You might sketch a graph of the table data or have another student share his or her graph.

Questions 3 and 4
Students are likely to have had difficulty developing a general rule for this situation; repeatedly subtracting 10 percent does not lend itself to a simple expression. The key to obtaining a general rule is to recognize that decreasing something by 10 percent is the same as multiplying it by 0.9.

You might ask, **How does the area at the end of each year compare to the area at the beginning of that year?** The relationship may be clearer if you have students consider a round number, such as 100,000 square miles, and ask how the area after a year compares to the initial area.

Another approach is to have them write the initial area as A and the new area as $A - 0.1A$, and then combine terms to get $0.9A$.

With help, students should be able to formulate the general expression $1{,}200{,}000 \bullet 0.9^X$ for the area after X years.

Finally, discuss how the rain forest problem relates to Alice. Students should figure out that the situation is essentially the same as that of Alice's beverage, except that Alice's height decreases by 50% per ounce while the rain forest decreases by 10% per year.

Key Questions

If 10 percent of the forest is destroyed each year, how many years will it take until it is all gone? Ten years, right?

How does the area at the end of each year compare to the area at the beginning of that year?

What is the current area multiplied by to figure the area for the next year?

Extending Exponentiation

Intent

In these activities, students will derive several rules for computing with exponents and extend their understanding of exponential expressions to include zero and negative integers.

Mathematics

In these activities, students will use the Alice metaphor and patterns in lists like $2^3 = 8$, $2^2 = 4$, and $2^1 = 2$ to derive a number of rules for working with exponents.

$$A^x \bullet A^y = A^{x+y}$$

$$\left(A^x\right)^y \bullet \left(A^y\right)^x = A^{xy}$$

$$A^0 = 1 \text{ if } A \neq 0$$

$$A^{-x} = \frac{1}{A^x} \text{ if } A \neq 0$$

Progression

The activities first give meaning to an exponent of zero and then develop the **additive law of exponents** and the **law of repeated exponentiation.** They conclude with the extension of exponents to include negative integers. In addition, students will present their work on the first POW of the unit.

Here Goes Nothing

A New Kind of Cake

Piece After Piece

When Is Nothing Something?

Many Meals for Alice

In Search of the Law

Having Your Cake and Drinking Too

Rallods in Rednow Land

Continuing the Pattern

Negative Reflections

Here Goes Nothing

Intent

Students will consider zero as the value of an exponent and use patterns, tables, and graphs to explain their findings.

Mathematics

Students consider the meaning of 2^0, initially through the Alice metaphor, and focus on the idea that multiplying by the quantity 2^0 must be equivalent to multiplying by 1, the multiplicative identity. In addition to rationalizing the definition that $2^0 = 1$ through the Alice situation, they look at the patterns of powers of 2 in graphical and tabular arguments.

Progression

Students collaborate in groups to consider the impact on Alice's height of eating no cake and what that implies for the meaning of 2^0. The follow-up discussion emphasizes $2^0 = 1$ as a mathematical definition. Later activities will extend this definition to other bases.

Approximate Time

30 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

Before students begin, you may want to review the formula developed earlier, in which eating *C* ounces of cake multiplies Alice's height by a factor of 2^C.

For Question 3, you may need to clarify that groups only need to substitute 0 for *C* in the expression 2^C—they don't need to evaluate the expression.

Discussing and Debriefing the Activity

Have volunteers from different groups answer Questions 1 to 4. They should be clear that eating 0 ounces of cake doesn't change Alice's height, which means her height is multiplied by 1.

The presenter for Question 2 might refer to the posted graph (from Question 1 of *Graphing Alice*) and show that if the graph is continued to the left, it might hit the *y*-axis at $y = 1$. Although the graph might not point clearly to a *y*-value of 1, at this time students need only to realize that this result is reasonable.

For Question 3, they should recognize that substituting 0 for *C* in the expression 2^C gives 2^0.

For Question 4, you may need to summarize what was said in Questions 1 to 3 so that the appropriate conclusion becomes clear: that it seems to make sense for 2^0 to be equal to 1.

Back to "A Wonderland Lost"

Reinforce the discussion by returning to the graph from students' work on *A Wonderland Lost.* Remind students of the rule they found for the rain forest situation ($1,200,000 \bullet 0.9^x$) and ask what value they want when $X = 0$. The graph should include the point (0, 1,200,000), which shows they want $1,200,000 \bullet 0.9^x$ to equal 1,200,000 when $X = 0$, meaning they want 0.9^0 to be equal to 1.

$2^0 = 1$ Is a Definition

It's very important that students realize that a definition is needed to give meaning to the expression 2^0. Bring out that it makes sense to say "2^3 means to multiply three 2s together" and "2^5 means to multiply five 2s together," but it doesn't make sense to say "2^0 means to multiply zero 2s together." So a decision has to be made—there needs to be a convention, or agreement, as to the value of 2^0.

Tell students that long ago, mathematicians agreed to *define* 2^0 as having a value of 1. The purpose of this activity is to show that this is the *most reasonable* definition, because it fits what happens to Alice and it fits the graph. (You can expect students both to resist and to forget this definition. The notion that any computation involving multiplication and zero gives a result of zero is a strong one, and it may take some students a while to let go of this idea.)

The Exponential Pattern

Use the pattern of powers of 2 to reinforce the idea that $2^0 = 1$. Make a list like this.

$$2^5 = 32$$
$$2^4 = 16$$
$$2^3 = 8$$
$$2^2 = 4$$
$$2^1 = 2$$

Get the class to articulate this pattern in various ways, such as "Each result is twice the one below it" or "Divide by 2 as the exponent goes down by 1."

What would be the next equation in this pattern? Students should recognize that the natural way to continue this pattern is with the equation $2^0 = 1$.

Remind students that $2^0 = 1$ is, ultimately, a definition. The usual definition of exponents in terms of repeated multiplication has broken down, because there are no 2s to multiply. So we must use some other method of defining the expression, and it makes sense to formulate the definition in a way that is consistent with other ideas.

What are the various reasons the definition "$2^0 = 1$" makes sense? Students should identify at least three reasons.

- It fits the rule that 2^C tells what to multiply Alice's height by when she eats C ounces of cake.
- It seems a reasonable way to extend what they already have of the graph of the equation $y = 2^x$.
- It fits the pattern of powers of 2.

As students realize that all these methods agree, they should become more satisfied with the definition. Soon they will begin developing the **additive law of exponents** and will find that this law provides another way to justify the definition.

Key Questions

What would be the next equation in this pattern?

What are the various reasons the definition "$2^0 = 1$" makes sense?

A New Kind of Cake

Intent

Students check their understanding of using zero as an exponent by extending their work from base 2 to other bases. In addition, they look at graphs of $y = 2^x$ and $y = x^2$ to explore the effects of switching base and exponent.

Mathematics

The activity provides review and reinforcement of ideas about the equation and graph of the exponential function and about zero as an exponent. In this instance, students use base 5. They also compare the graphs of $y = 2^x$ and $y = x^2$ with the primary intention of recognizing that the expression 2^x grows much faster than x^2 as x increases.

Progression

After some work on their own, students come together as a class to compare results. This activity transitions to the next activity, *Piece After Piece,* by focusing on the question "What is Alice's height multiplied by?" after she eats various amounts of cake.

Approximate Time

20 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Introduce this activity by telling students they will now consider what happens with another base.

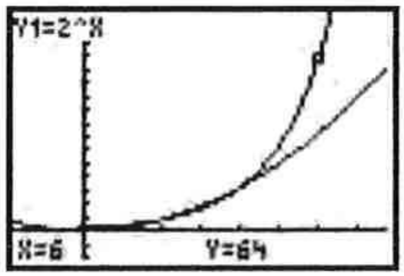
Discussing and Debriefing the Activity

You may want to have a volunteer present his or her ideas on Part I. The presenter should note that eating 0 ounces multiplies Alice's height by 1 and that this does seem to fit the graph obtained in Question 2. Because the scale on that graph covers such a wide range of values, however, this is not a very convincing argument for defining 5^0 as 1.

The key element of Question 3 is part c. The student might argue, for instance, that eating 2 ounces multiplies Alice's height by 25, which is 5^2, so eating 0 ounces should multiply her height by 5^0, which means 5^0 must equal 1.

The presenter for Question 4 should be able to use reasoning analogous to that in the discussion of powers of 2 from *Here Goes Nothing* to argue that it makes sense to define 5^0 as 1.

For Part II, students will presumably have graphed the two equations by plotting individual points. The main idea that needs to emerge is that the expression 2^x grows much faster than the expression x^2 as x increases. If students did not realize this, they probably did not take their graphs far enough out to the right. You might have the class choose some numbers between 10 and 20 and find the values of the expressions 2^x and x^2 for comparison, or graph the equations on a calculator.



Piece After Piece

Intent

Students use the metaphor of eating cake and drinking beverage to explore the additive law of exponents.

Mathematics

In the unit's opening activity, *Alice in Wonderland,* students developed the principle that eating C ounces of cake multiplies Alice's height by 2^C. This generalization assumes that eating C ounces is the same as eating 1 ounce C times—in other words, it doesn't matter whether Alice eats her cake all at once or one ounce at a time. Now students will make explicit use of this assumption to develop the additive law of exponents. The **additive law of exponents** says that when two exponential expressions with the same base are multiplied, this property holds:

$$A^X \bullet A^Y = A^{X+Y}$$

Progression

Working in small groups, students use the Alice metaphor to develop the additive law of exponents for base 2 cake and beverage. In the follow-up discussion, they use repeated multiplication to explain the additive law of exponents for base 2, and then they generalize the procedure. The next activity, *When Is Nothing Something?,* brings together their ideas about zero as an exponent and the additive law of exponents.

Approximate Time

40 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

The real question of this task—What happens if Alice doesn't eat all her cake in one sitting?—is suggested in the first sentence.

Have students read the activity. Encourage some conjecture, and then have them explore the task in their groups.

Discussing and Debriefing the Activity

Begin by having students answer Question 1. The presenter for Question 1a might describe what happens overall, saying something like, "Her height is multiplied by 256." If so, ask for an explanation of how the presenter arrived at this conclusion. **What are the stages of her height change?** Ensure that everyone understands

that Alice's height is multiplied first by 8 and then by 32. Bring out that 8 comes from the expression 2^3 and that 32 comes from the expression 2^5.
You might suggest students represent Alice's original height with a variable. **If Alice's initial height is *h* and she eats 3 ounces of cake, what will her new height be?** Write the response as $h \bullet 2^3$. Proceed similarly with the second stage, writing the next result as $(h \bullet 2^3) \bullet 2^5$. Then ask a similar question for the 8-ounce piece of cake.

If students used a specific initial height, they will certainly recognize that the overall result is the same as eating 8 ounces of cake. Focus on the explanation for this in terms of the arithmetic. **How does the arithmetic explain why the results are the same?** A reasonable justification might be, "Multiplying by 8 and then by 32 is the same as multiplying by 256."

The goal is to bring out what this means in terms of expressions with exponents. Essentially, it says that $2^3 \bullet 2^5$ is the same as 2^8.

This fact may seem obvious to some students but not to others. A good way to clarify the relationship is to make the individual factors of 2 explicit by asking, for instance, where the factor of 8 comes from. Return to the expression 2^3, and ask students to break it down into individual factors—that is, as $2 \bullet 2 \bullet 2$. Proceeding similarly with the factor 2^5, they should determine that $2^3 \bullet 2^5$ can be written as

$$(2 \bullet 2 \bullet 2) \bullet (2 \bullet 2 \bullet 2 \bullet 2 \bullet 2)$$

At the same time, they should note that the single expression 2^8 is equal to

$$2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$$

You can ask about the number of 2s in each expression—**How many 2s are in the expression $2 \bullet 2 \bullet 2$? In $2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$? In $2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$?**—and then return to the exponential forms of these expressions to develop the equation

$$2^3 \bullet 2^5 = 2^8$$

Question 2
This is a good time to look at some other examples. Ask for volunteers to share their work on Question 2, and conduct a brief version of the development above for one or two examples.

If students offer examples with small exponents, have the class write out the individual factors, as done here.

$$2^2 \bullet 2^3 = 2^5$$

$$(2 \bullet 2) \bullet (2 \bullet 2 \bullet 2) = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$$

Move from examples like this one, in which students can actually count the factors, to examples in which they simply add the exponents, such as

$$2^{17} \bullet 2^{32} = 2^{49}$$

Finally, ask for a generalization of this process. Students will probably be able to develop an equation such as

$$2^X \bullet 2^Y = 2^{X+Y}$$

Question 3

Have a volunteer present Question 3. If it doesn't arise in the presentation, ask explicitly how the reasoning would apply if the situation were about beverage instead of cake. Students should be able to develop a "beverage counterpart" to the last equation along the lines of

$$\left(\frac{1}{2}\right)^X \bullet \left(\frac{1}{2}\right)^Y = \left(\frac{1}{2}\right)^{X+Y}$$

The Additive Law of Exponents

As an additional stage in the development of the general **additive law of exponents,** ask students to make up similar examples of what would happen if Alice consumed another type of cake. For example, ask, **Does eating 3 ounces and then 5 ounces of base 7 cake have the same effect as eating 8 ounces of that cake? Why?** Students should be able to explain this with an expression such as

$$\underbrace{(7 \bullet 7 \bullet 7)}_{\text{3 factors}} \bullet \underbrace{(7 \bullet 7 \bullet 7 \bullet 7 \bullet 7)}_{\text{5 factors}}$$

pointing out that this gives a total of 8 factors, so $7^3 \cdot 7^5$ is equal to 7^8.

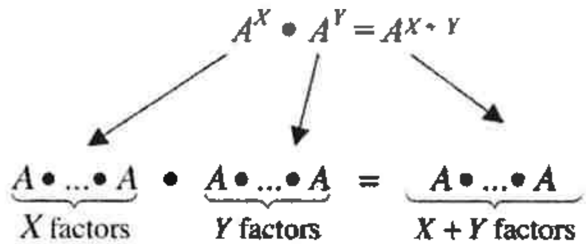
Depending on how students respond, you might go directly from such examples to the most general case, or you might have them develop generalizations for specific bases other than 2. For example, they might come up with the equation

$$7^X \bullet 7^Y = 7^{X+Y}$$

and generalize to derive the principle

$$A^X \bullet A^Y = A^{X+Y}$$

Help students understand this visually by encouraging them to write each exponential expression as a product of a group of factors. **What does A^X mean? What does A^Y mean? How would you write these expressions without using exponents?** Match the individual exponential expressions with their "written out" forms to get a display like this.



Identify the generalization $A^X \bullet A^Y = A^{X+Y}$ as the **additive law of exponents.** Point out that it involves a situation in which two things hold true:

- The bases are the same.
- The two exponential expressions are being multiplied.

Post the additive law of exponents together with the explanation.

Key Questions

What are the stages of Alice's height change?

If Alice's initial height is *h* and she eats 3 ounces of cake, what will her new height be?

How does the arithmetic explain why the results are the same?

How many 2s are in the expression 2 • 2 • 2? In 2 • 2 • 2 • 2 • 2? In 2 • 2 • 2 • 2 • 2 • 2 • 2 • 2?

Does eating 3 ounces and then 5 ounces of base 7 cake have the same effect as eating 8 ounces of that cake? Why?

What does A^X mean? What does A^Y mean? How would you write these expressions without using exponents?

When Is Nothing Something?

Intent

Students focus on the special meaning of zero as an exponent and then return to the additive law of exponents to explain zero as an exponent in one more way.

Mathematics

Zero is commonly referred to as "nothing" but rarely in mathematics does zero have no meaning. Zero has two general uses in mathematics: as a placeholder in our place-value number system and as the real number 0, located halfway between –1 and 1 on the number line.

Progression

Students work on this task individually. The brief follow-up discussion connects students' methods of justifying their understanding of zero as an exponent to the **additive law of exponents.**

Approximate Time

20 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

As a reminder, ask the class what 2^0 has been defined to equal. Tell students that in this activity, they will explore zero as an exponent in more depth, as well as consider when zero means something and when it means "nothing."

Discussing and Debriefing the Activity

Have several volunteers offer their explanations for Question 1. Questions 2 and 3 give students an opportunity to be both imaginative and reflective. There is no specific mathematical content that needs to emerge here, so students can shape the discussion by what they have to offer.

Zero as an Exponent in the Additive Law of Exponents

If the topic hasn't yet surfaced, ask, **How could you use 2^0 in the additive law of exponents?** Ask for an instance of the additive law of exponents that uses 2^0. Suppose, for example, students suggest the equation

$$2^0 \bullet 2^3 = 2^3$$

Ask, **What number does 2^0 act like in this equation?** To clarify this question, replace each 2^3 in the equation by the number 8 and the 2^0 by a box, as shown here, and ask, **What number should we put in the box to give a true equation?**

$$2^0 \bullet 2^3 = 2^3$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\square \bullet 8 = 8$$

Students should realize that the missing number must be 1, so 2^0 is acting like 1 in the equation $2^0 \bullet 2^3 = 2^3$.

Bring out that this analysis, based on the additive law of exponents, is yet another reason to define 2^0 as 1. Emphasize that this is consistent with the other three explanations students have found to justify this definition:

- Alice eating no cake
- The graph of $y = 2x$
- The pattern of exponential values

Key Questions

How could you use 2^0 in the additive law of exponents?

What number does 2^0 act like in this equation?

What number should we put in the box to give a true equation?

Many Meals for Alice

Intent

Students use the metaphor of eating cakes and drinking beverages to explore another general law about exponents, the law of repeated exponentiation.

Mathematics

When an exponential expression is raised to a power, the following rules hold:.

$$\left(A^D\right)^M = \left(A^M\right)^D = A^{DM}$$

Students use the Alice metaphor to derive and justify these rules, known as the **law of repeated exponentiation.**

Progression

Students work in small groups and then as a class to examine what happens when Alice eats several meals of the same size.

Approximate Time

35 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

Refresh students' memories about the activity *Piece After Piece,* in which they used the Alice metaphor to help derive a general law for exponents, and mention that they will discover another law of exponents today.

If groups have trouble generalizing from Question 2, you might suggest other specific examples for them to explore.

The main idea of this activity can be brought out based on students' work in Question 2; Questions 3 through 5 are primarily for groups who work more quickly. You may want to wait until all groups have at least begun Question 3 before starting the discussion.

Discussing and Debriefing the Activity

Have one or two students give their results for Question 2, referring to Question 1 as needed.

Students may express their answers to Question 1 as powers of 8, because each meal multiplies Alice's height by 2^3, or 8. In general, this gives 8^M as the multiplying factor for M meals of 3 ounces each.
For the general expression in Question 2, they will likely replace 8 by 2^D and get $\left(2^D\right)^M$ as the factor by which her height is multiplied after M meals of D ounces each.

Help the class obtain a general equation for simplifying exponential expressions by comparing two approaches for Question 2. In the approach just described, Alice ate M meals of D ounces each. A second approach is to recognize that Alice is eating of total of DM ounces, so her height is multiplied by 2^{DM}. If no one uses this second approach, ask such questions as, **How much cake was eaten altogether? What does that do to Alice's height?** For Question 1, the answers are "3*M* ounces" and "multiply by 2^{3M}." For Question 2, the answers are "*DM* ounces" and "multiply by 2^{DM}."

After establishing that the sequence in which the cake is eaten doesn't matter, comparing the two methods gives

$$\left(2^D\right)^M = 2^{DM}$$

Will this rule work with bases other than 2? Students will probably be able to generalize the rule as

$$\left(A^D\right)^M = A^{DM}$$

Once students have developed this general formula, ask them to explain the process in terms of repeated multiplication. **Can you write the process as a long-multiplication problem?** They might use a sequence of equalities such as those listed here to connect the two sides of the general equation.

$$\begin{aligned}
\left(A^D\right)^M &= \underbrace{\left(A^D \bullet A^D \bullet \ldots \bullet A^D\right)}_{M\text{ factors}} \\
&= \underbrace{\underbrace{\left(A \bullet A \bullet \ldots \bullet A\right)}_{D\text{ factors}} \bullet \underbrace{\left(A \bullet A \bullet \ldots \bullet A\right)}_{D\text{ factors}} \bullet \underbrace{\left(A \bullet A \bullet \ldots \bullet A\right)}_{D\text{ factors}}}_{M\text{ factors}} \\
&= \underbrace{\left(A \bullet A \bullet A \bullet \ldots \bullet A\right)}_{DM\text{ factors}} \\
&= A^{DM}
\end{aligned}$$

Post the general formula

$$\left(A^D\right)^M = A^{DM}$$

with an explanation like the preceding one , and label it the **law of repeated exponentiation.**

If time allows, use Questions 3 to 5 to develop the rule

$$\left(2^D\right)^M = \left(2^M\right)^D$$

Students might explain this principle using the fact that both expressions are equal to 2^{DM} by the reasoning in Question 2.

Key Questions

How much cake was eaten altogether? What does that do to Alice's height?

Will this rule work with bases other than 2?

Can you write the process as a long-multiplication problem?

In Search of the Law

Intent

In *Piece After Piece* and *Many Meals for Alice,* students derived general rules for operations involving exponential expressions using the Alice metaphor. They now use these rules to derive new ones.

Mathematics

In this activity, students derive the rules listed here by applying the Alice metaphor, the additive law of exponents, and the law of repeated exponentiation.

$$A^X \bullet B^X = (A \bullet B)^X$$

$$A^X \bullet A^X = A^{2X}$$

Progression

Students work on the activity individually. The follow-up discussion brings out justifications for several general principles of exponents and establishes the convention of calling 0^0 *undefined.* In the activity *Confusion Reigns,* students will reexamine these general principles so that they don't simply memorize rules.

Approximate Time

20 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Students have developed several rules for exponents. Tell them that they could derive many, many such rules. The rules identified so far use only one base. In this activity, students will look for rules that work when the base changes.

Discussing and Debriefing the Activity

Let students briefly collaborate and compare ideas in their groups, and then have them report. Focus the discussion on the explanations of what is happening with the factors in a given exponential expression. In the long run, having students understand specific examples will be more productive than having them memorize formulas.

Question 1

For Question 1, students should know how to regroup and pair the factors. If the presenter doesn't bring this out clearly, encourage the audience to ask for clarification. For instance, if the presenter is working with the example $3^7 \bullet 5^7$ given in the problem, drawing out a variety of ways to record this as repeated multiplication can yield patterns that suggest a "law." **In what ways can we write $3^7 \cdot 5^7$ as repeated multiplication?** Answering this should yield expressions such as these.

$$3 \bullet 3 \bullet 3 \bullet 3 \bullet 3 \bullet 3 \bullet 3 \bullet 5 \bullet 5 \bullet 5 \bullet 5 \bullet 5 \bullet 5 \bullet 5$$

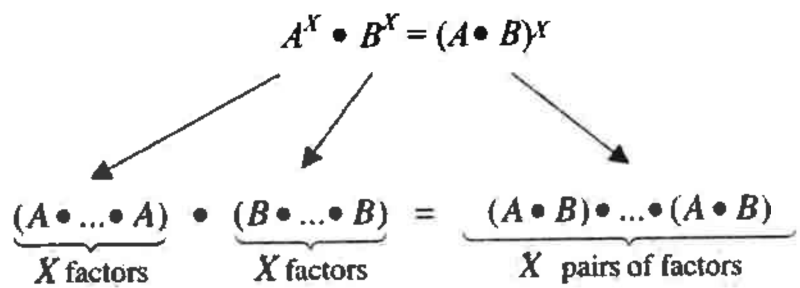
$$(3 \bullet 5) \bullet (3 \bullet 5) \bullet (3 \bullet 5) \bullet (3 \bullet 5) \bullet (3 \bullet 5) \bullet (3 \bullet 5) \bullet (3 \bullet 5)$$

If students don't suggest this second example, encourage them to realize that they can pair up and rearrange the factors by asking, **How might the fact that there are the same number of 3s as 5s be useful?** Students should notice that the expression $3^7 \bullet 5^7$ can be rewritten as $(3 \bullet 5)^7$.

If needed, have another student present a similar example. Work toward a generalization of the principle that any product of the form $A^X \bullet B^X$ can be rewritten as a single exponential expression using the equation

$$A^X \bullet B^X = (A \bullet B)^X$$

You may want to ask students to explain this general principle in terms of repeated multiplication, in a manner similar to that used for the specific examples. A display like the one here illustrates how to rearrange and regroup the factors. Post this principle, with its explanation.



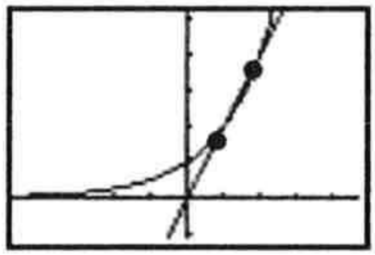
Question 2

The goal in Question 2 is to bring out that an expression of the form $A^X \bullet A^X$ can be simplified using either the additive law of exponents or the principle from Question 1 along with the law of repeated exponentiation explored in *Many Meals for Alice.* The two methods give equivalent results.

According to the additive law of exponents, $A^X \bullet A^X$ is equal to A^{X+X}, which equals A^{2X}. By the principle from Question 1, $A^X \bullet A^X$ is equal to $(A \bullet A)^X$, which equals $(A^2)^X$. And by the law of repeated exponentiation from *Many Meals for Alice,* $(A^2)^X$ is equal to A^{2X}. Thus the two approaches—the additive law of exponents and the principle from Question 1—lead to the same answer.

Question 3
Let students share their results on Question 3. It turns out that the only *positive integer* solutions to the equation $A^X = A \bullet X$ are

- $X = 1$ and A = any number (because $A^1 = 1 \bullet A$, or $A = A$)
- X = any positive number and $A = 0$
- $X = 2$ and $A = 2$ (the two solutions of the equations $Y = 2^X$ and $Y = 2X$; refer to the graph)



If X is a value other than 1, there will be a unique value of A that fits the equation, but unless $X = 2$, that value will be irrational. For instance, if $X = 3$, the value of A that fits the equation is $\sqrt{3}$.

What About 0^0?
The case for which X and A are both 0 presents a special problem. Bring out that in trying to define 0^0, there is a contradiction between two principles.

- On one hand, any power of 0 ought to be equal to 0. (You may need to look at examples such as 0^2 and 0^3 to clarify this principle.)
- On the other hand, we've defined expressions with an exponent of 0 to be equal to 1.

Because of this contradiction, the expression 0^0 is generally considered undefined. Calculators give an error message if you try to calculate 0^0.

Ask students, **Do you know of any other situations in which operations are undefined?** If no one thinks of any, remind them of division by zero and discuss why expressions like $5 \div 0$ are undefined. (The issue of undefined expressions will arise again when negative exponents are considered.)

Key Questions

In what ways can we write $3^7 \bullet 5^7$ as repeated multiplication?

How might the fact that there are the same number of 3s as 5s be useful?

Do you know of any other situations in which operations are undefined?

Having Your Cake and Drinking Too

Intent

Using the Alice metaphor and the rules they have derived so far, students discover why it makes sense to define expressions using negative exponents (such as 2^{-3}) as representing fractions.

Mathematics

For the sake of consistency, we define 2^0 as 1 and 2^{-B} as $\frac{1}{2^B}$. With these definitions, the rule $A^X \bullet A^Y = A^{X+Y}$ works even when $Y < 0$ and $|Y| \geq X$. In addition, the pattern that begins $2^3 = 8$, $2^2 = 4$, $2^1 = 2$ can continue, with $2^0 = 1$ and $2^{-1} = \frac{1}{2}$.

Based on their experience in this activity, students establish the principle that

$2^C = \left(\frac{1}{2}\right)^B = 2^{C-B}$ and use this principle to establish the definition $2^{-B} = \frac{1}{2^B}$.

Progression

Working in groups, students examine what Alice's height is multiplied by if she consumes both cake and beverage. Then they experiment with what happens if she consumes more beverage than cake. This results in a need to consider negative exponents, which are introduced during the discussion of the activity.

Approximate Time

60 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

Introduce this task by posing the first question aloud to the class or as a question for groups to work on for a short period of time. **What is Alice's height multiplied by if she consumes the same number of ounces of cake and beverage?** Discuss students' ideas and record the equation they develop. Then have groups work on the remaining questions.

For Question 4, if a group comes up with the expression

$$2^C = \left(\frac{1}{2}\right)^B$$

encourage the members to look at their examples from Questions 2 and 3 to find a way to rewrite their answer as a power of 2 with a single exponent.

Discussing and Debriefing the Activity

You can begin the discussion after all groups have at least started Question 4. If some groups don't get to Question 5, you can deal with the issue it raises in the whole-class discussion.

If you haven't already talked about Question 1 as a class, begin by getting students to articulate the idea that equal amounts of cake and beverage "cancel out." They may be able to write a general equation to explain this cancellation, such as

$$2^N \bullet \left(\frac{1}{2}\right)^N = 1$$

Questions 2 and 3

For Question 2, students should determine that any combination in which the number of ounces of cake is 3 more than the number of ounces of beverage will work. They should find similar results for the two parts of Question 3.

Through their work on Questions 2 and 3, which start with a desired result and ask for combinations that yield that result, students will probably also recognize how to go in the other direction. Ask them to articulate the arithmetic they do when presented with a combination of cake and beverage. **How do you get from the number of ounces of cake and beverage to the effect on Alice's height?** They might say, for example, "Subtract the number of ounces of beverage from the number of ounces of cake, and take 2 to that power."

Question 4

Ask students to report their observations about individual cases as a general expression. **If *C* is the number of ounces of cake and *B* is the number of ounces of beverage, what happens to Alice's height?** They should be able to state that Alice's height is multiplied by 2^{C-B}.

Different groups may offer equivalent expressions that you can show to be equal. Or you may want to ask for an equation that shows the separate effects of the cake and the beverage in a single expression, such as

$$2^C = \left(\frac{1}{2}\right)^B = 2^{C-B}$$

Before going on, post the expression 2^{C-B} for combining cake and beverage. Students will refer to it as the basis for defining negative exponents.

Build confidence in the expression 2^{C-B} to help motivate the definition of negative exponents, as well as to reinforce earlier work with zero as an exponent. Ask, **Does the expression 2^{C-B} work if *B* and *C* are equal?** You might offer a specific example, such as Alice eating 4 ounces of cake and washing it down with 4 ounces of beverage.

Students should recognize that if $B = C$, Alice's height doesn't change; in other words, it is multiplied by 1. Because the expression gives 2^{4-4} and we have defined 2^0 as equal to 1, the expression does work when *B* is equal to *C*.

Question 5: The Case of Negative Exponents

To introduce the idea of negative exponents, ask, **What does the expression 2^{C-B} say in cases in which *B* is greater than *C*?**

Look at some specific cases. For example, **What does the formula say about the case in which Alice eats 0 ounces of cake and drinks 3 ounces of beverage?** There are two aspects to this question, and the key is bringing them together.

- Drinking 3 ounces of beverage multiplies Alice's height by $\left(\frac{1}{2}\right)^3$, or $\frac{1}{8}$.
- Substituting 0 for *C* and 3 for *B* in the expression 2^{C-B} tells us that Alice's height is multiplied by 2^{0-3}, which simplifies to 2^{-3}.

Once both aspects have been brought out, ask what this says about defining 2^{-3}. You may want to remind students that, like 2^0, the expression 2^{-3} cannot be defined in the usual way, in terms of repeated multiplication.

Students' work in developing the definition of 2^0 will probably lead them to conclude that it makes sense to define 2^{-3} as $\frac{1}{8}$. (*Rallods in Rednow Land* offers a different approach to the same conclusion.)

Present one or two more numeric examples, and then ask, **How can we generalize these results? In other words, how should we define 2^{-B}?** On one hand, this formula says to multiply Alice's height by 2^{0-B}, or 2^{-B}. On the other hand, drinking *B* ounces of beverage while eating no cake will multiply Alice's height by $\frac{1}{2^B}$.

Students should agree that, for the sake of consistency, it makes sense to define 2^{-B} as $\frac{1}{2^B}$. Emphasize that, as with the zero exponent, this is a definition. Students may prefer to write $\left(\frac{1}{2}\right)^B$ instead of $\frac{1}{2^B}$, which is fine.

Key Questions

What is Alice's height multiplied by if she consumes the same number of ounces of cake and beverage?

How do you get from the number of ounces of cake and beverage to the effect on Alice's height?

If *C* is the number of ounces of cake and *B* is the number of ounces of beverage, what happens to Alice's height?

Does the expression 2^{C-B} work if *B* and *C* are equal?

What does the expression 2^{C-B} say in cases in which *B* is greater than *C*?

What does the formula say about the case in which Alice eats 0 ounces of cake and drinks 3 ounces of beverage?

How can we generalize these results? In other words, how should we define 2^{-B}?

Rallods in Rednow Land

Intent

This activity will help students to appreciate the fact that quantities that grow exponentially grow very quickly.

Mathematics

The key mathematical idea in this task, in which students add the number of coins on each square of a chessboard, and the number of coins on each square is double the number on the previous square, is exponential growth. By the 31st square, the number of coins on that square alone has reached 2^{30}, or 1,073,741,824, which is more than the first choice for reward. The total number on the first 31 squares is $2^{31} - 1$, or 2,147,483,647. And there are still 33 squares to go.

To fully answer the question, students must devise a way to compute the sum of the geometric sequence

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^n = \sum_{i=0}^{n} 2^i$$

Using patterns, they might realize that

$$1 + 2 = 3, \text{ or } 2^2 - 1$$
$$1 + 2 + 2^2 = 7, \text{ or } 2^3 - 1$$
$$1 + 2 + 2^2 + 2^3 = 15, \text{ or } 2^4 - 1$$

In general,

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$$

Or, using the summation notation students have encountered before,

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Progression

Working individually, students examine a classic problem about exponential growth by considering intuitively how the sum of a geometric series might compare to a very big number (1 billion). In the follow-up discussion, students share ideas about how to compute a geometric sum. (It is not intended that they develop a formal method.)

Approximate Time

5 minutes for introduction
20 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Students are likely to encounter scientific notation on their calculators in their work on this activity. Some may have encountered this notation before; others may be unfamiliar with it. If they raise questions about the "strange" results on their calculators, offer a brief explanation and reassure them that they will learn more about this notation later in the unit.

Tell students you wish to share an interesting mathematical problem—related to exponents—that can be traced far back in history, and then describe the situation and the adviser's two choices. Ask students to share their initial, intuitive guesses as to which would be the better choice.

Discussing and Debriefing the Activity

Have students gather in their groups to compare ideas on Question 2. They do not need to find the sum of the sequence $1 + 2 + 4 + \cdots + 2^{63}$ to answer the question, because the 31st square by itself already has more than a billion rallods.

You might ask students to provide an expression for the number of rallods on the *N*th square to clarify that it's 2^{N-1} and not 2^N.

Have a volunteer or two present their ideas on Question 3. One approach is to keep adding terms until the sum reaches 1 billion. Another is to recognize that each term is 1 more than the sum of the previous terms and then look for a term that is over half a billion. A third approach is to add the two or three largest terms and assume that this estimate is close enough. All three approaches show that 30 squares give just over a billion rallods.

Supplemental Activity

More About Rallods (extension) asks students to find a general formula for the sum of the first *n* powers of 2 and then has them explore other geometric sequences.

Continuing the Pattern

Intent

Students use a pattern approach to understand the definition of negative exponents. The activity sets the stage for a summative lesson on several recent topics.

Mathematics

Having explored the definition of negative exponents in several ways, students will now approach the topic by inspecting the pattern of decreasing exponents on a constant base. They will also reason through the equivalence of negative exponents on fractional bases to the positive (opposite) exponents on the corresponding reciprocal base. For example,

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16} = 2^{-4}$$

In this context, students simplify complex fractions. They also consider calculator usage with regards to zero and negative exponents, and they return to the question of zero as a base.

Progression

Students work individually, and then as a whole class, to confirm that the pattern of exponential expressions agrees with the definition of exponentiation for negative exponents derived in *Having Your Cake and Drinking Too*. You then elaborate on the patterns, demonstrating ways to record equivalences involving complex fractions. The activity concludes with a discussion of calculators with regards to negative and zero exponents.

In their work in this unit, students are given several ways to approach a fairly abstract concept: that expressions with negative exponents are defined in terms of fractions. The activity *All Roads Lead to Rome* will review the various ways to think about extending the definition of exponential expressions beyond positive integer exponents.

Approximate Time

20 minutes for activity (at home or in class)
40 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Tell students that in this activity they will discover another way of thinking about negative exponents.

Discussing and Debriefing the Activity

Begin the discussion by reviewing the sequence of results in Question 1. Ask students to describe the pattern, which can be articulated in various ways, such as, "Each value is half the one above it" or "Divide by 2 as you go down the list."

Encourage students to write the results involving negative exponents in several ways: as simplified fractions, as powers of $\frac{1}{2}$, and in the form $\frac{1}{2^B}$.

$$2^{-1} = \frac{1}{2}$$

$$2^{-2} = \frac{1}{4} = \left(\frac{1}{2}\right)^2 = \frac{1}{2^2}$$

$$2^{-3} = \frac{1}{8} = \left(\frac{1}{2}\right)^3$$

$$2^{-4} = \frac{1}{16} = \left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$$

Ask them to relate the conclusions here to their previous work with negative exponents. **How do these results relate to your work on *Having Your Cake and Drinking Too?*** Use these examples to review and generalize the principle that we define 2^{-B} as $\frac{1}{2^B}$. Students should realize that the pattern just found for powers of 2 is another reason it makes sense to define expressions with negative exponents as we do. For many students, this is the most convincing and memorable way to think about negative exponents.

For Question 2, students should recognize that a similar pattern holds for any whole-number base and that, in general, it seems to make sense to define A^{-B} as equal to $\frac{1}{A^B}$.

For Question 3, they should realize that they have reversed the pattern from Question 1. Ask, **Does the general principle that A^{-B} is defined as $\frac{1}{A^B}$ apply**

when A is $\frac{1}{2}$? Help students to understand that if this principle holds, we are saying, for example, that $\left(\frac{1}{2}\right)^{-4}$ should be defined as

$$\frac{1}{\left(\frac{1}{2}\right)^{-4}}$$

which suggests that $\left(\frac{1}{2}\right)^{-4}$ is equal to 16 (which is 2^4). This will require students to think about how to simplify a *complex fraction*—that is, a fraction in which the numerator or denominator is itself a fraction.

You may want to have students first look at the simpler case of

$$\frac{1}{\frac{1}{2}}$$

Here are two ways to think about simplifying this expression, both of which can be applied to any complex fraction of the form

$$\frac{\frac{a}{b}}{\frac{c}{d}}$$

- Multiply both the numerator (1) and the denominator $\left(\frac{1}{2}\right)$ by 2, giving

$$\frac{2 \bullet 1}{2 \bullet \frac{1}{2}}$$

which simplifies to $\frac{2}{1}$, which equals 2. Bring out that this is the same process students always use to create equivalent fractions, such as in expressing $\frac{1}{2}$ as $\frac{2}{4}$.

- Interpret the fraction as division, that is, as $1 \div \frac{1}{2}$, and use the "invert and multiply" rule to rewrite it as $1 \bullet \frac{2}{1}$.

Summing Up

Before using the additive law of exponents to confirm the definition, you may want to have a volunteer summarize the class's conclusions about defining expressions with negative integer exponents.

Students may initially give either a numeric example, such as $2^{-2} = \frac{1}{4}$, or a general principle, such as $A^{-B} = \frac{1}{A^B}$. Either way, help the class to state the general principle clearly, perhaps in this way:

> *An exponential expression with a negative exponent is defined by the equation* $A^{-B} = \frac{1}{A^B}$.

Post this principle for reference.

Confirming the Definition Using the Additive Law of Exponents

Choose a particular expression with a negative exponent, such as 2^{-2}, and ask the class, **What is an example of the additive law of exponents that uses the expression 2^{-2}?** Ask for an example in which the exponent on the right side comes out positive, such as

$$2^{-2} \bullet 2^5 = 2^{-2+5}, \text{ or } 2^3$$

Have students verify that the new definition is consistent with the additive law of exponents by substituting the value of each expression: $\frac{1}{4}$ for 2^{-2}, 32 for 2^5, and 8 for 2^3. With these values substituted, the equation becomes the true equation

$$\frac{1}{4} \bullet 32 = 8$$

Then have students try an example or two in which all the exponents are negative, such as

$$2^{-3} \bullet 2^{-2} = 2^{-3+(-2)}, \text{ or } 2^{-5}$$

Substituting values based on the new definition gives the true equation

$$\frac{1}{8} \bullet \frac{1}{4} = \frac{1}{32}$$

A good special case to examine is one in which the exponents are opposites, such as $2^5 \bullet 2^{-5}$. Bring out that the exponents "cancel out" to give a sum of 0, while the exponential expressions themselves also cancel out to give a product of 1.

More Examples of Cake and Beverage

Review the general phenomenon of positive and negative exponents in terms of the Alice metaphor, perhaps by presenting several combinations of cakes and beverages—some with more cake, some with more beverage, some with equal amounts—and having students analyze the effect on Alice's height in two ways.

- Work with the two types of food sequentially (for example, first the cake, then the beverage) to determine the effect on Alice's height.
- Combine the cake and beverage into a single amount of one or the other, using the intuitive idea that "equal amounts of cake and beverage cancel each other." (Students may recognize that cake and beverage are analogous to hot and cold cubes, introduced in the Year 1 unit *Patterns.*)

Bring out that the two approaches give the same results. Help the class to verbalize the notion that in combining cake and beverage, they are treating the beverage as a kind of "negative cake." Therefore, the effect of B ounces of beverage, which is to multiply Alice's height by $\frac{1}{2^B}$, should be the same as the effect of $-B$ ounces of cake. According to the general formula students have developed, eating $-B$ ounces of cake should multiply Alice's height by 2^{-B}. In other words, $\frac{1}{2^B}$ and 2^{-B} should be equal.

Graphing $y = 2^x$

During the activity *Graphing Alice,* you posted a graph of points from the equation $y = 2^x$ for positive integer values of x. As students worked with zero as an exponent, they saw that defining 2^0 to be 1 seemed consistent with this graph. Now it's time to extend the graph to include negative integer values for x.

In that discussion, students speculated about what would happen if they extended the graph of $y = 2^x$ to include negative values of x. Ask volunteers to add new points to that graph, first for x with a value of 0 and then for negative integer values of x.

Although the scale of the earlier graph may make it hard to plot these points precisely, students should observe that the general shape of the first quadrant portion of the graph is consistent with these new points. As x moves from larger to smaller positive values, the y-values decrease in a way that fits smoothly with new points at (0, 1), $\left(-1, \frac{1}{2}\right)$, $\left(-2, \frac{1}{4}\right)$, and so on.

Negative and Zero Exponents on Calculators
Ask students to verify that their calculators give results for zero and negative exponents that agree with the definitions developed so far.

For negative exponents, calculators will give results in decimal form. Students will need to verify that these are equal to the common-fraction values they have used in the definition. For example, if they evaluate 2^{-3}, they will get 0.125, which is equal to $\frac{1}{8}$.

Zero as a Base
Ask, **What happens if you use 0 as the base with a negative exponent? Why?** Students should get an error message. If time permits, this can lead to a good discussion of the issue of division by zero and why division by zero is undefined. At the same time, you can review the situation of 0^0 described in *In Search of the Law.*

Key Questions

How do these results relate to your work on *Having Your Cake and Drinking Too*?

Does the general principle that A^{-B} is defined as $\frac{1}{A^B}$ apply when A is $\frac{1}{2}$?

What is an example of the additive law of exponents that uses the expression 2^{-2}?

What happens if you use 0 as the base with a negative exponent? Why?

Negative Reflections

Intent

Students carefully review and summarize what they know about negative exponents in preparation for the upcoming topic of fractional exponents.

Mathematics

The work in *Extending Exponentiation* has extended the possible values for exponents from positive whole numbers—which can be interpreted as repeated addition—to zero and negative integers. This work has relied on the Alice metaphor, patterns, and the systematic development of rules using prior rules. In this activity, students pause to look back over and summarize this work.

Progression

Working individually, students summarize their work with integer exponents by writing an explanation for the definition along with some examples. They share their explanations with an adult, and a brief class discussion of these conversations concludes the activity.

Approximate Time

20 minutes for activity (at home)
5 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Remind students that the past few activities have extended the definition for exponents from simple repeated multiplication for positive exponents ($2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$) to also make sense for zero and negative values.

Discussing and Debriefing the Activity

You may want to invite a few students to briefly share how the adults they spoke with reacted to the ideas.

This is a good activity for assessing students' understanding. You will probably benefit from reading at least a selection of students' explanations, even if you choose not to grade the assignment. Students' explanations in Question 1 in particular will give you a good idea of how well they have understood the principles involved in extending the definition of exponentiation.

Curiouser and Curiouser!

Intent

Through these activities, students extend their growing understanding of exponents to include rational exponents.

Mathematics

In an approach similar to the development of meaning for zero and negative integer exponents, students' understanding of rational exponents will not be based on the idea that an exponent indicates repeated multiplication. Instead, students will use the Alice metaphor, numeric patterns, rules they developed in previous activities, and graphs of exponential functions to develop an understanding of this rule:

For all integer values x and y ($y \neq 0$), and for $b > 0$,

$$b^{x/y} = \sqrt[y]{b^x} = \left(\sqrt[y]{b}\right)^x$$

Progression

In *Curiouser and Curiouser!,* students first consider exponents that are unit fractions and then work with the more general case of any fraction. They then review all their rules for exponents and do an activity that sets up the study of logarithms in *Turning Exponents Around.* In addition, students begin work on the second POW of the unit.

A Half Ounce of Cake

It's in the Graph

POW 13: A Digital Proof

Stranger Pieces of Cake

Confusion Reigns

All Roads Lead to Rome

Measuring Meals for Alice

A Half Ounce of Cake

Intent

Students use the Alice metaphor to think about the meaning of fractional exponents. The discussion of the activity also provides some standard terminology related to exponents and checks that students are using their calculators appropriately.

Mathematics

In this activity, students learn the meaning of fractional exponents by equating them to what they already know about roots and to the additive law of exponents.

$$2^{1/2} = \sqrt{2} \text{ because } 2^{1/2} \bullet 2^{1/2} = 2^{1/2+1/2} = 2$$

The class discusses the language and notation of roots (which may be a review for some students), including the idea that the notation represents an exact value, whereas what a calculator reports is most often a rounded value.

This activity deals only with *unit fractions*—fractions with a numerator of 1 and a denominator that is a positive integer. More general fractional exponents will be explored in *Stranger Pieces of Cake.*

Progression

Students work on the activity in small groups and share their ideas with the whole class. A teacher-led discussion then provides conventions for exponent notation, connections to ideas about roots, and a review of working with roots and fractional exponents on the calculator.

Approximate Time

50 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

To introduce the activity, ask, **What do you think would happen to Alice if she ate cake, but less than a whole ounce?** Students should quickly agree that she will grow, but to something less than double her original height.

Exactly how much would she grow? Say, for example, she eats exactly one half ounce. By what factor would her height be multiplied? One likely suggestion will be a growth factor of 1.5.

Let's say that piece was so good, she wants the rest of the ounce of cake. By what factor will she grow when she eats the second half ounce? Help students to recognize that if Alice eats another half ounce, her height will again be multiplied by the same factor. Encourage discussion so that students invest themselves in establishing this principle. Then put groups to work on the activity. **In your groups, design a way to check whether our initial guesses are correct.**

As they begin work, remind them to think about what Alice's height is *multiplied by* if she eats half an ounce of cake.

As soon as students realize that 1.5 doesn't work, many will begin searching for an appropriate multiplying factor by testing different numbers on the calculator.

One approach is to ask them to create an example of the **additive law of exponents** that shows Alice eating half an ounce and then another half an ounce, as suggested in Question 1. If someone suggests that this situation can be represented by the equation $2^{1/2} \bullet 2^{1/2} = 2^1$, suggest that students replace each instance of $2^{1/2}$ with a box and replace 2^1 with 2, and ask what number should go in each box (the same number in both) to make the equation true.

$$\square \bullet \square = 2$$

As needed, help students realize that they are looking for a number that if multiplied by itself gives a result of 2—that is, the square root of 2. They should be able to identify the number being described as $\sqrt{2}$.

You may want to ask the first groups that finish to prepare presentations.

Discussing and Debriefing the Activity

If a group has prepared to present Question 1, ask a representative to make that presentation. Emphasize that the presenter should talk about their group's exploration as well as their solution, rather than simply showing the number.

Remind the class of the general formula—that eating *C* ounces of cake multiplies Alice's height by 2^C—and ask how that formula applies to this situation. Be sure students recognize that the answer to Question 1 reveals how they should define $2^{1/2}$.

If no one suggests using the additive law of exponents, ask for an explanation defining $2^{1/2}$ based on that principle. Although students may have found that eating half an ounce of cake multiplies Alice's height by about 1.4, they may not make a connection between this number and the use of a fractional exponent.

Students will probably recall the term *square root.* Review the notation $\sqrt{2}$ if needed. They should be able to estimate $\sqrt{2}$, both by using the square-root key on a calculator and by guess-and-check, and thus realize that $2^{1/2}$ equals $\sqrt{2}$.

Question 2

After students have grasped that for half an ounce of cake they need a number whose square is 2, they should easily extend the idea to other fractional pieces of cake. Thus they should realize that if Alice eats a piece of cake that weighs a third of an ounce, her height will be multiplied by the number whose *third* power is 2. They can then use guess-and-check on their calculators to find a solution to the equation $x^2 = 2$.

The Language and Notation of Roots

Point out that the solution to $x^2 = 2$ is called the *square root of 2*. The solution to $x^3 = 2$ is called the *cube root of 2.* Introduce the notation $\sqrt[3]{2}$ for this number.

Follow up by introducing the notation $\sqrt[5]{2}$. Students should recognize that this is the solution to $x^5 = 2$. Use the phrase *fifth root of 2* for this number.

Point out that we could write $\sqrt[2]{2}$ for the square root of 2, but we don't. Also mention that the symbol $\sqrt{\ }$ is called the *radical sign.* (The word *radical* comes from a Latin word root that means "root.")

If students wonder why 1.41 is not sufficient for representing $\sqrt{2}$, explain that there is no decimal they can write whose square is exactly 2, so if they want to represent the number exactly, they need to use the symbol $\sqrt{2}$. (The issues of approximation and rounding are discussed in the unit *Do Bees Build It Best?,* especially in the activity *Falling Bridges.*)

Defining Fractional Exponents

You can now return to extending students' knowledge of fractional exponents. Review the general principle about Alice—that eating *C* ounces of cake multiplies her height by 2^C—and ask, **What happens if Alice eats half an ounce of cake?** Help students to understand that according to the general principle, Alice's height is multiplied by $2^{1/2}$, but their work in the activity shows that it should be multiplied by $\sqrt{2}$.

Remind students that as with negative and zero exponents, the "repeated multiplication" definition doesn't work for fractional exponents, so we need to *define* the expression $2^{1/2}$ by some other method. Students should recognize that their work suggests it makes sense to define $2^{1/2}$ as $\sqrt{2}$ and, similarly, $2^{1/3}$ as $\sqrt[3]{2}$ and $2^{1/5}$ as $\sqrt[5]{2}$.

You will probably want to get a summary of these ideas and post a general principle such as

$$A^{1/n} = \sqrt[n]{A}$$

Roots and Fractional Exponents on Calculators

Have students check that their calculators agree with the definition of $2^{1/2}$ as $\sqrt{2}$ by verifying that the calculators give the same answer for both.

Then have them explore how to find the value of roots and expressions using exponents that are unit fractions.

Some calculators use the symbol ^ (called a *caret*) for exponentiation, writing 2^5 for 2^5. It's often necessary to use parentheses around a fraction used as an exponent. For example, a calculator will probably interpret the expression 2^1/2 as $(2^1) \div 2$, which equals 1. The sure way to get the value of $2^{1/2}$ is to enter 2^(1/2). Some scientific calculators have a $\boxed{\sqrt[x]{y}}$ key (often as a "second function" with the $\boxed{y^x}$ key). To represent $\sqrt[3]{4}$ using the $\boxed{\sqrt[x]{y}}$ key, you enter the y-value, press the $\boxed{\sqrt[x]{y}}$ key, and then enter the x-value. Thus a key sequence like

$$\boxed{4}\ \boxed{\text{2ndF}}\ \overset{\sqrt[x]{y}}{\boxed{y^x}}\ \boxed{3}$$

is used. (On some calculators, you would enter the x-value first.)

For calculators without such a key (as with many graphing calculators), one way to find roots other than square roots is by using fractional exponents. Another way is to use the MATH menu, with a key sequence like

$$\boxed{3}\ \boxed{\text{MATH}}\ \boxed{\sqrt[x]{\ }}\ \boxed{4}.$$

Help students make the connection between roots and fractional exponents. They should understand that the expression $a^{1/n}$ means the same thing as $\sqrt[n]{a}$.

Key Questions

What do you think would happen to Alice if she ate cake, but less than a whole ounce?

Exactly how much would she grow? If she eats exactly one half ounce, by what factor would her height be multiplied?
By what factor will she grow when she eats the second half ounce?

It's in the Graph

Intent

By using a graphical approach to make meaning of expressions like $2^{1/2}$, students reinforce their understanding of fractional exponents.

Mathematics

Students will compare graphs of the exponential function $y = 2^x$ and the linear function $y = x + 1$ to recognize from a visual perspective that, by the curved nature of $y = 2^x$, the value of y when $x = \frac{1}{2}$ must be less than 1.5. (Refer to the graph in "Discussing and Debriefing the Activity.") Students graph a few other exponential functions with similar questions in mind.

Progression

Students work on this activity individually, share ideas and questions in their small groups, and review their findings in a class discussion.

Approximate Time

20 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Materials

It's in the Graph blackline master (transparency)

Doing the Activity

To introduce the activity, ask the class to restate what $2^{1/2}$ means in the Alice situation. Encourage at least two students to interpret the symbols in their own words.

Tell students that they will now consider what the graph of $y = 2^x$ suggests for the decimal value of $2^{1/2}$.

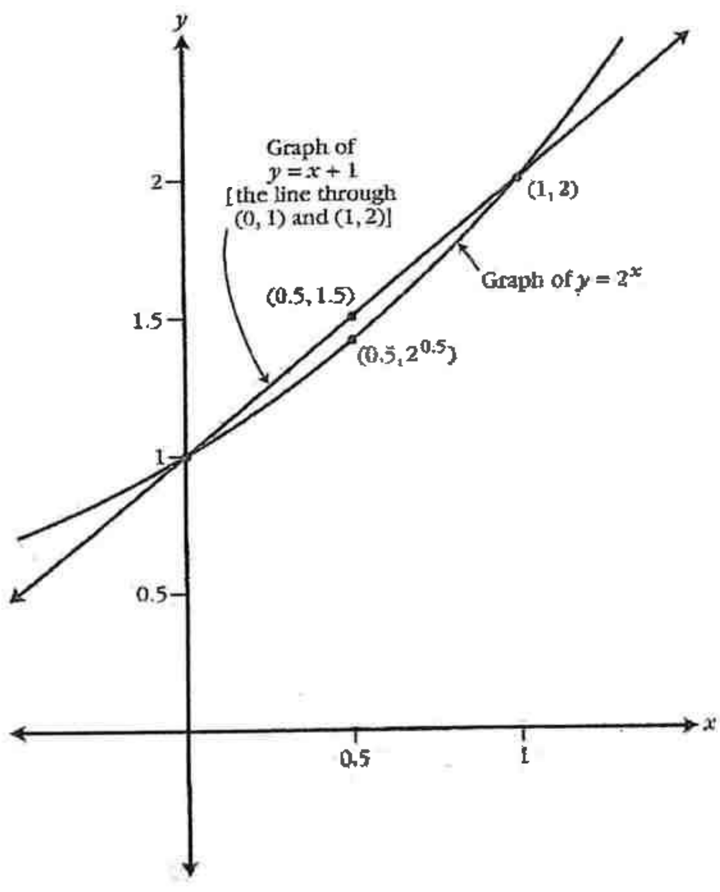
Discussing and Debriefing the Activity

The discussion of this activity will confirm the reasonableness of the definition that has been developed for an exponent of $\frac{1}{2}$. Students should realize that the estimates provided by their graphical analyses are consistent with this definition.

Begin with the comparison of graphs in Question 1, which will bring out graphically that $2^{1/2}$ is less than 1.5.

Students might also use a graphing calculator to graph the function $y = 2^x$ and use the Trace feature to get a more precise value for $2^{0.5}$ by finding the y-coordinate of the point for which x equals 0.5. The ZOOM menu will allow them to choose as much precision as they want. Make sure they realize that the y-coordinate has the numeric value of $\sqrt{2}$ and that their reading is an approximation of this.

Ask for a volunteer to present Question 2. He or she should note that the graph of $y = x + 1$ goes through the point (0.5, 1.5). Students have already observed that the graph of the equation $y = 2^x$ is curved, so when x is equal to 0.5, this graph is below the straight line through (0, 1) and (1, 2).



If time allows, use Question 3 to review the general graphs of other exponential functions. In particular, bring out that they all go through the point (0, 1). For part b, students should note that the graph of $y = 9^x$ seems to confirm that $9^{1/2} = 3$.

Finally, discuss the difference between the graph in part c and those in the other examples.

POW 13: A Digital Proof

Intent

To accomplish a proof for this problem, students must recognize key structures used to develop the puzzle.

Mathematics

The task for students is to determine all possible solutions to a number puzzle, which challenges them to think beyond simply guessing and checking until they find a solution that works. They complete the task by communicating an argument that their solution is unique. The deductive logic required is similar to that used in *POW 12: Logic from Lewis Carroll.*

Progression

This POW has two distinct phases of student work, followed by write-ups and presentations. Initially, students explore the puzzle, trying to find or narrow in on a solution. The second phase involves identifying structures of the puzzle that define the uniqueness of the solution. From these structures, students can write a proof.

Approximate Time

30 minutes for introduction
1–3 hours (at home)
20 minutes for presentations

Classroom Organization

Individuals and groups, follow by whole-class presentations

Doing the Activity

Read the activity as a class, and make sure students understand the directions. Have a volunteer explain the significance, for example, of putting a 3 in the box labeled 4. **What must happen if you put a 3 in Box 4?** Students should understand that this means there must be exactly three 4s used altogether in the boxes.

Have groups begin work on the problem. As they start to come up with possible solutions, encourage them to confirm aloud that each of the five digits they've placed in the boxes satisfies the requirements. Many will struggle to find a solution that works, failing to carefully consider the meaning of one or two numbers.

In their groups, students will begin to identify some of the constraints of the puzzle, which will help them develop a systematic process to narrow down possible solutions. Remind them that their main task is to prove they have found all possible solutions.

If groups find a solution, encourage them to identify what they learned about the puzzle that helped guide them toward that solution. These discoveries will likely be key parts of the proof they will write.

To help lead students to develop the argument they will need in the proof, ask, **Have you found all possible solutions? How do you know?**

The day before the POW is due, identify three students to prepare presentations.

Discussing and Debriefing the Activity

Have the three selected students make their presentations. Focus the presentations and discussion on the proof that the solution given is the *only* solution.

In one sense, the proof that the solution is unique consists of eliminating all other possibilities. However, because there are so many possible ways to fill the boxes (5^5, or 3,125), the other cases must be eliminated in an organized and systematic way.

This is an excellent problem for talking about proof. Students should recognize that saying "I couldn't find another solution" is not a proof that there are no others.

Key Questions

What must happen if you put a 3 in Box 4?

Have you found all possible solutions? How do you know?

Supplemental Activity

Ten Missing Digits (extension) expands the puzzle in this POW to ten missing digits.

Stranger Pieces of Cake

Intent

In this activity, students examine exponential expressions in which the exponent is not a unit fraction.

Mathematics

For all integer values x and y ($y \neq 0$), and for $b > 0$,

$$b^{x/y} = \sqrt[y]{b^x} = \left(\sqrt[y]{b}\right)^x$$

This rule makes sense given the other rules developed to this point in the unit. For example, using the law of repeated exponentiation,

$$b^{x/y} = \left(b^x\right)^{1/y} = \sqrt[y]{b^x} \text{ and } b^{x/y} = \left(b^{1/y}\right)^x = \left(\sqrt[y]{b}\right)^x$$

Progression

Working individually, students consider what an exponential statement like $2^{3/5}$ might mean, and then they generalize to a definition for fractional exponents. The follow-up discussion also explores negative fractional exponents and the idea that students have now articulated meaning for any rational number exponent for a positive base.

Approximate Time

5 minutes for introduction
20 minutes for activity (at home or in class)
35 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Read through Question 1 of the activity with students to orient them to the task.

Many students will struggle with Question 1, especially if they don't recognize $\frac{3}{5}$ as "three one-fifths." If time allows, have them begin tackling the question in their groups prior to working on their own.

Discussing and Debriefing the Activity

Ask for volunteers to explain their work on Question 1. Students typically approach this by saying that eating a $\frac{3}{5}$-ounce piece of cake is the same as eating three $\frac{1}{5}$-ounce pieces. Because each $\frac{1}{5}$-ounce piece multiplies Alice's height by $\sqrt[5]{2}$, eating three such pieces will multiply her height by $\left(\sqrt[5]{2}\right)^3$. Thus it makes sense to define $2^{3/5}$ as $\left(\sqrt[5]{2}\right)^3$.

An alternate approach is to imagine Alice eating five pieces of cake, each weighing $\frac{3}{5}$ ounce. This is a total of 3 ounces of cake, and would thus multiply Alice's height by 2^3, or 8. Because five $\frac{3}{5}$-ounce pieces eaten together multiply her height by 8, one such piece would multiply her height by $\sqrt[5]{8}$.

The first approach interprets $2^{3/5}$ as $\left(\sqrt[5]{2}\right)^3$. The second approach interprets $2^{3/5}$ as $\left(\sqrt[5]{2}\right)^3$. Students can verify that these values are equal.

If no student suggests this idea, you might offer it yourself. Observing that $\sqrt[5]{2^3}$ should be equivalent to $\left(\sqrt[5]{2}\right)^3$ gives insight into the meaning of the notation.

One way to prove that $\left(\sqrt[5]{2}\right)^3$ is equal to $\sqrt[5]{2^3}$, based on the general laws of exponents, is to use the sequence of equalities

$$\left[\left(\sqrt[5]{2}\right)^3\right]^5 = \left(\sqrt[5]{2}\right)^{15} = \left[\left(\sqrt[5]{2}\right)^5\right]^3 = 2^3$$

which shows that $\left(\sqrt[5]{2}\right)^3$ is the fifth root of 2^3.

Question 2: Defining $2^{p/q}$

If students define $2^{3/5}$ as $\left(\sqrt[5]{2^3}\right)^3$, they will likely have little trouble generalizing to the idea that eating a piece of cake weighing $\frac{p}{q}$ ounces is like eating p pieces that each weigh $\frac{p}{q}$ ounces.

Because each $\frac{1}{q}$-ounce piece multiplies Alice's height by $\sqrt[q]{2}$, p such pieces should multiply her height by $\left(\sqrt[q]{2}\right)^p$. Therefore, it makes sense to define $2^{p/q}$ as $\left(\sqrt[q]{2}\right)^p$.

A student may ask, "What if q is 0? What is the 0th root of 2?" If so, point out that the exponential form would then be $2^{\frac{p}{0}}$, and the fraction $\frac{p}{0}$ is undefined. For similar reasons, the 0th root of 2, or of any number, is undefined.

Defining Negative Fractional Exponents

To extend the definition of exponentiation to include all rational numbers as exponents, ask whether anyone has an idea about this. **How should we define $2^{-1/2}$?** If needed, suggest there might be a way to use Alice's beverage for assistance. Remind students that drinking 3 ounces of beverage multiplies Alice's height by 2^{-3}, or $\left(\frac{1}{2}\right)^3$, and a whole ounce multiplies it by 2^{-1}, or $\frac{1}{2}$. Consequently, half an ounce should multiply her height by $2^{-1/2}$, or $\sqrt{\frac{1}{2}}$.

Does this definition fit the earlier principle that $A^{-B} = \frac{1}{A^B}$?

Work with students to help them recognize that

$$\sqrt{\frac{1}{2}} = \frac{1}{2^{1/2}}$$

In *Simply Square Roots* from the unit *Do Bees Build It Best?*, students worked with the general principle

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

which shows that

$$\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

The denominator $\sqrt{2}$ is equal to $2^{1/2}$, so

$$\sqrt{\frac{1}{2}} = \frac{1}{2^{1/2}}$$

The General Exponential Function

Another important idea that should come out of this discussion is that the function $y = 2^x$ makes sense for *any* number x. At this stage, students may have only a hazy notion that there are such things as irrational numbers, so you won't be able to discuss the "complete" exponential function, with a real number domain, in a formal sense. But they should realize that they have defined it for all rational exponents (for a positive base), and that should persuade them that they can make sense of any exponential expression with a positive base.

Bring out this point by asking, **How could an expression like $2^{0.562}$ be interpreted in terms of the Alice situation?** Students should be able to articulate that this number is what Alice's height would be multiplied by if she ate 0.562 ounce of cake. They should also recognize that theoretically they could find this number by thinking of this as 562 pieces, each weighing $\frac{1}{1000}$ ounce, and that each of those pieces would multiply Alice's height by $\sqrt[1000]{2}$, so that

$$2^{0.562} = \left(\sqrt[1000]{2}\right)^{562}$$

Similarly, $2^{-0.562}$ represents the factor by which Alice's height is multiplied if she drinks 0.562 ounce of beverage.

After exploring such examples, ask explicitly, **Does the expression 2^x make sense for all values of x?** Students should realize that it at least makes sense when x is a rational number. If they raise questions about irrational exponents, explain that the definition can be extended using repeated approximations and the concept of a *limit.*

The Complete Graph

This is a good occasion to look once again at the graph of $y = 2^x$ as a whole. Have students graph this function on their calculators and use the ZOOM menu and the Trace feature to check that the coordinates of points on the graph are consistent with the definitions they have formulated in extending the operation of exponentiation to include zero, negative, and fractional exponents.

The General Base and Exponent

Finally, bring out that the work just done with base 2 applies to any positive base. You might discuss a "random" example such as $0.416^{-6.78}$ to illustrate how such a general definition would work. Students can describe this as the factor by which Alice's height is multiplied if she drinks 6.78 ounces of base 0.416 beverage. Thus this number is equal to

$$\frac{1}{\left(\sqrt[100]{0.416}\right)^{678}}$$

Key Questions

How should we define $2^{-1/2}$?

Does this definition fit the earlier principle that $A^{-B} = \frac{1}{A^B}$?

How could an expression like $2^{-0.562}$ be interpreted in terms of the Alice situation?

Does the expression 2^x make sense for all values of x?

Supplemental Activities

Exponential Graphing (reinforcement) offers students more opportunities to examine the graphs of exponential functions.

Basic Exponential Questions (extension) raises challenging questions about inequalities involving exponential expressions. Question 2 is intentionally trivial (as it involves base 1). Question 3 follows up with a similar but more complicated problem. Question 4 is quite difficult to solve in general. The only whole-number solutions are the cases in which X is 2 and Y is 4, and vice versa. Students may find explanations for why there are no other solutions.

Confusion Reigns

Intent

Students review and explain the general principles for exponents developed during the unit. They come to the recognition that such laws eventually boil down to understanding what exponents are and that the laws can be re-created by writing out the exponential expressions using repeated multiplication.

Mathematics

Students review three laws of exponents, developed in the activities *Piece After Piece, Many Meals for Alice,* and *In Search of the Law,* with an emphasis on the justification for each law.

$$A^x \bullet A^y = A^{x+y}$$

$$A^x \bullet B^x = (A \bullet B)^x$$

$$(A^x)^y = A^{xy}$$

Progression

Students work individually to evaluate a number of proposed rules for working with exponents. In the follow-up discussion, they confirm their understanding and identify an associated law of exponents (or "nonlaw") for each question.

Approximate Time

20 minutes for activity (at home or in class)
20 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Acknowledge that students have discovered many general laws about exponents that always hold. Tell them that in this activity, they will review some of those laws.

Discussing and Debriefing the Activity

You may want to let students spend some time in groups sharing ideas. They will be able to verify which equations are numerically correct by doing the arithmetic, so the focus of the discussion can be on explanations and finding general rules. Although the principles needed for Questions 2 and 3 were discussed earlier (in *Many Meals for Alice* and *In Search of the Law*), this is a good opportunity to review them.

The comments listed here summarize the key mathematical ideas to keep in mind during the discussion.

Question 1
There are no simple rules for adding numbers with exponents, whether the bases are the same or different. This "nonrule" may be worth remembering.

Question 2
The general principle for multiplying when the exponents are the same (whether the bases are the same or different) is

$$A^B \bullet C^B = (A \bullet C)^B$$

which is in line with Lara's idea. Have someone use repeated multiplication to explain in detail how Lara's example works. As discussed in connection with *In Search of the Law,* students should be able to write $2^3 \bullet 5^3$ as a product of individual factors, like this

$$2 \bullet 2 \bullet 2 \bullet 5 \bullet 5 \bullet 5$$

and then rearrange the factors in pairs, like this

$$(2 \bullet 5) \bullet (2 \bullet 5) \bullet (2 \bullet 5)$$

to show that this is equal to 10^3.

Question 3
Help students relate this question to their work on *Many Meals for Alice.* They should realize that Jen has the right idea by finding a general rule like

$$\left(A^B\right)^C = A^{BC}$$

Have them demonstrate how this works using a numeric example and repeated multiplication. For instance, $(7^2)^3$ means $7^2 \bullet 7^2 \bullet 7^2$, and each factor of 7^2 is equal to $7 \bullet 7$, so

$$(7^2)^3 = 7^2 \bullet 7^2 \bullet 7^2 = (7 \bullet 7) \bullet (7 \bullet 7) \bullet (7 \bullet 7) = 7^6$$

The "outside exponent" of 3 means there are three sets of 7s, and the "inside exponent" of 2 means there are two 7s in each set. Students should be able to explain that three sets of 7s with two 7s in each set is a total of six 7s, as $2 \bullet 3$ equals 6. And because the six 7s are multiplied together, the result is equal to 7^6.

So what's important here? It will be nice if students use either of these equations when they come across such situations:

$$A^B \bullet C^B = (A \bullet C)^B$$

$$(A^B)^C = A^{BC}$$

What's more important are the realizations that such laws of exponents eventually boil down to understanding what exponents are and that they can be re-created if one takes the time to write out the exponential expressions using repeated multiplication.

All Roads Lead to Rome

Intent

Students have used several approaches to gain an understanding of the extension of exponentiation beyond positive integer exponents. This activity gives them a chance to review and reflect on this variety of perspectives and will give you information on their ability to synthesize various approaches.

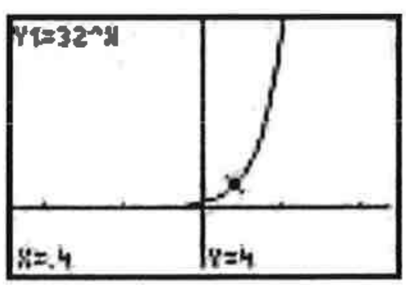
Mathematics

Students summarize how each of several approaches can be used to extend exponentiation to zero, negative, and fractional exponents. For example, to find $32^{2/5}$, they can use the rules derived in this unit to conclude

$$32^{2/5} = \left(32^{1/5}\right)^2 = \left(\sqrt[5]{32}\right)\left(\sqrt[5]{32}\right) = 2 \bullet 2 = 4$$

The unit provides a number of ways to make sense of this computation. According to the Alice metaphor, we want to know what to multiply Alice's height by if she eats $\frac{2}{5}$ ounce of base 32 cake. For 1 ounce, we would multiply by 32. For $\frac{1}{5}$ ounce, we would multiply by 2, as doing this 5 times would result in multiplying by 32. She will be eating $\frac{1}{5}$ ounce twice, so we should multiply by 2 • 2, or 4.

- Using the additive law of exponents, $32^{1/5}$ is the number we would multiply by itself 5 times to get 32, so $32^{1/5} = 2$. Then $32^{2/5} = 2^2 = 4$.
- Using the graph of $y = 32^x$, we can trace to find that y is 4 when x is $\frac{2}{5}$, or 0.4.



Progression

Students will explore this activity in small groups, with each student recording her or his own responses. A short conclusion organized around group presentations confirms their findings. Students' work in this activity will be included in their unit portfolios.

Approximate Time

50 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

Tell students that this activity will be an opportunity to review and summarize their recent work with exponents. The reports they produce will be part of their unit portfolios.

Tell students that they do not have to explain every problem in Question 2 using all four methods. If they get bogged down with a particular explanation, they should move on and return to it later.

Discussing and Debriefing the Activity

Although this activity reviews the various approaches, it is not intended to lead to a detailed discussion of each problem. If time is limited, you might omit discussion of Questions 2c and 2d.

Ask for groups to present their ideas. You might begin with two presenting groups for the first problem or two and then reduce to one presentation as students feel comfortable.

Measuring Meals for Alice

Intent

In this final activity of *Curiouser and Curiouser!,* students look for "missing exponents" in the Alice setting. Their work sets the stage for introducing the concept of a logarithm.

Mathematics

If Alice is 1 foot tall, how much base 2 cake should she eat to grow to 10 feet? This question translates into the equation

$$1(2^x) = 10$$

The task is to find the exponent x that will produce a result of 10. In other words, given the function $y = 2^x$, find x for a given y.

This "undoing" activity sets students up to understand the concept of a **logarithm,** because if $y = 2^x$, then $x = \log_2 y$. That is, the inverse of the exponential function is the logarithmic function. At this point in the unit, students will guess-and-check using their calculators to approximate the solutions.

Progression

Working individually, students write an equation to represent each problem and solve for the unknown exponent using guess-and-check methods. In a class discussion, they share a variety of strategies for solving the problems.

Approximate Time

5 minutes for introduction
20 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Introduce this activity by posing a question to remind students how to set up an exponential equation for these situations. **If Alice begins at 5 feet tall and wants to grow to 40 feet, how much base 2 cake will she need to eat?**

Even though students may be able to recognize quickly that 3 ounces of cake will do the trick, ask them to work in their groups to write an equation with the desired amount of cake as the unknown. Have them find a solution or show that their "obvious" solution works. Then explain that this next activity involves this same idea but in a more complicated form.

Discussing and Debriefing the Activity

You might ask students to share their ideas in groups and then report to the class, emphasizing their thought processes behind writing the equations and finding the solutions.

Students should find that the answer to Question 1 is 3.3 ounces. Before exploring the relationship between this and Question 2a, get an explanation of how the presenter derived the solution. He or she might recognize it as the solution to the equation $2^x = 10$. Presumably students will have found the numeric value by guess-and-check. (Logarithms will be introduced in *Sending Alice to the Moon.*) If the class had difficulty solving the problem, you might begin by asking for a rough approximation. **Between which two whole numbers does the answer lie, and why?**

The main focus of Question 2 should be on the connection between it and Question 1. Students should be able to explain, perhaps with some support from you, why the answer to Question 2 should be exactly twice that for Question 1.

Encourage a variety of explanations. Here are two possible approaches, one based on the Alice metaphor and one on laws of exponents.

- If eating 3.3 ounces of cake multiplies Alice's height by 10, eating this amount twice will multiply her height by 10 and then by 10 again. Therefore, eating 6.6 ounces multiplies her height by 100.
- The answer from Question 1 reveals that $2^{3.3}$ is approximately equal to 10, which means that $2^{6.6} = 2^{3.3 + 3.3} = 2^{3.3} \bullet 2^{3.3} \approx 10 \bullet 10 = 100$.

If students have given clear explanations for Questions 1 and 2, you may choose to discuss Questions 3 and 4 only briefly. In Question 3, Alice should drink about 1.6 ounces of beverage. In Question 4, she will become 3.8 feet tall.

Key Questions

If Alice begins at 5 feet tall and wants to grow to 40 feet, how much base 2 cake will she need to eat?

Between which two whole numbers does the answer lie, and why?

Supplemental Activity

Alice's Weights and Measures (extension) explores issues of approximation. When a measurement is only an approximation, what effect does it have on computations that make use of that measurement? This activity makes a good follow-up to *Measuring Meals for Alice*.

Turning Exponents Around

Intent

These final activities raise two important mathematical ideas related to exponents, as well as ask students to reflect over their work throughout the unit.

Mathematics

Students focus on two important mathematical ideas in these activities.

- The inverse of the exponential function $y = b^x$ is the logarithmic function $x = \log_b y$. The logarithmic function can be used to find the missing value of the exponent in an exponential equation. The graph of the logarithmic function is the reflection of the graph of the exponential function across the line $y = x$.
- Exponents can be used to write large and small numbers in **scientific notation,** as a number between 1 and 10 multiplied by an integer power of 10.

Progression

Students begin these activities by defining and exploring exponential functions. They then delve into scientific notation. They compile unit portfolios and take end-of-unit and end-of-year assessments. Finally, they participate in a class discussion of the big mathematical ideas in Year 2 of the curriculum.

Sending Alice to the Moon

Alice on a Log

Taking Logs to the Axes

Base 10 Alice

Warming Up to Scientific Notation

Big Numbers

An Exponential Portfolio

All About Alice Portfolio

Sending Alice to the Moon

Intent

Students search for "missing exponents" in the Alice setting, working particularly with base 10. The follow-up discussion leads to the development of the concept of a logarithm, to which students are introduced as a tool to solve for an unknown exponent. Students also identify some of the convenient logarithmic properties of working in base 10.

Mathematics

Drawing on their work in *Measuring Meals for Alice,* students begin this activity by solving a couple of exponential equations, in base 10 this time, which leads to the introduction of **logarithms.**

When confronted with equations of the form $10^x = 239{,}000$, students must find the power of 10 that results in the given value—in this case, 239,000. The answer is the base 10 logarithm of 239,000, which is written $\log_{10} 239{,}000$. And

$$\log_{10} 239{,}000 \approx 5.378 \text{ because } 10^{5.378} \approx 239{,}000$$

This activity includes an introduction to the terminology and notation of logarithms. It emphasizes the importance of base 10 logarithms and also brings out that numbers between 0 and 1 have negative logarithms.

Progression

Working individually, students are expected to solve these questions involving powers of 10 using guess-and-check on their calculators. The subsequent class discussion briefly resolves any questions and then moves into a mini-lecture introducing logarithms.

Approximate Time

20 minutes for activity (at home or in class)
40 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

To convey a sense of what the questions are asking, you might open the discussion by asking students between which two integers each answer lies. Save the answers to these questions for use in the discussion of logarithms that follows. In Question

1, Alice needs about 5.38 ounces of cake. In Question 2, she needs about 9.56 ounces of beverage.

Introducing Logarithms

Bring out that the questions in this activity and Questions 1 to 3 in *Measuring Meals for Alice* are quite similar. They all ask what power to raise a particular base to in order to get a certain result. Mention that this type of question gets asked a lot, in many different contexts, so people have devised special terminology and notation for it.

Explain that the solution to the equation $10^x = 239{,}000$, from Question 1, is represented by the expression

$$\log_{10} 239{,}000$$

Explain that this expression is read, "log, base 10, of 239,000" (or "log of 239,000 to base 10") and that *log* is an abbreviation for *logarithm.*

Ask, **What would this number mean to Alice?** Students should be able to articulate that it tells the amount of base 10 cake Alice needs to eat to multiply her height, which started as 1 mile, by 239,000.

Introduce a simpler example. **What does $\log_2 8$ mean to Alice? What does it mean as an exponential equation?** Students should realize that the expression represents how much base 2 cake Alice must eat to multiply her height by 8, as well as the solution to the exponential equation $2^x = 8$.

Using the Alice metaphor or the equation itself, students should figure that $\log_2 8 = 3$. Display this equation, and review how to read logarithm expressions.

Ask students to make up some logarithm expressions of their own and interpret them in terms of exponents. They should be able to articulate that an expression like $\log_a b$ represents "the power you raise *a* to in order to get *b*." That is, it represents the solution for *x* in the equation $a^x = b$.

What does the expression $\log_a b$ mean in Alice's situation? Students should recognize that *a* describes the type of cake (so we read the "*a* portion" of the expression as "base *a*") and *b* describes what is happening to Alice's height. The expression answers the question, "How much base *a* cake should Alice eat to multiply her height by *b?*"

Base 10 Logarithms

Tell students that because our number system is a base 10 system, logarithms for base 10 are widely used in mathematics and science, and most calculators have a key that gives base 10 logarithms.

Have students find some base 10 logarithms on their calculators, starting with verifying their answer to Question 1.

You might mention that logarithms to other whole-number bases are less commonly used and are in this unit only to provide more examples of logarithms. Point out that for whole-number bases other than 10, students cannot get the logarithm from a single key, so they should use guess-and-check, as they probably did for base 10 on the activity.

Estimating Base 10 Logarithms

To clarify the significance of using base 10 logarithms, you might ask, **If the base 10 logarithm of a number is between 4 and 5, how big is the number?** Students should conjecture that the number is between 10^4 and 10^5, or at least 10,000 and less than 100,000, which means it is a five-digit number. It may take several examples for them to make the connection between the size of the logarithm and the number of digits.

Then turn the process around and have students estimate the logarithm of a given whole number. **Between which two whole numbers is the logarithm of 273,189?** Students should realize that because this is a six-digit number, its logarithm must be between 5 and 6. As they develop this awareness, be sure they realize that this principle applies only to logarithms in base 10.

Work your way down to numbers between 1 and 10 as preparation for the discussion of logarithms for numbers between 0 and 1. Students should recognize that a number between 1 and 10 has a base 10 logarithm between 0 and 1.

Negative Logarithms

Build on the pattern students have just noted by asking about the base 10 logarithms for numbers between 0 and 1. **What's the approximate value of $\log_{10} 0.25$?** Emphasize that you only want to know between which two integers the answer lies.

Students may realize intuitively that the answer should be between −1 and 0. If not, you might return to the pattern established for large whole numbers and ask, **We know that a number between 10 and 100 has a logarithm between 1 and 2 and that a number between 1 and 10 has a logarithm between 0 and 1. What would you expect for the logarithm of a number between 0.1 and 1?** You should be able to establish that $\log_{10} 0.25$ is negative.

Ask students to find $\log_{10} 0.25$ more precisely on their calculators. If they do this using the base 10 logarithm key (and get approximately −0.6), ask, **How could you confirm this result using exponents?** The goal is to get them to recognize that the statement $\log_{10} 0.25 \approx -0.6$ is equivalent to the statement $10^{-0.6} \approx 0.25$.

The primary goal here is for students to discover that numbers between 0 and 1 have negative logarithms and that the smaller the number, the more negative the logarithm.

Logarithms of Negative Numbers?
Take this opportunity to bring out that only positive numbers have logarithms. You might ask about a specific example, such as, **What is $\log_{10}$ -2.7?** If students try to do this on a calculator, they should get an error message (unless the calculator is set to deal with complex numbers).

Why doesn't -2.7 have a base 10 logarithm? If necessary, ask students to write an appropriate exponential equation. They should recognize that they are trying to solve the equation $10^x = -2.7$ and that this equation has no solution, because exponential expressions with base 10 (or any other positive base) give only positive values.

Logarithms and the Graph of $y = 10^x$
You can clarify all of these ideas by referring to the graph of the equation $y = 10^x$. Bring out that y-values between 0 and 1 on this graph correspond to negative x-values and that there are no points on this graph with negative y-values. (The graph of the base 10 logarithm function is discussed in the activity *Taking Logs to the Axes.*)

The Alice Approach
Follow up the discussion of logarithms for numbers less than 1 by relating the issue of logarithms back to Alice's situation. If needed, review the general idea that asking for a logarithm is similar to asking how much cake Alice should eat.

Then take one of the examples the class has discussed, such as $\log_{10} 0.001$, and ask, **How much base 10 cake should Alice eat to multiply her height by 0.001?** Students may respond that eating cake will make her taller, not shorter. If so, use that response to review the idea that drinking the beverage is similar to eating "negative cake."

Based on this principle, students can find $\log_{10} 0.001$ by recognizing that Alice needs to drink 3 ounces of beverage to multiply her height by 0.001 and then interpreting this as the same as eating "−3 ounces" of cake. Thus it makes sense to define $\log_{10} 0.001$ as equal to −3.

Key Questions

What does $\log_2 8$ mean to Alice? What does it mean as an exponential equation?

What does the expression $\log_a b$ mean in Alice's situation?

If the base 10 logarithm of a number is between 4 and 5, how big is the number?

Between which two whole numbers is the logarithm of 273,189?

What's the approximate value of $\log_{10} 0.25$?

A number between 10 and 100 has a logarithm between 1 and 2, and a number between 1 and 10 has a logarithm between 0 and 1. What would you expect for the logarithm of a number between 0.1 and 1?

What is $\log_{10} -2.7$? Why doesn't −2.7 have a base 10 logarithm?

How much base 10 cake should Alice eat to multiply her height by 0.001?

Supplemental Activities

A Little Shakes a Lot (reinforcement) explores an interesting real-world use of logarithms: earthquakes.

Who's Buried in Grant's Tomb? (extension) offers another setting in which students can explore the relationship between exponents and logarithms.

Alice on a Log

Intent

Students explore logarithms in the context of Alice and her cake and beverage.

Mathematics

In the Alice metaphor, the equation $10^x = y$ is understood to mean, "If Alice eats x ounces of base 10 cake, her height will be multiplied by y." In this activity, the questions reinforce a similar interpretation of $\log_{10} y = x$: "x is the number of ounces of base 10 cake Alice needs to eat to multiply her height by y."

The focus of the activity is on the relationship between logarithmic and exponential equations. Students convert exponential situations into logarithmic equations and then estimate the solutions by guessing and testing with the associated exponential equations.

Progression

Students begin this activity in the classroom, with the opportunity to work with peers, and then complete it individually. They return to their groups to share ideas and ask questions before reviewing the concepts explored in the activity as a class.

Approximate Time

5 minutes for introduction
20 minutes for activity (at home or in class)
15 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Encourage students to read through the activity and then begin working in their groups, possibly jumping right to Question 2. After they have a start on the activity, they can complete it on their own.

Discussing and Debriefing the Activity

Give groups a brief time to come to a consensus about the answers before having students report on each question.

Ask presenters to include appropriate exponential equations in their presentations. For example, the class should note that Question 2c is equivalent to asking for the solution to the equation $10^x = 50$, and the presenter should give the expression $\log_{10} 50$ to represent the answer.

For Questions 2e and 2f, students might use either 10 or $\frac{1}{10}$ as the base. For instance, they might interpret Question 2f in either of two ways:

- As asking for the solution to the equation $10^{-x} = \frac{1}{4}$. Then $-x$ is $\log_{1/10}\left(\frac{1}{4}\right)$,

  and x is $-x = \log_{1/10}\left(\frac{1}{4}\right)$.

- As asking for the solution to the equation $\left(\frac{1}{10}\right)^x = \frac{1}{4}$. Then $x = \log_{1/10}\left(\frac{1}{4}\right)$.

You might try to elicit both approaches, both of which give approximately 0.60 for x.

Finally, tell students that they will sometimes encounter just "log" rather than "$\log_{10}$." When no base is written, the assumption is that it is base 10.

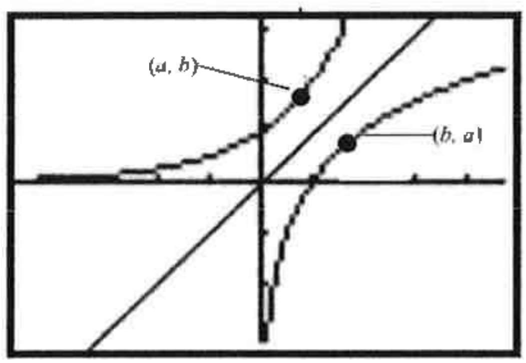
Taking Logs to the Axes

Intent

Students sketch graphs of logarithmic functions and compare them with each other and with graphs of exponential functions. The graphs of both types of functions are not carefully examined in this unit, but students will become aware of some general properties.

Mathematics

The logarithmic and exponential functions are inverses, so their graphs are reflections across the line $y = x$. That is, if the point (a, b) is on the graph of $y = 2^x$, then the point (b, a) is on the graph of $y = \log_2 x$.



In this activity, students examine the graphs of logarithmic functions and their connections to those of exponential functions. In particular, they investigate how the graph changes as the base changes and recognize the symmetrical relationship between the graph of a logarithm function and that of the corresponding exponential function.

Progression

This activity opens with students guessing what the graph of a logarithmic function might look like. They then work individually or in small groups to explore such graphs, comparing graphs of different bases as well as graphs of related exponential and logarithmic functions.

Approximate Time

20 minutes for activity
10 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

Doing the Activity

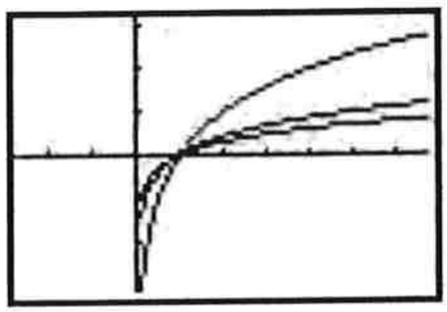
This activity requires little or no introduction.

If students work on the activity in groups, bring the class together for discussion when most groups have done at least some work on Question 4.

As groups finish Questions 1a, 1b, and 2, you can choose a group to present the graph for each part, including discussing the scales.

Discussing and Debriefing the Activity

Have students briefly present the graphs for Questions 1a, 1b, and 2, or jump right to Questions 3 and 4. Encourage a variety of comparisons. For instance, they might point out that as the base for a logarithm function increases, the portion of the graph for values of x greater than 1 gets "flatter."



If no one points out the relationship between the graphs of a logarithm function and the corresponding exponential function (they are reflections across the line $y = x$), try to elicit this idea by having students identify specific points on each. For example, if they used the point (8, 3) for the graph of $y = \log_2 x$, bring out that this point fits this equation because $2^3 = 8$, which means (3, 8) is on the graph of the function $y = 2^x$. Using examples like this, help students realize that if (a, b) is on the graph of $y = \log_2 x$, then (b, a) is on the graph of $y = 2^x$ and the two graphs are symmetrical with respect to the line $y = x$. (Refer to the graph in the "Mathematics" section above.)

Base 10 Alice

Intent

This activity offers a series of base 10 problems in the Alice context designed to introduce students to standard ways of recording numbers in scientific notation.

Mathematics

Scientific notation is a standard method for writing very large or very small numbers by expressing them as a number between 1 and 10 multiplied by an integer power of 10. For example, the mass of the earth is estimated to be about $5.97 \bullet 10^{24}$ kilograms (that huge number starts with 597 and ends with 22 zeros), and today's computer processor transistors are about $4.5 \bullet 10^{-10}$ meter to $6.5 \bullet 10^{-10}$ meter across (less than a one billionth of a meter). In each case, the number between 1 and 10 expresses the significant digits of the number (the number's *precision*), and the power of ten expresses its **order of magnitude.**

In the context of this unit, the number between 1 and 10 can be thought of as Alice's initial height. The order of magnitude can be thought of as the number of ounces of cake she eats.

Progression

Students work on the activity individually, followed by some time in groups to share ideas. The follow-up discussion introduces scientific notation. The next activity, *Warming Up to Scientific Notation,* provides additional examples for students to explore.

Approximate Time

20 minutes for activity (at home or in class)
30 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

Give groups a few minutes to discuss their results and then bring the class together for a discussion.

Let volunteers explain their answers to the three parts of Question 1. For part c, it would be good to get both "50 trillion" and "5 with 13 zeros after it" as ways of describing the answer.

Then ask, **How could you write your answers to Question 1 using exponents?** For example, students should be able to write the answer to part c as $5 \cdot 10^{13}$. Point out that this is much easier to write and to work with than the value 50,000,000,000,000.

Have one or two students share their answers to Questions 2 and 3, with the goal of getting statements involving powers of 10 and "counting zeros." For example, a student might answer Question 2 by saying that the height Alice grows to must be 5 times a particular power of 10 or must be 5 followed by a number of zeros.

To solidify this basic idea, ask students to express a variety of numbers using powers of 10. **How would you write 23,000 as a whole number times a power of 10?** Solicit more than one answer, such as $23 \cdot 10^3$ and $230 \cdot 10^2$.

Connect these expressions with the Alice situation by asking, **What does each of these expressions mean in terms of an initial height for Alice and eating a whole number of ounces of base 10 cake?** For example, $23 \cdot 10^3$ could represent a starting height of 23 feet and eating 3 ounces of cake.

Point out that the different ways to write a given number as "whole number times a power of 10" also represent different starting heights and amounts of cake. This is also a good time to bring out that certain combinations involve initial heights that are not whole numbers. For example, help students to realize that 23,000 could also represent a starting height of 2.3 feet and eating 4 ounces of cake.

Scientific Notation

Use the discussion of this activity to introduce the idea of **scientific notation.** The introduction described here is based on the assumption that students have little or no previous exposure to the concept. Because you may have students who do know something about scientific notation, begin by asking if anyone is familiar with the term and have volunteers describe what they know. Build on that foundation to elicit any key concepts not introduced by the volunteers.

Tell students that it is useful in mathematics and science to have a standard way to write numbers using powers of 10. That standard way is like having Alice start at a height of at least 1 foot and less than 10 feet.

Ask, **How would you write 162,000 in this standard form?** They should be able to come up with the answer $1.62 \cdot 10^5$. Tell them that this is called scientific notation and that, for convenience, we will informally refer to the notation 162,000 as "ordinary" notation.

Similarly, the two components of a number written in scientific notation, such as 1.62 and 10^5 in the example, might be informally referred to as the "number part" and the "power-of-ten part."

Order of Magnitude

Tell students that in science and other disciplines, we often want a very rough idea of how big something is. The exponent in scientific notation helps with this approximation.

Two numbers that have the same exponent when written in scientific notation or that differ by a factor of less than 10 are said to have the same **order of magnitude.** For example, the population of California (about 38 million, as of 2008) has the same order of magnitude as the population of New York (about 19 million), because both numbers would be written in scientific notation as $a \bullet 10^7$, where a is a number between 1 and 10.

On the other hand, the population of New York has a different order of magnitude than the population of either the whole United States (about 300 million, or $3.0 \bullet 10^8$) or Nevada (about 2.4 million, or $2.4 \bullet 10^6$).

Questions 4 and 5

Return to Questions 4 and 5, and ensure that the numeric answers are clearly understood. For example, students should be able to explain why, if Alice starts out 5 feet tall and drinks 4 ounces of base 10 beverage, she ends up 0.0005 foot tall.

Ask students to articulate the mathematics involved in this rewriting. Focus their attention on the fact that moving the decimal point one place within a number changes the place value for each digit by a factor of 10 or $\frac{1}{10}$, depending on the direction of the move.

Writing Decimals Less than 1 Using Powers of 10

Ask students, **How would you write the number 0.0005 using a power of 10?** The context of Questions 4 and 5 may elicit the answer $5 \bullet \left(\frac{1}{10}\right)^4$. If so, acknowledge that this is equal to 0.0005, but state that you want students to use 10 instead of $\frac{1}{10}$ as the base. This may require some reminders about negative exponents and the idea that beverage is like "negative cake." Students should be able to express the result of starting at 5 feet and drinking 4 ounces of beverage with the expression $5 \bullet 10^{-4}$.

Go through a series of beverage problems, such as asking how Alice might end up 0.0035 foot tall. Students should realize that she could have started at 35 feet and drunk 4 ounces of beverage, or started at 3.5 feet and drunk 3 ounces of beverage, and so on.

Ask students how they think a height of 0.0035 foot should be written using scientific notation. They should recognize that $3.5 \bullet 10^{-3}$ is the standard form.

Key Questions

How could you write your answers to Question 1 using exponents?

How would you write 23,000 as a whole number times a power of 10?

What does each of these expressions mean in terms of an initial height for Alice and eating a whole number of ounces of base 10 cake?

How would you write 162,000 in this standard form?

How would you write the number 0.0005 using a power of 10?

Warming Up to Scientific Notation

Intent

Students practice writing and interpreting numbers in scientific notation and identify general principles for doing arithmetic with numbers in scientific notation.

Mathematics

Scientific notation is a way to write numbers—especially very large or very small numbers—as a number between 1 and 10 multiplied by an integer power of 10. When computing with numbers in this form, we take advantage of the rules for computation with exponents that students have been deriving in this unit. For example,

$$(3 \bullet 10^4) \bullet (2 \bullet 10^7) = 3 \bullet 2 \bullet 10^4 \bullet 10^7 = 6 \bullet 10^{4+7} = 6 \bullet 10^{11}$$

$$(9 \bullet 10^3) \div (3 \bullet 10^{-4}) = 9 \div 3 \bullet 10^3 \div 10^{-4} = 3 \bullet 10^{3-(-4)} = 3 \bullet 10^7$$

In this activity, students will derive these procedures.

Progression

Students work on their own to practice using scientific notation and then work individually or in small groups to devise methods for computing with numbers in scientific notation. The activity concludes with a whole-class discussion focusing on general principles about multiplying and dividing scientific numbers.

Approximate Time

10 minutes for introduction
20 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Begin by having students do Questions 1 and 2 on their own. In Questions 3 and 4, they will begin to derive some general principles for working with scientific notation. Read the instructions to Question 3 aloud, and emphasize that students should do these problems without a calculator. Have students begin with part a, encouraging them to share their methods, and continue as time allows.

Ask for volunteers to share their techniques for simplifying the problem in part a. Students can complete the remainder of the activity on their own.

Discussing and Debriefing the Activity

Have students go over Questions 1 to 3 in their groups.

Ask for volunteers to share ideas on Question 4. Encourage them to provide examples and to talk about what they did to notice the patterns they are reporting. Students are likely to arrive at ideas like these.

- To multiply numbers in scientific notation, multiply the "number parts" and add the exponents of the "power-of-ten parts."
- To divide numbers in scientific notation, divide the "number parts" and subtract the exponents of the "power-of-ten parts."

Students may not realize that after applying these principles, they sometimes need to make an adjustment to standardize their answer. For example, following the first rule for Question 3b gives $35 \bullet 10^3$, but the correct scientific notation is $3.5 \bullet 10^4$. If necessary, ask, **What do you need to do to adjust your answer to put it in scientific notation?**

For Question 5, ask whether anyone has questions about how to work with scientific notation on the calculator. If there are questions, you might pair students up to work through the difficulties.

Key Question

What do you need to do to adjust your answer to put it in scientific notation?

Big Numbers

Intent

This final mathematical activity of the unit offers further practice with techniques for multiplying and dividing numbers in scientific notation and places scientific notation in several real-world contexts.

Mathematics

This activity draws together work with exponents and laws derived for exponents as students solve a set of practical problems. The mathematical focus is on estimation and **order of magnitude.**

Progression

After creating techniques for multiplying and dividing scientific numbers in *Warming Up to Scientific Notation,* students work in small groups to apply these techniques to some challenging problems. It is crucial not that they solve all the problems, but that they gain experience in working with and recording solutions to problems involving very big and very small numbers.

Approximate Time

40 minutes

Classroom Organization

Small groups, followed by whole-class discussion

Materials

To enhance the activity, you may want to obtain the video *Powers of 10* and related videos, which dramatically emphasize the magnitude of change related to powers of 10.

Doing the Activity

Have students begin working in their groups. If they are frustrated, you may want to do Question 1 as a class. If necessary, review the facts that a mile is 5280 feet (Question 3), a kilogram is 1000 grams (Question 5), and a meter is 1000 millimeters (Question 8).

Discussing and Debriefing the Activity

You might ask members of different groups to present each answer. Students may do these problems in a variety of ways, and it is worthwhile to elicit other approaches after the presentations, at least for a few of the problems.

Students might answer Question 1 by writing 30 as $3 \cdot 10^1$ and expressing the answer as the quotient $(3 \cdot 10^1) \div (5 \cdot 10^{-7})$. This gives $0.6 \cdot 10^8$, which can be rewritten in scientific notation as $6 \cdot 10^7$.

Alternatively, they might reason that because the computer does a computation in $5 \bullet 10^{-7}$ second, it can do 10^7 computations in 5 seconds. Because 30 seconds is 6 times as long as 5 seconds, the computer can do $6 \bullet 10^7$ computations in 30 seconds.

For Question 2, you might have students write an expression that represents the answer without actually doing any arithmetic. For example, you could express the number of seconds per year as

$$60 \bullet 60 \bullet 24 \bullet 365$$

and the number of gallons per year as

$$(60 \bullet 60 \bullet 24 \bullet 365) \div 76{,}000$$

You can then have students consider ways to estimate this product, such as rewriting it as approximately

$$(6 \bullet 10^1) \bullet (6 \bullet 10^1) \bullet (2 \bullet 10^1) \bullet (4 \bullet 10^2) \div (8 \bullet 10^4)$$

You might note that as 24 has been rounded down and 365 has been rounded up, there is some balancing out.

Students might then multiply $6 \bullet 6$ to get 36, approximate $36 \bullet 2$ as 70 and $70 \bullet 4$ as 300, and then approximate $300 \div 8$ as 40. The powers of 10 combine to give 10^1, for a final estimate of $40 \bullet 10^1$, which is 400. (A more exact answer is about 415 gallons. If it's a leap year, the answer is just over 416 gallons.)

Here are approximate answers to the remaining questions.

- Question 3: about $2.53 \bullet 10^8$ years (or about 253 million years)
- Question 4: about $29,500
- Question 5: about $5 \bullet 10^{25}$ atoms
- Question 6: about $3.3 \bullet 10^5$ or 330,000 earths
- Question 7: about $3.7 \bullet 10^{17}$ inches
- Question 8: about $4.7 \bullet 10^{15}$ grains of sand (or about 4.7 quadrillion grains of sand)

Supplemental Activity

Very Big and Very Small (extension) asks students to identify and investigate more situations involving very big and very small numbers in contexts they find intriguing.

An Exponential Portfolio

Intent

Students begin compiling their portfolios by recording and justifying all the laws about exponents derived during the unit.

Mathematics

In this summative activity, students should recall the following laws about exponents and justify why each is true.

$$A^x \cdot A^y = A^{x+y} \text{ for any rational numbers } x \text{ and } y$$

$$\left(A^x\right)^y = \left(A^y\right)^x = A^{xy} \text{ for any rational numbers } x \text{ and } y$$

$$A^0 = 1 \text{ if } A \neq 0$$

$$A^{-x} = \frac{1}{A^x} \text{ if } A \neq 0$$

$$b^{x/y} = \sqrt[y]{b^x} = \left(\sqrt[y]{b}\right)^x \text{ if } b > 0,\ x \text{ and } y \text{ are integers, and } y \neq 0$$

Progression

In one of the first steps toward creating their unit portfolios, students work on their own to review their notes in order to summarize what they have learned about one of the major mathematical topics of the unit: laws about exponents. Afterward, they can compare their lists in groups or with the whole class.

Approximate Time

30 minutes for activity (at home or in class)
10 minutes for discussion

Classroom Organization

Individuals, followed by small-group or whole-class discussion

Doing the Activity

This activity requires no introduction.

Discussing and Debriefing the Activity

Have students share their work in their groups or as a whole class. If done as a class, let volunteers offer general laws about exponents. As each law is suggested, have the rest of the class decide whether the statement is true. Solicit as many different explanations as possible.

All About Alice Portfolio

Intent

Students compile their unit portfolios and write their cover letters.

Mathematics

This activity helps students to review the main mathematical ideas of the unit: extending the operation of exponentiation, laws of exponents, graphing, logarithms, and scientific notation. In particular, students select work that demonstrates understanding of the operation of exponentiation, laws of exponents, and graphing.

Progression

Students begin by reviewing their work and notes for the unit in class and then complete the activity by writing a cover letter summarizing the mathematics of the unit and their personal growth during the unit and Year 2 of IMP.

Approximate Time

20 minutes for introduction
25 minutes for activity (at home)

Classroom Organization

Individuals

Doing the Activity

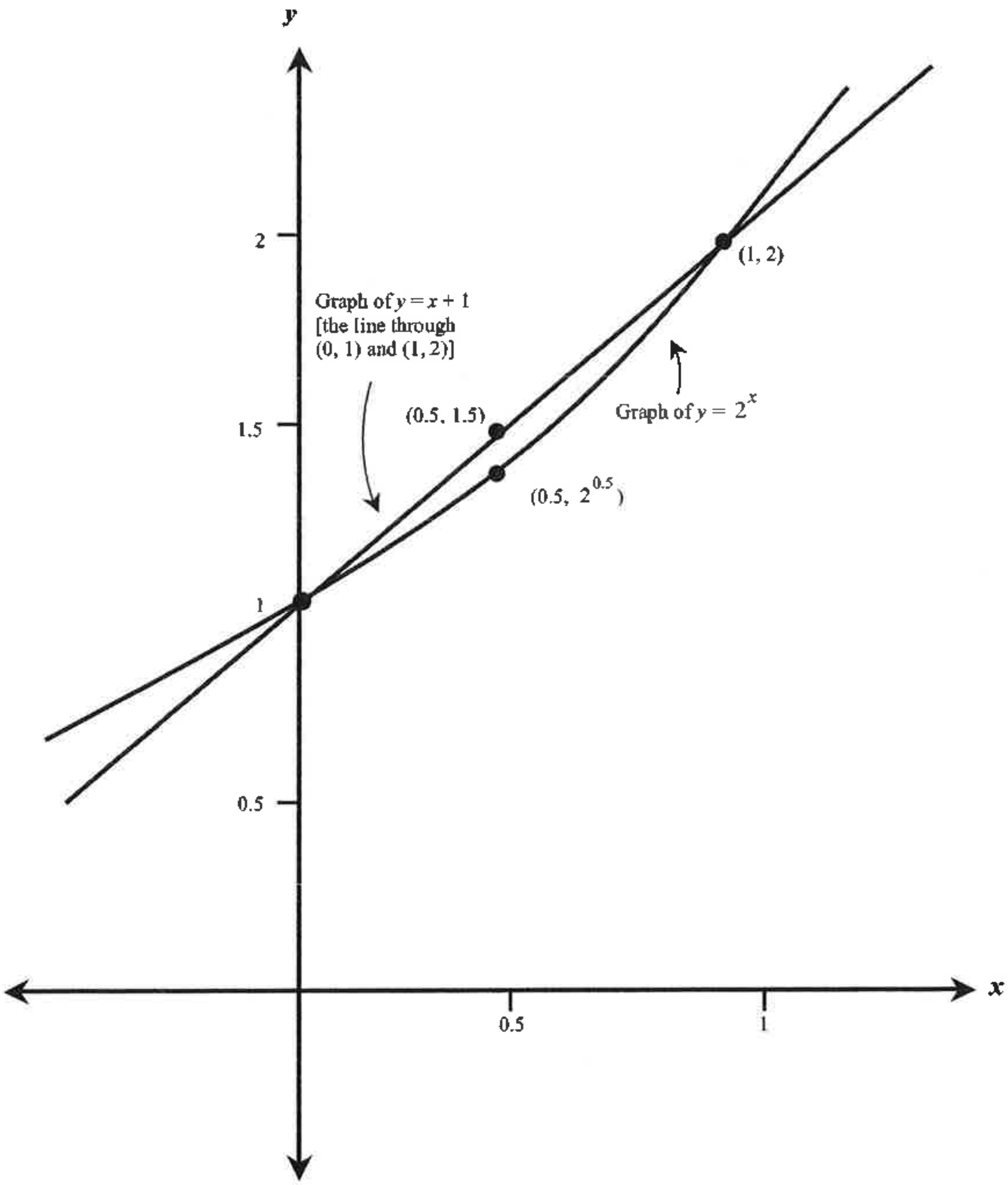
Have students read the instructions in the student book carefully. Review your expectations for their portfolios.

Discussing and Debriefing the Activity

You may want to have students share their portfolios in their groups, comparing what they wrote about in their cover letters and the activities they selected.

Blackline Master

It's in the Graph



In-Class Assessment for *All About Alice*

Seven equations are given here. Some of the equations are true and some are false. Do parts a and b for each equation.

a. State whether the equation is true or false.

b. Explain your answer.

- If you think the equation is true, explain why. If possible, state and explain a general principle that the equation illustrates.
- If you think the equation is false, change the right side of the equation to make it true. Then explain why the new equation is true.

1. $10^5 \bullet 10^{12} = 10^{17}$
2. $\frac{3^8}{3^2} = 3^4$
3. $\sqrt{10^{16}} = 10^4$
4. $\left(5^2\right)^3 = 5^6$
5. $\frac{1.4 \bullet 10^8}{0.7 \bullet 10^4} = 2 \bullet 10^4$
6. $5^3 + 5^2 = 5^5$
7. $\log_2 8 = 3$

Take-Home Assessment for *All About Alice*

Part I: Graph It

1. Sketch the graph of the equation $y = 1.5^x$ from $x = -3$ to $x = 3$. Label at least five points with their coordinates.
2. Explain and show how to use your graph to estimate these values.
   a. $1.5^{-0.5}$
   b. $\log_{1.5} 2$

Part II: Far, Far Away

Give your answers to these questions in scientific notation, and explain clearly how you got them.

3. Light travels very fast, at approximately $1.86 \bullet 10^5$ miles per second. A *light-year* is the distance light travels in a year.
   a. About how many miles are there in a light-year?
   b. The star Betelgeuse is about $2.5 \bullet 10^{15}$ miles from the earth. About how many light-years is this distance?
4. An *astronomical unit* is the distance from the earth to the sun, which is approximately $9.3 \bullet 10^7$ miles.
   a. About how many astronomical units from the earth is Betelgeuse?
   b. Which is bigger: an astronomical unit or a light-year? About how many of one equals the other?

Part III: All Roads Lead to Understanding

The next two equations show how certain expressions with exponents are defined. For each equation, explain in as many different ways as you can why the expressions are defined the way they are. Give at least two explanations for each definition.

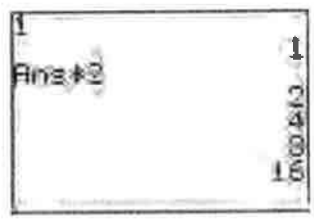
5. $4^0 = 1$
6. $5^{-3} = \dfrac{1}{125}$

All About Alice Guide for the TI-83/84 Family of Calculators

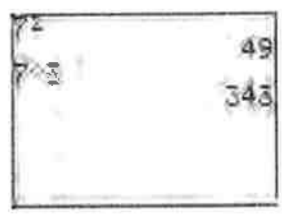
This guide gives suggestions for selected activities of the Year 2 unit *All About Alice.* The notes that you download contain specific calculator instructions that you might copy for your students. NOTE: If your students have the TI-Nspire handheld, they can attach the TI-84 Plus Keypad (from Texas Instruments) and use the calculator notes for the TI-83/84.

The primary calculator topics in *All About Alice* are exponents, logarithms, and scientific notation. As students begin to explore exponential functions, both the calculator's graphing feature and its ability to repeatedly execute the previous command will be very useful in illustrating the concepts involved.

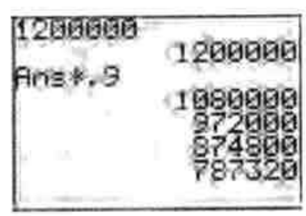
Alice in Wonderland: The calculator's ability to recall and use a previous entry can provide another way to illustrate what happens as Alice eats several ounces of cake. Begin by entering a value for her initial height (1 meter is convenient) into the calculator and pressing ENTER. Now press × 2 ENTER to see the results of eating the first piece of cake. After the first ounce, every time you press ENTER, the calculator will display Alice's height after eating another ounce.



Be sure students learn how to use exponents both on the graphing calculator and on their personal calculators. On the graphing calculator, 7^2 can be entered simply as 7 x^2, and 7^3 is entered as 7 ^ 3, whereas on many personal calculators, 7^3 is entered as 7 y^x 3.



A Wonderland Lost: The calculator can help students determine the rule for Question 3. After you have reached the point in the discussion at which students recognize that multiplying the area by 0.9 is the same as subtracting 10 percent, you might use the next sequence of questions while illustrating on the overhead graphing calculator. Enter the initial area of 1,200,000 and press ENTER. Press × . 9 ENTER and ask what this represents. Press ENTER several times, asking what the result represents after each time. Finally, ask what was done to 1,200,000 to get the last number that is displayed and how this could be written in a shorter form.



Here Goes Nothing: When discussing Question 4, let students enter **2^0** into their calculators to see that it really does give a result of 1. Some students will have done this right away, and you may need to remind them to explain why their answer makes sense in terms of the conclusions from the first three questions.

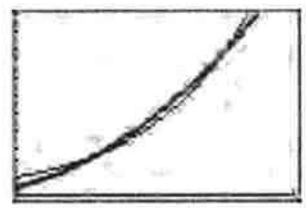
A New Kind of Cake: Question 5 is bound to stimulate some discussion of how the two functions $y = 2^x$ and $y = x^2$ compare for small values of x, because the curves cross each other a couple of times. Students will probably not have sketched their graphs in enough detail to see this clearly, and it is tricky to find a viewing window on the calculator that shows the two curves distinctly. These window settings work fairly well:

Xmin=1

Xmax=5

Ymin=0

Ymax=20



A heavier line style can be chosen for one of the functions, making it easier to differentiate between them. After this part of the curves has been discussed, change **Xmax** to **7** and **Ymax** to **100** to give a clear picture of how rapidly the two functions will diverge.

Piece After Piece: The effect of eating the cake in two stages can be illustrated quite clearly on the calculator during the discussion of this activity. Use a procedure similar to that described previously for *Alice in Wonderland,* but pause after 3 ounces and again after 5 more ounces. It will be very clear that the result would be the same without the pauses.

In Search of the Law: The discussion in the subsection "What about 0^0?" in the Teacher's Guide for this activity represents another opportunity to reinforce the vocabulary terms *range* and *domain.* Have students try 0^0 on the graphing calculator, and the calculator will display **ERR:DOMAIN.**

Rallods in Rednow Land: Students will probably encounter scientific notation on their calculators when working on this activity. Scientific notation will be covered later in this unit. At this time, students simply need to know that a display of, suppose, **1.844674407E19** means that the decimal belongs 19 places to the right of where it now is.

Keeping a running total of the number of coins on the board so far presents an interesting problem in Question 3. Writing down and reentering intermediate answers into the calculator is cumbersome and prone to error. Some of your students may enjoy the challenge of writing a program to answer Question 3 or to find the number of squares necessary to yield any given sum. The calculator entries listed here may also be used to do the calculation:

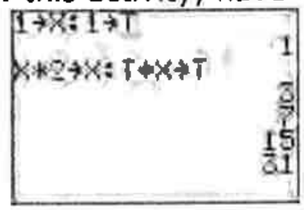
1→x:1→T [ENTER] displays the total for the first square;

X*2→X:T+X→T [ENTER] displays the total for the first two squares.

(X represents the number of coins on a square; T represents the total number of coins so far.)

Each time ENTER is pressed, the coins on one more square will be added. Simply count the number of times you press ENTER (including the first two times) until a sum greater than one billion is displayed.

Having Your Cake and Drinking Too: After the discussion of this activity, have students verify that the calculator gives the same result for 2^{-3} as it does for $\frac{1}{2^3}$. Remind them that when entering negative exponents, they must use the negative key, not the subtraction key.



It's in the Graph: After students have decided what the graph of $y = 2^x$ should look like when it is extended for negative values of x, have them confirm their findings with a graph on the calculator.

There are two ways to set the calculator up to trace only to integer values of x. One way is with 8:**ZInteger** from the **Zoom** menu. This is used exactly like **Zoom In**, with the cursor and ENTER key used to define the center of the new screen. Another way is entering **1→ΔX** on the home screen and using an integer for **Xmin**. Press VARS, select the **Window** menu, and then choose **ΔX**. Have students trace to an x-value of zero and to negative values of x.

Ask students to return to the home screen by pressing 2ND [QUIT] and then to enter **0^–2**. They will get an error statement. Have them explain why they cannot use a negative power of zero.

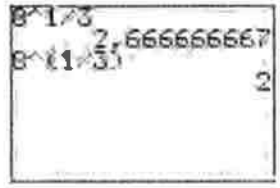
Then ask what the graph of $y = 0^x$ will look like. Have students graph it on the calculator. They will seem to get a blank graph screen. The actual graph is covered by the axis. Turn the axes off by pressing 2ND [FORMAT], highlight **AxesOff**, and pressing ENTER.

Press TRACE to return to the graph, which will appear as a horizontal ray along the positive x-axis. Tracing along the graph yields y-values of zero except where x is zero or negative. For $x \le 0$, the calculator does not display a value for y, because it is undefined.

A Half Ounce of Cake: If students try to solve this activity by entering $2^{(1/2)}$ into the calculator, ask them to show another way of finding that answer. Explain that the object of this activity is not really to find the number by which Alice's height is multiplied, but to find it in a way that will lead to understanding what that fractional exponent really means.

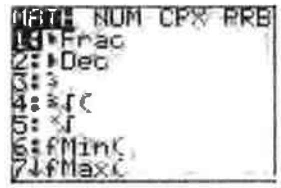
Stranger Pieces of Cake: After the discussion of roots and fractional exponents, students need to become familiar with how to handle these on both the graphing calculator and their own calculators. Suggest that they use an expression whose value they know, such as $8^{1/3}$, to explore how each calculator works.

When entering a fractional exponent on the graphing calculator, students will have to be careful to use the parentheses. Otherwise, they will get the result of $(8^1)/3$ instead of $8^{(1/3)}$.



Many calculators have a $\sqrt[x]{y}$ function, often as a secondary operation of a y^x key. Students will need to experiment to determine in which order they need to enter the values for x and y on those calculators.

The graphing calculator has root functions under the **MATH** menu accessed by pressing [MATH]—there is a cube-root function and a $\sqrt[x]{\ }$ function.



Have students compare the results of each of the following methods of finding the cube root of 8.

1. Enter **8^(1/3)**.
2. Press [MATH], select **4:** $\sqrt[3]{\ }$ from the menu, and press [8] [ENTER].
3. Enter [3], press [MATH], select **5:** $\sqrt[x]{\ }$ from the menu, and press [8] [ENTER].

Measuring Meals for Alice: A nice alternative to guess-and-check on Question 1 is to graph the function $y = 2^x$ and trace to find the value of x that yields $y = 10$.

Alice on a Log: It will again be necessary for students to experiment with their own calculators to see how the logarithm function works and in which order the keys must be pressed. Have them use a logarithm for experimentation that they can easily interpret, such as $\log_{10} 1000$, which equals 3.

The graphing calculator has a [LOG] key with a secondary operation of 10^x. Enter the keystrokes in the same order in which you would write them. To find $\log_{10}1000$, press [LOG] [1] [0] [0] [0] [ENTER].

Inform students that when they write "log" without specifying a base, it is assumed to be base 10. The [LOG] key on the calculator will only find base 10 logarithms, so the base is never entered into the calculator. Base 10 logarithms and natural logarithms (which will be introduced in Year 3) are the only logarithms supported by the graphing calculator.

Taking Logs to the Axes: Question 2 asks students to use the graphing calculator to draw the graph of $y = \log_{10} x$. Choosing the viewing window appropriately will take some thought. Ask students to think carefully about what happens to y as x increases. Which will be larger, x or y? By how much? They will find that the range they select for x will have to be much greater than that for y in order to get a meaningful graph. Ask them to try to find a viewing window that shows large values of x as well as what happens when x is between 0 and 1 .

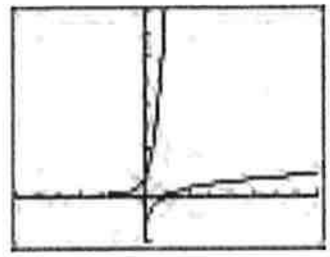
Question 4 of is difficult to illustrate on the graphing calculator—being limited to base 10 causes the graphs to be so steep that the curves appear to merge with the axes with many window settings. These settings below yield a window that allows a reasonable illustration for the functions $y = \log x$ and $y = 10^x$:

Xmin=−6

Xmax=8

Ymin=−2

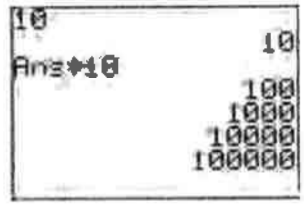
Ymax=8



Warming Up to Scientific Notation: The Calculator Note "Scientific Notation on the Calculator" will introduce your students to how the graphing calculator handles scientific notation. They will still need to explore how scientific notation is handled on their personal calculators, but most are very similar.

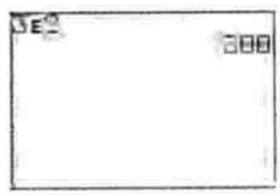
Scientific Notation on the Calculator

You have probably already encountered scientific notation displayed on your calculator, but you may not have realized what it was. When a number is very large (or very small), the calculator automatically shifts to scientific notation. An easy way to note how it works is to enter the sequence of operations listed here: Enter the number 10 and press ENTER. Now multiply the result by 10 by pressing × 1 0 ENTER. Now continue to multiply the result by 10 by simply pressing the ENTER key. Do this until the calculator shifts into scientific notation.



Notice that a display of **1E10** does not mean 1^{10} (which would only be 1), but $1 \bullet 10^{10}$ (which is 10 billion). (Similarly, **1E12** means $1 \bullet 10^{12}$. The base, 10, is fixed.) This notation with an **E** in the display is commonly used in calculators, but it is not conventionally used for results written on paper. When recording your answer from the calculator on paper, use the full scientific notation, which includes writing the base of 10 and its exponent.

You can enter numbers into the calculator in scientific notation using **EE** (enter exponent). It is the secondary operation of the comma key, which is located in the row above the 7 key. For example, enter $3.5 \bullet 10^{12}$ by pressing 3 . 5 2ND [EE] 1 2 ENTER.



If you enter a smaller number in scientific notation, like $3 \bullet 10^2$, and then press ENTER, the calculator will switch back to standard notation to display the answer. You can force the calculator to display all answers in scientific notation by going to the MODE screen, highlighting **Sci,** and pressing ENTER. Try entering $3 \bullet 10^2$ again and note what happens.

If you have a personal calculator that is not a graphing calculator, experiment with scientific notation on that calculator as well. Pay special attention to the order of keystrokes necessary to enter a number in scientific notation with a negative exponent. Some calculators require that the exponent be entered first and then be made negative, while others require pressing the negative key before entering the number in the exponent. Some display the power of 10 without the **E**.

In-Class Assessment for *All About Alice*

Seven equations are given here. Some of the equations are true and some are false. Do parts a and b for each equation.

a. State whether the equation is true or false.

b. Explain your answer.

- If you think the equation is true, explain why. If possible, state and explain a general principle that the equation illustrates.
- If you think the equation is false, change the right side of the equation to make it true. Then explain why the new equation is true.

1. $10^5 \bullet 10^{12} = 10^{17}$
2. $\frac{3^8}{3^2} = 3^4$
3. $\sqrt{10^{16}} = 10^4$
4. $(5^2)^3 = 5^6$
5. $\frac{1.4 \bullet 10^8}{0.7 \bullet 10^4} = 2 \bullet 10^4$
6. $5^3 + 5^2 = 5^5$
7. $\log_2 8 = 3$

Take-Home Assessment for *All About Alice*

Part I: Graph It

1. Sketch the graph of the equation $y = 1.5^x$ from $x = -3$ to $x = 3$. Label at least five points with their coordinates.
2. Explain and show how to use your graph to estimate these values.
   a. $1.5^{-0.5}$
   b. $\log_{1.5} 2$

Part II: Far, Far Away

Give your answers to these questions in scientific notation, and explain clearly how you got them.

3. Light travels very fast, at approximately $1.86 \bullet 10^5$ miles per second. A *light-year* is the distance light travels in a year.
   a. About how many miles are there in a light-year?
   b. The star Betelgeuse is about $2.5 \bullet 10^{15}$ miles from the earth. About how many light-years is this distance?
4. An *astronomical unit* is the distance from the earth to the sun, which is approximately $9.3 \bullet 10^7$ miles.
   a. About how many astronomical units from the earth is Betelgeuse?
   b. Which is bigger: an astronomical unit or a light-year? About how many of one equals the other?

Part III: All Roads Lead to Understanding

The next two equations show how certain expressions with exponents are defined. For each equation, explain in as many different ways as you can why the expressions are defined the way they are. Give at least two explanations for each definition.

5. $4^0 = 1$
6. $5^{-3} = \frac{1}{125}$

IMP Year 2
First Semester Assessment

I. *Cheerful Cows*

The Cheerful Cow Dairy Company has designed a new milk container. It is a right prism with a base that is an equilateral triangle. The sides of the base are 5 inches in length and the container is 10 inches tall.

1. How much cardboard is needed to build each container? Assume there is no wasted cardboard and no overlap.
2. What is the volume of each container?
3. Assume a cow can be milked at a rate of 25 squirts per minute and each squirt contains 0.5 cubic inch of milk. How long will it take to obtain enough milk to fill one of the new containers?

II. *Equalities and Inequalities*

1. Ling is in charge of buying sodas for the choir members this week. Each of the 34 members gets one soda. However, people often complain that there are too many or too few diet sodas.

   When Denzel was in charge last week, there were no complaints. Ling asks Denzel how many sodas he bought of each type. Denzel doesn't remember, so Ling looks in the choir account book and finds that Denzel spent $24.51. Diet sodas cost 69¢ each and regular sodas are 74¢ each.

   Set up and solve a system of equations to figure out how many of each kind Denzel bought.

2. Consider the inequality $2x + 3y \leq 10$.

   a. Graph this inequality. Explain in detail the method and reasoning you used to construct your graph. Be sure to explain the relationship between the graph and the inequality.

   b. Make up a real-life problem the inequality could be used to represent.

III. Explaining Area

You have seen that the area of any triangle can be found as half the product of its base and its altitude. Explain why this is true.

IV. Mini-POW

Solve this problem. Then write a thorough and clear explanation of the process you used to solve it. Your explanation should convince a reader that your answer is correct and is the only answer possible.

Vanessa invited 17 guests to her party. She assigned each person a number from 2 to 18, keeping the number 1 for herself. At one point, all the people were dancing in pairs. Vanessa noticed that the sum of the numbers for each pair was the square of a whole number.

What number did Vanessa's dancing partner have?

IMP Year 2
Second Semester Assessment

I. Some Algebra Basics

1. Write an expression without parentheses that is equivalent to 6(x + 5). Then explain why the expressions are equivalent using each of these methods.
   - An area model
   - Repeated addition
   - A numeric example

2. Solve each equation for the given variable.

   a. $3x + 28 = 43$ b. $15 - (3d + 1) = 8$

   c. $9 + 2(y - 3) = 19$ d. $3m + 8 = 5m + 10$

3. Semi Fast is training for the relay team. She wants to analyze her latest 400-meter race to find places where she can improve. She would like to be as fast as Speedy.
   After studying the video of her race, she comes up with the function
   $$m(t) + 0.097t^2 + 3t$$
   to describe the distance she ran (in meters), $m(t)$, as a function of t, the time elapsed (in seconds).

   a. Make and In-Out table showing the distance Semi had run after 5, 20, 25, 20, 25, and 30 seconds.
   b. Make a graph (on paper) that represents this situation.
   c. Explain how to use your graph to find how far Semi had run after 23 seconds. State how far she had run.
   d. How long did it take Semi to finish the 400-meter race? Explain how you arrived at your answer.
   e. What is the value of $m(14)$?
   f. Solve the equation $m(\text{t}) + 250$ for t.

4. Answer these questions about the graph of the quadratic equation $y = x^2 + 8x - 10$.

   a. Find the vertex of the graph. Explain your reasoning.
   b. Find the x- and y-intercepts of the graph to the nearest tenth.
   c. Sketch the graph, labeling the vertex and the intercepts.
   d. Give the x-intercepts of the graph exactly, using square roots.

II. Let's Talk About It

The monkeys at a zoo sometimes throw banana peels at the people watching them. The staff is concerned about the monkeys and the zoo patrons. One of the zookeepers feels that if he spends time talking to the monkeys, he might be able to decrease the amount of this behavior.

To test his theory, the zookeeper talks to 26 of the 63 monkeys but does not talk to the other 37. After a few weeks, he has people watch the two groups of monkeys on a busy Sunday. Of the 26 monkeys he has been talking to, 5 were observed throwing banana peels at the visitors. Of the 37 monkeys he has not talked to, 16 were observed throwing banana peels at the visitors.

1. State a null hypothesis for this situation. Define the appropriate populations clearly.
2. State the zookeeper's hypothesis precisely, in terms of the populations described in Question 1.
3. Make a table that will help you calculate the χ^2 statistic for this survey.
4. Compute the χ^2 statistic.
5. Do you think the results of this survey support the zookeeper's hypothesis? Explain your answer carefully, using the χ^2 probability table on the next page.
6. What flaws might there be in this study? How might they be corrected?

Value of the χ^2 statistic	Probability of getting a χ^2 statistic this large or larger when the null hypothesis is true
0.0	1.0000
0.2	.6547
0.4	.5271
0.6	.4386
0.8	.3711
1.0	.3173
1.2	.2733
1.4	.2367
1.6	.2059
1.8	.1797
2.0	.1573
2.2	.1380
2.4	.1213
2.6	.1069
2.8	.0943
3.0	.0832
3.2	.0736
3.4	.0652
3.6	.0578
3.8	.0513
4.0	.0455
4.2	.0404
4.4	.0359
4.6	.0320
4.8	.0285
5.0	.0254
5.2	.0226
5.4	.0201
5.6	.0180
5.8	.0160
6.0	.0143
6.2	.0128
6.4	.0114
6.6	.0102
6.8	.0091
7.0	.0082
7.2	.0073
7.4	.0065
7.6	.0058
7.8	.0052
8.0	less than .005

III. Power Power

1. Examine the In-Out table.
   a. Write a rule for the table.
   b. Make a graph based on the table.
   c. Use your graph to find the approximate value for the *Out* when the *In* is 2.5. Explain your reasoning.

2. Find the value of each expression and justify your answers. That is, explain why each expression has the value it does.
   a. $\log_3 9$
   b. 5^0

3. Write each number in standard scientific notation.
   a. 38,000
   b. 0.00506
   c. $0.16 \bullet 10^4$

IV. Mini-POW

Solve this problem and write up your results in POW style. Describe your process, solution, and any extensions or generalizations.

You are going to the baseball game with your sister. When you get there, you find an empty row with 30 seats. To save room for friends, you decide to leave at least three empty seats between the two of you.

Given this requirement, how many different choices do you have for where you and your sister can sit in the row?

Mathematics Unit Plan

Unit Title: Patterns

Designed by: D. Fendel, D. Resek, L. Alper, S. Fraser, 2011, *Interactive Mathematics Year 1, Second Edition* (Emeryville, CA: Key Curriculum Press).

Grade: 8/9

Time Frame (Number of Lessons): 13 days

Summary of Unit

This unit is an essential introduction for students to the variety of ways for working on and thinking about mathematical problems. Students are introduced to general learning skills and methods that are developed and used throughout the four-year IMP curriculum and that form the foundation of the learning process through which students will build mathematical ideas. Here is a summary of these learning skills and methods.

- Working in groups to analyze problems
- Learning about group cooperation and roles in group learning
- Expressing mathematical ideas orally and in writing
- Making presentations in small groups and to the class
- Developing strategies for solving problems
- Using concrete mathematical models
- Doing investigations in which the task is not clearly defined
- Becoming familiar with alternative forms of assessment, such as self-assessment and portfolios
- Learning about the use of graphing calculators.

Patterns emphasizes extended, open-ended exploration and the search for patterns. Important mathematics introduced or reviewed in *Patterns* include: In-Out tables, functions, variables, positive and negative numbers, and basic geometry concepts related to polygons.

Proof, another major theme, is developed as part of the larger theme of reasoning and explaining.

UNIT OVERVIEW

This unit addresses the following Common Core Standards for Math:

MATHEMATICAL PRACTICE STANDARDS

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

CONTENT STANDARDS

- CC.A-SSE.1: Interpret expressions that represent a quantity in terms of its context.
- CC.F-IF.1: Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
- CC.F-IF.3: Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.
- CC.F-BF.1: Write a function that describes a relationship between two quantities.
- CC.F-BF.1A: Determine an explicit expression, a recursive process, or steps for calculation from a context.
- CC.F-BF.2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two lines.
- CC.G-C0.1: Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Big Idea// Unit Essential Question:

Mathematics has been described as a "science of patterns." Patterns play multiple roles in mathematics -- in describing functions, creating conjectures and developing proof.

Unit Enduring Understanding(s):

✓ Math as the science of patterns.

✓ Introduction on how to work on and think about mathematical problems.

✓ Using concrete mathematical models.

Knowledge and Skills:

- Find, analyze, and generalize geometric and numeric patterns
- Analyze and create In-Out tables
- Use variables in a variety of ways, including to express generalizations
- Develop and use general principles for working with variables, including the distributive property
- Work with order-of-operations rules for arithmetic
- Use a concrete model to understand and do arithmetic with positive and negative integers
- Apply algebraic ideas, including In-Out tables, in geometric settings
- Develop proofs concerning consecutive sums and other topics

Assessments:

Teachers will have a variety of opportunities to formatively assess student understanding, in addition to end of unit summative assessments. Each of these assessment tools are included within the lesson activities or at the end of this unit.

- ✓ Problem of the Week
- ✓ Written/Oral Assignments:
    - Calculator Exploration
    - Pulling Out Rules
    - You're the Chef
    - Consecutive Sums
    - Angular Summary
    - Border Varieties
- ✓ End of Unit In Class
- ✓ End of Unit Take Home

- ✓ Student Portfolio

Accommodations/Differentiation

IMP offers both extension and reinforcement activities to address the varying needs of students in the classroom. All of these activities are noted and including in the lesson activity pages.

Patterns

Functions, Reasoning, and Problem Solving

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Contents

Introduction

Patterns Unit Overview

Intent

This unit is an essential introduction for students to the variety of ways for working on and thinking about mathematical problems. Students are introduced to general learning skills and methods that are developed and used throughout the four-year IMP curriculum and that form the foundation of the learning process through which students will build mathematical ideas. Here is a summary of these learning skills and methods.

- Working in groups to analyze problems
- Learning about group cooperation and roles in group learning
- Expressing mathematical ideas orally and in writing
- Making presentations in small groups and to the class
- Developing strategies for solving problems
- Using concrete mathematical models
- Doing investigations in which the task is not clearly defined
- Becoming familiar with alternative forms of assessment, such as self-assessment and portfolios
- Learning about the use of graphing calculators and, if available, appropriate computer software

Mathematics

Patterns emphasizes extended, open-ended exploration and the search for patterns. Important mathematics introduced or reviewed in *Patterns* include In-Out tables, functions, variables, positive and negative numbers, and basic geometry concepts related to polygons. Proof, another major theme, is developed as part of the larger theme of reasoning and explaining. Students' ability to create and understand proofs will develop over their four years in IMP; their work in this unit is an important start. This unit focuses on several mathematical ideas:

- Finding, analyzing, and generalizing geometric and numeric patterns
- Analyzing and creating In-Out tables
- Using variables in a variety of ways, including to express generalizations
- Developing and using general principles for working with variables, including the distributive property
- Working with order-of-operations rules for arithmetic

- Using a concrete model to understand and do arithmetic with positive and negative integers
- Applying algebraic ideas, including In-Out tables, in geometric settings
- Developing proofs concerning consecutive sums and other topics

Read "Patterns: A Mathematical Commentary" by Eric Robinson, Professor of Mathematics at Ithaca College, Ithaca, New York.

Progression

In *The Importance of Patterns,* the unit opens with an introduction to functions and their representations. Students begin to build their ability to tackle novel mathematical problems and have their first experiences with graphing calculators. The activities in *Communicating About Mathematics* build on the strands begun in *The Importance of Patterns,* focusing on the written and oral communication of mathematical ideas. In *Investigations,* students explore several rich mathematical tasks while employing the tools and techniques they have developed so far. Finally, in *Putting It Together,* students bring all of their new mathematical tools and techniques, as well as their developing identity as a learning community, to bear on a group of summary activities.

Pacing Guides

50-Minute Pacing Guide (22 days)

Day	Activity	In-Class Time Estimate
The Importance of Patterns		
1	Welcome students	10
	What's Next?	30
	Homework: *Past Experiences*	10
2	Discussion: *Past Experiences*	10
	POW 1: The Broken Eggs	35
	Homework: *Who's Who?*	5
3	Discussion: *Who's Who?*	15
	The Standard POW Write-up	20
	Homework: *Inside Out*	15
4	Discussion: *Inside Out*	45
	Homework: *POW 1: The Broken Eggs*	5
5	Discussion: *POW 1: The Broken Eggs*	10
	Calculator Exploration	40
	Homework: *Pulling Out Rules*	0
6	Discussion: *Pulling Out Rules*	20
	Lonesome Llama	30
Communicating About Mathematics		
	Homework: *POW 1: The Broken Eggs* (continued)	0
7	Discussion: *POW 1: The Broken Eggs* (continued)	5
	Role Reflections	15
	Marcella's Bagels	30
	Homework: *POW 1: The Broken Eggs* (continued)	0
8	Discussion: *POW 1: The Broken Eggs* (continued)	15
	1-2-3-4 Puzzle	30
	Homework: *Uncertain Answers*	5
9	Discussion: *Uncertain Answers*	15

	Extended Bagels	15
	Discussion: Expectations for presentations	15
	Homework: *POW 1: The Broken Eggs* (continued)	5
10	Presentations: *POW 1: The Broken Eggs*	15
	The Chefs' Hot and Cold Cubes	35
	Homework: *Do It the Chefs' Way*	0
11	*POW 2: Checkerboard Squares*	10
	Discussion: *Do It the Chefs' Way*	30
	Reflection: First 11 days of class	5
	Homework: *You're the Chef*	5
Investigations		
12	Discussion: *You're the Chef*	10
	Consecutive Sums	30
	Homework: *Add It Up*	10
13	Discussion: *Add It Up*	10
	Consecutive Sums (continued)	30
	Homework: *Group Reflection*	10
14	*Consecutive Sums* (continued)	25
	Discussion: *Group Reflection*	15
	Discussion: *POW 2: Checkerboard Squares* (continued)	10
	Homework: *That's Odd!*	0
15	Discussion: *That's Odd!*	10
	Pattern Block Investigations	40
	Homework: *Degree Discovery*	0
16	Discussion: *Degree Discovery*	15
	Polygon Angles	30
	Homework: *An Angle Summary*	5
17	Discussion: *An Angle Summary*	10
	Presentations: *POW 2: Checkerboard Squares*	20

Putting It Together		
	Squares and Scoops	20
	Homework: *Another In-Outer*	0
18	Discussion: *Another In-Outer*	30
	Homework: *Diagonally Speaking*	20
19	Discussion: *Diagonally Speaking*	20
	The Garden Border	25
	Homework: *Border Varieties*	5
20	Discussion: *Border Varieties,* including equivalent expressions and the distributive law	30
	Homework: *Patterns* Portfolio	20
21	*In-Class Assessment*	40
	Homework: *Take-Home Assessment*	10
22	Discussion: *In-Class Assessment* and *Take-Home Assessment*	30
	Unit Reflection	20

90-minute Pacing Guide (13 days)

Day	Activity	In-Class Time Estimate
The Importance of Patterns		
1	Welcome students	10
	What's Next?	30
	POW 1: The Broken Eggs	30
	The Standard POW Write-up	15
	Homework: *Who's Who?* and *Past Experiences*	5
2	Discussion: *Who's Who?* and *Past Experiences*	30
	Inside Out	60
	Homework: *POW 1: The Broken Eggs*	0
3	Discussion: *POW 1: The Broken Eggs*	10
	Calculator Exploration	40
	Lonesome Llama	30
	Role Reflections	5
	Homework: *Pulling Out Rules*	0
	Homework: *POW 1: The Broken Eggs* (continued)	5
4	*Role Reflections* (continued)	15
	Discussion: *Pulling Out Rules*	20
Communicating About Mathematics		
	Marcella's Bagels	30
	Discussion: *POW 1: The Broken Eggs* (continued)	20
	Homework: *POW 1: The Broken Eggs* (continued)	5
5	Discussion: Expectations for presentations	15
	Presentations: *POW 1: The Broken Eggs*	20
	Extended Bagels	15
	The Chefs' Hot and Cold Cubes	35
	Homework: *Do It the Chefs' Way*	5
6	Discussion: *Do It the Chefs' Way*	35
	Reflection: First 5 days of class	5

	1-2-3-4 Puzzle	30
	POW 2: Checkerboard Squares	15
	Homework: *Uncertain Answers* and *You're the Chef*	5
7	Discussion: *Uncertain Answers* and *You're the Chef*	20
Investigations		
	Consecutive Sums	20
	Add It Up	30
	Consecutive Sums (continued)	15
	Homework: *Group Reflection*	5
8	*Consecutive Sums* (continued)	50
	Pattern Block Investigations	35
	Homework: *That's Odd!* and *Degree Discovery*	5
9	Discussion: *That's Odd!*	10
	Pattern Block Investigations (continued)	10
	Discussion: *Degree Discovery*	15
	Polygon Angles	30
	Discussion: *Group Reflection*	15
	Discussion: *POW 2: Checkerboard Squares*	10
	Homework: *An Angle Summary*	0
10	Discussion: *An Angle Summary*	10
	Presentations: *POW 2: Checkerboard Squares*	20
Putting It Together		
	Squares and Scoops	20
	Diagonally Speaking	40
	Homework: *Another In-Outer*	0
11	Discussion: *Another In-Outer*	30
	Diagonally Speaking (continued)	15
	The Garden Border	25
	Discussion: Introduce portfolios	15
	Homework: *Border Varieties*	5
12	Discussion: *Border Varieties,* including equivalent expressions and the distributive law	40
	Patterns Portfolio	35

	Homework: *Patterns* Portfolio (finish)	5
	Homework: *Take-Home Assessment*	10
13	*In-Class Assessment*	45
	Discussion: *In-Class Assessment* and *Take-Home Assessment*	30
	Unit Reflection	15

Materials and Supplies

All IMP classrooms should have a set of standard supplies and equipment. Students are expected to have materials available for working at home on assignments and at school for classroom work. Lists of these standard supplies are included in the section "Materials and Supplies for the IMP Classroom" in *A Guide to IMP*. There is also a comprehensive list of materials for all units in Year 1.

Listed below are the supplies needed for this unit. General and activity-specific blackline masters are available for presentations on the overhead projector or for student worksheets. The masters are found in the *Patterns* Unit Resources.

Patterns

- Four tubs of pattern blocks
- A set of overhead pattern blocks
- A bag of beans or similar manipulative to serve as counters
- Cubes of two different colors to represent positive and negative numbers
- Sets of *Lonesome Llama* cards (one set for each group; included in the *Patterns* Blackline Masters)

More About Supplies

- Graph paper is a standard supply for IMP classrooms. Blackline masters of 1-Centimeter Graph Paper, ¼-Inch Graph Paper, and 1-Inch Graph Paper are provided so that you can make copies and transparencies for your classroom. (You'll find links to these masters in "Materials and Supplies for Year 1" of the Year 1 guide and in the Unit Resources for each unit.)

Assessing Progress

Patterns concludes with two formal unit assessments. In addition, there are many opportunities for more informal, ongoing assessment throughout the unit. For more information about assessment and grading, including general information about the end-of-unit assessments and how to use them, see "Assessment and Grading" in *A Guide to IMP*.

End-of-Unit Assessments

Each unit concludes with in-class and take-home assessments. The in-class assessment is intentionally short so that time pressures will not affect student performance. Students may use graphing calculators and their notes from previous work when they take the assessments. You can download unit assessments from the Patterns Unit Resources (or you can find them in the Blackline Masters).

Ongoing Assessments

Assessment is a component in providing the best possible ongoing instructional program for students. Ongoing assessment includes the daily work of determining how well students understand key ideas and what level of achievement they have attained in acquiring key skills.

Students' written and oral work provides many opportunities for teachers to gather this information. Here are some recommendations of written assignments and oral presentations to monitor especially carefully that will offer insight into student progress.

- Presentations on *Calculator Exploration:* These presentations will give you information on how comfortable students are with calculators and open-ended investigation.
- *Pulling Out Rules:* This activity will help you gauge how well students understand the basic ideas of In-Out tables and evaluate their ability to write rules to describe tables.
- *You're the Chef:* This summary activity will tell you how well students understand the arithmetic of positive and negative integers.
- Presentations on *Consecutive Sums:* These presentations will indicate how students are developing the ability to conduct independent mathematical investigations.
- *An Angular Summary:* This activity will help you gauge students' understanding of the sum of the angles in a polygon and related formulas.
- *Border Varieties:* This activity will reflect students' understanding of the use of variables.

Supplemental Activities

Patterns contains a variety of activities at the end of the student pages that you can use to supplement the regular unit material. These activities fall roughly into two categories.

- **Reinforcements** increase students' understanding of and comfort with concepts, techniques, and methods that are discussed in class and are central to the unit.
- **Extensions** allow students to explore ideas beyond those presented in the unit, including generalizations and abstractions of ideas.

The supplemental activities are presented in the teacher's guide and the student book in the approximate sequence in which you might use them. Below are specific recommendations about how each activity might work within the unit. You may wish to use some of these activities, especially the later ones, after the unit is completed.

Keep It Going (reinforcement) Students use patterns to find the next few terms of four number sequences and then describe the patterns they found.

The Number Magician (reinforcement) In essence, this activity describes a multistep In-Out machine. Students determine the original number that produces one particular answer and analyze the method used to determine the original number so quickly.

Whose Dog Is That? (extension) This logic puzzle, much like *Who's Who?,* gives students another opportunity to use organized thinking and to write clear explanations. Students are given several interlocking conditions and must use logical reasoning to determine a set of conclusions. Give students several days to work on the activity and to write up their results.

A Fractional Life (reinforcement) This activity is part of *The Greek Anthology,* a group of problems collected by ancient Greek mathematicians. It will help reinforce students' work with In-Out tables and can be used any time after In-Out tables are introduced.

Counting Llama Houses (extension) Students identify the ways in which the houses in *Lonesome Llama* differed and then determine how many different houses could have been created using these variations.

It's All Gone (reinforcement) In a variation on *Marcella's Bagels,* a man goes from store to store getting and spending money, winding up with no money in the end. Students are asked to determine how much money he had when he started.

1-2-3-4 Varieties (reinforcement) This activity adds a rule to those used in *1-2-3-4 Puzzle:* now the digits must appear in numeric order. In addition to finding expressions for the first 25 whole numbers, students are asked to find the greatest possible answer given these rules and to make up their own variations to the original activity.

Positive and Negative Ideas (extension) This activity extends students' work with hot and cold cubes. It asks them to consider other ways they might model integer arithmetic.

Chef Divisions (extension) This activity extends ideas introduced through the "hot-and-cold-cubes" model. While modeling division with hot and cold cubes, students think more deeply about the model and the reasoning involved in working with negative numbers.

More Broken Eggs (extension) In *The Broken Eggs,* students found a possible number of eggs the farmer might have had when her cart was knocked over. The task now is to look for other solutions, to find and describe a pattern for obtaining all the solutions, and to explain why all the solutions fit that pattern.

Three in a Row (extension) Students explore sums of three consecutive numbers as well as sums of other lengths. The activity is appropriate following the discussion of *That's Odd!*

Any Old Sum (extension) In this variation on the *Consecutive Sums* investigation, students examine sums that are not consecutive. In addition to extending ideas in *Consecutive Sums,* this activity gives students more experience with open-ended problems.

Getting Involved (reinforcement) Several activities in this unit—such as Role Reflections and Group Reflection—ask students to reflect on the process of working in groups. In this related activity, students are asked to reflect on a situation in which one person in a group is not contributing.

The General Theory of Consecutive Sums (extension) Students explore consecutive sums of integers. You may want to allow students several days to work on this challenging activity.

Infinite Proof (extension) Students are asked to prove that the square of every odd number is odd and that every prime number greater than 10 must have 1, 3, 7, or 9 as its units digit. The activity gives students the opportunity to see that proofs are possible in situations involving infinitely many cases.

Different Kinds of Checkerboards (extension) In this follow-up to *POW 2: Checkerboard Squares,* students find the number of squares on nonsquare checkerboards and search for a general rule for checkerboards of dimensions m by n.

Lots of Squares (extension) In this substantial investigation, students are asked to divide a square into different numbers of smaller squares. The goal is to determine which numbers of smaller squares are impossible and which are possible, and to prove their results. Assign the activity after students have worked on developing proofs.

A Protracted Engagement (reinforcement) In this open-ended activity, students are asked to decode a message created using angles of different sizes to correspond to different letters of the alphabet, and then to code a message of their own.

A Proof Gone Bad (reinforcement) Students are asked to explain the contradictions in another student's proof. Assign the activity after students have worked on developing proofs.

From Another Angle (extension) This activity extends students' work with pattern blocks and generalizes ideas in *An Angular Summary*.

From One to N (extension) The task in this activity, which is a natural outgrowth of the ideas in *Squares and Scoops,* is to find a simple expression in terms of *n* that allows one to find a sum without repeated addition. If students find such an expression, they look for a proof that their answer is correct.

Diagonals Illuminated (extension) This follow-up activity to *Diagonally Speaking* draws the distinction between recursive and closed-form rules and asks students to develop a closed-form rule for the number of diagonals of any polygon. Students are then asked to explain why this rule makes sense.

More About Borders (extension) This activity contains variations on the *Border Varieties* activity.

Programming Borders (extension) Building on ideas in the supplemental activity *More About Borders,* this activity asks students to write a program that answers some or all of the questions posed in *More About Borders.*

Patterns: A Mathematical Commentary

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The goal of this commentary is to situate the mathematics in the unit *Patterns: Functions, Conjectures, and Proof* within the larger context of mathematics as a discipline. In my opinion, the material in this unit lies within the very *core* of modern mathematics. (What a perfect way to begin the curriculum program!)

Of course, part of the reason this material is placed within the mathematical core is because the unit deals with fundamental mathematical concepts: numbers, shapes (polygons), and functions. (I will mention more about functions later in the commentary.) However, as a mathematician, I want to focus on what gets me really excited about this unit: the environment it provides for students to engage in mathematical thinking and reasoning. And while mathematics has many facets, at its *center* lies mathematical thinking and reasoning: a methodology that includes exploring (investigating), conjecturing, explaining, and justifying certain phenomena. It is from mathematical thinking and reasoning activity that the body of mathematical facts emanates and grows—often building upon itself—and that mathematical algorithms, procedures, and structures are created.

Moreover, I think one of biggest motivations—if not the biggest—for most mathematicians to devote high energy to mathematics (and what obsesses them in particular instances—sometimes for years) is the chance to engage in mathematical inquiry: the search for answers to questions such as (1) What is going on here (mathematically)? (2) Why is it happening? (3) Can I be certain this always will happen? (4) Under what conditions will it happen this way? (That is, is this generalizable or extendable?) The first of these questions usually leads to a conjecture. The second stimulates a search for insight. The third motivates a search for validation (proof). Answers to the fourth extend the generality and applicability of a known result. Mathematicians do not have the luxury of knowing whether a conjecture is true or false before trying to prove it. They often look for evidence by exploring multiple examples and they probably will try a range of reasoning strategies in pursuit of their goal to confirm or reject the conjecture.

The point is that mathematicians' work centers on asking and answering questions. Now, if mathematicians answer questions, a mathematics textbook should be filled with rich questions—similar to those mentioned above. The *Patterns* unit (and, for that matter, the whole curriculum) is.

It is very heartening in this unit to see students engage in the search for mathematical insight, including the search for mathematical patterns, the formation of mathematical conjectures, verification or refutation of these conjectures, looking for generalizations and extensions of results, and abstracting properties from particular instances or examples. This is what mathematicians do. The reader easily can find instances of each of the aforementioned mathematical activities in multiple places within the *Patterns* unit. Now, the term "proof" commonly refers to (logical) justification of a result. So, narrowly construed, the word does not capture the rich mathematical reasoning aspects described above that accompany the search for proof. Fortunately, the *Patterns* unit begins to capture these mathematical reasoning characteristics.

Mathematics has been described as a "science of patterns." At the very least, this description is intended to emphasize the prominent place of patterns within the discipline. Indeed, patterns play multiple roles in mathematics. Here, we mention a few of theses roles. First, the search for patterns is a fundamental approach to creating conjectures. There are many open conjectures in current mathematics that relate to patterns. One open example (as of this writing) that is relatively easy to explain has to do with twin primes. Now, all prime numbers except the number 2 are odd. So, two different prime numbers greater than 2 must be at least two units apart. Some pairs of primes are exactly two units apart. For instance, 3 and 5 are two units apart; so are 5 and 7, 11 and 13, and 17 and 19. Two prime numbers (greater than 2) that are exactly two units apart are called twin primes. Many pairs of twin primes have been found, some quite a way along in the natural numbers, but seemingly with increasing scarcity. In the latter half of the 19th century it was conjectured that the pattern of existence of twin primes is unending. That is, there are infinitely many pairs of twin primes. But no one has been able to prove it—yet. (Twin primes are mentioned later in the IMP curriculum.) Study of twin primes has led to the discovery of many interesting patterns and results related to these numbers! (http://mathworld.wolfram.com/TwinPrimes.html)

In the *Patterns* unit under consideration, the activity *Consecutive Sums* is a good example of the role of patterns in developing conjectures. The request for explanation in that activity and the associated activity *That's Odd!* helps bring home the idea that finding patterns is only the first part of the mathematical inquisition; there is eventually a need to verify that the pattern persists and, hopefully, one will be able to answer why and how the pattern works.

Patterns are studied in many mathematical domains. They occur in the domains of numbers, shapes, and other mathematical structures. Sometimes patterns develop when they aren't expected and mathematicians can get excited when they see them. Michael Barnsley's "Chaos Game" provides this experience. (http://www.jgiesen.de/ChaosSpiel/ChaosEnglish.html) When the game is "played," a highly patterned deterministic fractal emerges as the *limit* of a random process.

While a pattern can serve as input for conjectures, a pattern can often be interpreted as the output of many a theorem. Almost any theorem can be thought of as a statement about patterns. The theorem in Euclidean geometry stating that the base angles of any isosceles triangle are always equal in measure is a statement of a persistent pattern in isosceles triangles.

How much data is necessary to make a conjecture about a pattern? Sometimes a little and sometimes a lot. To an experienced mathematician, sometimes one carefully constructed example can suggest a generalized pattern worth pursuing. On the other hand, sometimes conjectures with too little data turn out to be false. The great Pierre de Fermat (1601–1665) computed the numbers of the form $2^{2^n} + 1$ for $n = 0, 1, 2, 3$, and 4. The results were 3, 5, 17, 257, and 65,537—all prime numbers. Fermat confidently concluded that all numbers of the form $2^{2^n} + 1$ are prime, where n is a non-negative integer. Fermat was wrong in this case. In fact, the very next case, $2^{2^5} + 1 = 4{,}294{,}967{,}297$, is not a prime number. It was laborious to check primality in numbers this large by hand in Fermat's day and Fermat obviously did not check. In 1732, Leonard Euler reasoned correctly that 641 is a factor of $2^{2^5} + 1$.

So, in searching for patterns, it is important to note that some data can suggest a definitive pattern and be correct and some conjectured results from the data about a pattern can fail (as happened for Fermat). It is also important to note that some data can suggest multiple possible patterns. This latter fact is nicely illustrated in the IMP activities *Inside Out* and *Pulling Out Rules*. Activities like *Pulling Out Rules* also hint at the general mathematical notion of curve fitting: finding functions that "fit" data sets. (Where the term *fit* is further defined—often in relation to the kinds of data sets or functions involved.)

Now, we have segued into the third major word in the title of the unit: functions. The unit treats functions as rules of assignment; as inputs paired with outputs. As such, functions can be described verbally, suggested in table form, represented graphically, or coded symbolically. Each representation has its advantages and disadvantages. The first two types of representations are encountered early in the unit. But, as the unit progresses, the foundation and motivation to express rules symbolically is clearly laid. Of course, this is one of the fundamental characteristics of algebraic thinking: using symbols (variables) to code a general relationship.

Functions appear in nearly all major branches of mathematics and they serve a variety of roles. For example, they may help us to represent change (as in differential calculus) or what appears to be the opposite: invariance (as elements of transformation groups used to describe symmetry patterns of geometric objects).

Finally, as illustrated by the study of symmetry groups just mentioned, patterns, functions, conjectures, and proof are not isolated objects, but rather are part of the interwoven fabric of mathematics. Indeed, this interconnectivity is yet another central aspect of modern mathematics. This interconnectedness is well illustrated in the *Patterns* unit and, for that matter, in the curriculum as a whole.

The Importance of Patterns

Intent

The activities in *The Importance of Patterns* introduce a crucial mathematical idea—functions and their representations—that will weave its way through the entire curriculum. Students begin to build their ability to tackle novel mathematical problems, a way of doing mathematics that permeates IMP. In addition, these activities begin to establish expectations for students' classroom interactions—as a whole class, in small groups, and individually—and written work throughout the course. Finally, students have their first experiences using graphing calculators.

Mathematics

The central mathematical idea in *The Importance of Patterns* is the concept of function, one of the fundamental unifying principles in mathematics. Functions are introduced using numeric and nonnumeric examples, with an emphasis on looking for patterns and describing those patterns verbally. The term **function** is introduced in the context of describing the *Out* as "a function of" the *In*. The discussion introduces the principle that a function cannot have more than one output for a given input as well as the concepts of the **domain** and **range** of a function. Students use **variables** and *algebraic expressions* to describe numeric functions. They also apply *In-Out tables* to mathematical problems and see the distinction between tables arising from context and tables that are simply collections of number pairs. In-Out tables are a standard method for representing functions and are central to this unit and the curriculum.

Variables play a vital role throughout mathematics, and they represent a major step toward mathematical abstraction. One major goal in *Patterns* is for students to use the symbolic language of algebra as shorthand to describe patterns, particularly arithmetic patterns in In-Out tables. Students use both their verbal description of a table's rule and the pattern of arithmetic for finding specific outputs in order to develop an algebraic expression that describes the rule. As part of this work, they begin to use the terms *variable* and *algebraic expression*. The concept of **equivalent expressions** is introduced in the context of seeing that different expressions give the same results for an In-Out table.

Progression

What's Next?

Past Experiences

POW 1: The Broken Eggs

Who's Who?

The Standard POW Write-up

Inside Out

Calculator Exploration

Pulling Out Rules

Lonesome Llama

Role Reflections

What's Next?

Intent

This first activity in the unit engages students in a series of questions for which there may be no familiar procedure or algorithm and for which there might be many solutions. It is students' first opportunity to do mathematics together.

Mathematics

This activity introduces the mathematical idea of a **sequence**. Students are asked to find patterns that fit a given sequence and then to use these patterns to predict the next few terms of the sequence. The search for patterns is a recurring theme in this unit and throughout the IMP curriculum. These early activities also build a foundation for the concept of **function**, one of the truly big ideas of algebra.

Progression

After you have introduced IMP to students as a somewhat different kind of textbook, and have made students aware that their classroom working environment will have certain characteristics (see "Characteristics of the IMP Classroom" in the Overview to the Interactive Mathematics Program), this activity will be their first experience of IMP and the "IMP classroom." Students will work on this activity in a small group of peers. They will be encouraged to be creative in describing their patterns and to share ideas with group members. The activity concludes with students discussing some of the patterns they identified.

Approximate Time

30 minutes

Classroom Organization

Small groups, followed by whole-class discussion

Doing the Activity

You might begin by asking one or more volunteers to read the instructions aloud. Emphasize that students are to do more than simply find the next few terms of each sequence; they are also to give a description of the pattern they see.

Offer some ideas about expectations for group collaboration and interaction—namely, that groups do both. Suggest a few methods to do so, such as occasionally asking what a neighbor found to compare to your result or what someone sees when you haven't been able to notice a pattern. Emphasize that all students are expected to ask for help when stuck and to help others when asked.

Have groups begin writing down their ideas. Students might work individually on the first two questions and then discuss those in their groups before moving on to the next pair of questions. Stopping to share ideas lets the groups hear what everyone is thinking and see that there is more than one possible pattern or approach. Sharing also helps students learn to value each other's thinking and to collaborate.

Circulate as students work and listen in on the discussions, limiting interventions in order to encourage students to rely on their own thinking and to work collaboratively within their groups. You might ask groups that need help some probing questions, such as those below.

In your own words, what is the pattern you found?

Does the pattern fit the terms of the sequence?

Does someone else have another way to describe this pattern?

Did you find other patterns that start the same way?

Can there be more than one correct pattern?

Remind students, if needed, to look for more than one possible pattern for each sequence or more than one way to describe a given pattern. If many groups seem stuck, you might interrupt and have a class discussion of the first question or two to clarify what is being asked.

When the majority of groups have finished Questions 1 through 6, you might bring the class together for discussion. As time allows, you can then have them turn to Questions 7 and 8.

Discussing and Debriefing the Activity

On the first day of the unit, it is important to establish a classroom climate where student thinking is valued and where it is safe to take the risk of sharing ideas.

Ask presenters to describe the patterns they found and the next terms these patterns led to. These descriptions can be very informal. Then ask for comments from the class. Have the class work together to make the pattern descriptions clear. (Over the course of this unit, you will ask students to make more precise statements, including algebraic descriptions of some patterns.)

It is important that students see different ways to describe a given pattern as well as different patterns that fit a given initial sequence. Ask if anyone found other ways to describe a given pattern. What other ways did you find to describe this pattern? For example, some students may describe the sequence in Question 4 (1, 2, 4, . . .) as "Double each term to get the next term," while others may say "Add each term to itself to get the next term." Although these two descriptions are different, they lead to the same continuation of the pattern, with the next three terms being 8, 16, and 32.

Did you find other patterns that start the same way? Some students may have discovered different patterns that fit the same opening terms of a given sequence. For instance, in Question 4 (1, 2, 4, . . .), the pattern could be "Add 1, then add 2, then add 3, and so on." In this pattern, the next three terms would be 7, 11, and 16 (rather than the 8, 16, and 32 for a doubling pattern).

Question 5 also offers more than one option. For example, the sequence could repeat the opening terms, 1, 3, 5, 7, 5, 3, over and over again (so it goes 1, 3, 5, 7, 5, 3, 1, 3, 5, 7, 5, 3, 1, 3, . . .), or it could follow 1, 3, 5, 7, 5, 3 with 1, 3, 5, 7, 9, 7, 5, 3 and then 1, 3, 5, 7, 9, 11, 9, 7, 5, 3, and so on.

When students present more than one description of a pattern or more than one pattern, ask if both can be right. Bring out the idea that any description or pattern that fits the opening terms of a sequence is as correct as any other.

As time allows, ask students to present sequences that their groups created for Questions 7 and 8. You might display these on the board or have groups try to figure out each other's patterns.

Key Questions

In your own words, what is the pattern you found?

Does the pattern fit the terms of the sequence?

Does someone else have another way to describe this pattern?

Did you find other patterns that start the same way?

Can there be more than one correct pattern?

Supplemental Activity

Keep It Going (reinforcement) asks students to use patterns to find the next few terms of four number sequences and to describe the patterns they found.

Past Experiences

Intent

This individual activity is best used as homework, the first assignment of the school year. Including a writing assignment like this one will establish several expectations for the course.

Purposeful homework will be assigned every day, and students' work on these assignments will be an important part of the course.

Students will be asked to put their thinking—about mathematics and about themselves as learners of mathematics—to paper.

All students' thoughts and ideas about the mathematics they are learning are crucial to the success of the course.

Successful collaboration to do and learn mathematics is a key feature of this course.

Mathematics

At first glance, this assignment does not look particularly mathematical. However, a growing body of research suggests that successful mathematical problem solvers are reflective thinkers. They know mathematics, and they know about mathematics as a discipline. They are aware of themselves as mathematics learners, and they can think about their own thinking—monitoring progress, evaluating strategies, choosing among skills and tools—while doing mathematics. Psychologists call this *metacognition*, and it is a hallmark of the thinking of effective problem solvers. In this activity, students are asked—perhaps for the first time (and certainly not the last time in this program)—to reflect on some of their experiences as mathematics students.

Progression

This activity is designed to be done as homework after the first class and to be discussed, in small groups and as a whole group, in the next class.

Approximate Time

10 minutes for introduction

20 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Whole class, then individuals, followed by small groups

Doing the Activity

Take the time to share your expectations for this assignment and homework in general, including what you expect from students and what students can do if they don't understand an assignment. Telling students that you want to learn more about them and their backgrounds, and that you will not be grading their essays, but just recording whether they completed the assignment, may encourage them to do the assignment and to share honestly. One important goal of the first few homework assignments is to help students establish a pattern of doing their homework regularly.

Also impress upon students that they need to save their work throughout the unit, as they will be asked to include their written work on this assignment and others in the portfolios they will create at the end of this unit.

For the next day's discussion, you might want students to share their essays in their groups. If you plan to follow this suggestion, let students know now that other students will be reading their written work.

Discussing and Debriefing the Activity

Students can read the essays of the other members of their groups. You might suggest that after reading each other's thoughts and experiences, students answer the Key Questions listed below, perhaps displaying these or similar discussion questions on a transparency. Then students can share with the class the themes their groups encountered.

This is a good opportunity to reiterate that class participation—written, oral, and physical; in groups, individually, and with the whole class—is essential for success.

Key Questions

What are some of the important mathematical ideas you have studied?

How are your group's ideas about your most and least helpful learning experiences similar? How are they different?

How are your experiences, thoughts, and feelings about working with others similar? How are they different?

POW 1: The Broken Eggs

Intent

As the first POW, or Problem of the Week, *The Broken Eggs* is students' first opportunity to work on a substantial problem over several days and communicate the results of their work in writing, using a format that will carry across the four years of the program. (See "Problems of the Week" in the Overview to the Interactive Mathematics Program.)

Mathematics

This POW is a version of a well-known problem in number theory. Here is a translation from a seventh-century text written by the Hindu mathematician Brahmagupta: *An old woman goes to market, and a horse steps on her basket and crushes the eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember the exact number, but when she had taken them out two at a time, there was one egg left. The same happened when she picked them out three, four, five, and six at a time, but when she took them out seven at a time they came out even. What is the smallest number of eggs she could have had?* A similar problem was posed by the Chinese scholar Sun Tsu Suan-Ching in the third century: *There are certain things whose number is unknown. Repeatedly divided by 3, the remainder is 2; by 5 the remainder is 3; and by 7 the remainder is 2. What will be the number of things?*

In the activity, students search for numbers divisible by 7, but when divided by each of numbers 2 through 6 leave a remainder of 1. To find solutions to this problem, students must examine multiples of 7 and remainders when dividing by 2 through 6, and reason about patterns in these results. *The Broken Eggs* problem has many solutions, creating a complex task that will allow any high school student to begin to work on the question and all to pursue it as far as their interest (and time) allows (see "About Solutions to Activities" in the Overview to the Interactive Mathematics Program).

Progression

Students will work on this POW primarily outside of class. This unit is carefully designed to support student success, especially with this first long-term, problem-solving and writing project. The problem is posed early in *The Importance of Patterns* and revisited at several points over the next few class meetings. Three students will present their solutions to the class, and all will turn in their written work.

Approximate Time

5 minutes for introduction

30 minutes for groups to begin exploration

20 minutes for introducing POW write-ups

30 minutes for individuals (homework; do and write Process)

5 minutes for small-group discussion

30 minutes for individuals (homework; complete Process)

15 minutes for discussion findings and of next phases of write-up

15 minutes for discussion of presentation expectations

30 minutes for individuals (homework; complete POW write-up)

15 minutes for presentations

Classroom Organization

Individuals and small groups, concluding with whole-class presentations and class discussion

Materials

Presenters will need presentation materials, such as transparencies and pens, a few days prior to the due date.

Doing the Activity

Please be thoughtful about the extended timeframe over which the activity occurs. The more support you offer students in this, their first long-term and significant problem-solving and writing activity, the more successful they will be in the subsequent writing activities they will encounter.

The commentary below divvies the focus of this supportive work into several "phases." This is not meant to indicate sequential days; depending on your scheduling situation, consider these the segments necessary to support student work, however you parse it into your daily plans.

Introduce the Problem

You might begin this activity by having one or more volunteers read it aloud, through the section Your Task. Tell students that they will be working on this problem in groups, sharing ideas and insights. They should keep notes on how they and their groups work on the problem, as they will be discussing these things in their final write-ups.

Begin Exploring

Have students work on the problem in groups. Although some students might find a solution during this initial work, this is not an expectation for the first day.

If groups are stuck, ask what they have tried. It is important that students have time to reach conclusions at their own pace. To clarify whether they understand the problem, you might ask why 49 is not a correct answer. You might also suggest they consider this simplified problem: Suppose the farmer remembered that only when she put the eggs in groups of either two or five, there was one egg left over. What would be some possibilities for the number of eggs in that situation?

If any groups find the answer 301 today, you can urge them to look for other solutions. If groups need further challenges, they can look for a general solution or a description of how to find other solutions, and then for an explanation of how they know their general solution includes all possibilities.

The Standard POW Write-Up

Have students read *The Standard POW Write-Up.* Most POWs will have a Write-up section that uses the basic components listed in *The Standard POW Write-Up.* The write-up instructions will often simply refer to these components by name, giving additional details only when the write-up differs from the basic model provided here. You may want to post the five write-up components on the wall.

Have students work individually for a while to create problem statements for the POW. Then have them share their ideas in their groups, and have each group use these ideas to create the best problem statement possible.

Ask one or two groups to share their problem statements with the class. In the discussion, bring out that the problem statement should not simply repeat the problem as originally stated, but should try to focus on the essentials. You might work with students to distinguish between the "story" aspects of the problem and its mathematical core. With this first POW, students might include both aspects in their problem statements, but over the course of the curriculum, they should gradually move toward an emphasis on the mathematical elements of the POWs.

You might remind students that taking notes as they work will help with the Process part of their write-ups. Encourage them to collaborate with classmates. You might mention explicitly that you do not consider it cheating to work with someone else on a homework assignment or POW, as long as students acknowledge that collaboration in their write-ups. On the other hand, students should not simply copy each other's work or allow others to copy from theirs. You might also offer advice about how to help each other, such as by giving a hint or asking a leading question. Point out that if they give a friend the answer, they deprive the friend of much of the learning experience.

Begin Writing the Process

Two or three days after assigning this first POW, assign for homework simply working on the POW for 20 to 30 minutes. Tell students they should come to class tomorrow with a portion of their Process written. If they are using word-processing software for their write-ups, they should bring a printout of this draft.

Encourage students to keep notes about ideas they have and things they try and to pause occasionally to add to the Process portion of their write-ups. A structure such as work 7 minutes, write for 3 minutes might help students achieve this goal.

When students return to class, encourage them to share ideas in their groups. Then explicitly instruct them to share what they have done in writing the Process. It can be valuable to have students pass their write-ups around their groups to see how others are recording their solution methods.

Finish Writing the Process

On the very next day (the day the homework above is due), have students spend 20 to 30 minutes working on the problem and completing the write-up of the Process. Again, they should bring a draft to class.

In class, again encourage groups to share findings from their investigations the previous evening, including reading one another's Process, perhaps in pairs this time.

Follow this with a class discussion. Ask students to describe what they think is in the Process section, as they have written it over the past few days. Return to the *The Standard POW Write-Up* reference page, and ask students to compare their impressions with the description of the Process section here.

Remind students that they have three sections left to write: Solution, Extensions, and Self-assessment.

Completion and Presentation Preparation

Use the contexts of *Marcella's Bagels* and *Extended Bagels* to remind students of the write-up structures and expectations.

As this will be the first POW presentation, ask for three volunteers to present their work. You may find it easier to get volunteers if you mention that this first group of presenters will get some extra guidance. (For future POWs, either select students at random, choosing from among those who have not yet done POW presentations, until everyone has had a turn, or ask for volunteers, again explaining the expectation that everyone will present once before cycling through again.)

Discuss with the class what will be expected of presenters and of the audience. Emphasize that presentations are to be discussions about *ideas*. It is important that presenters prepare to share what they learned about the problem and not feel pressured to present "the" answer.

Audience members should listen to discover what presenters have figured out, how they approached the problem, and the reasoning behind their conclusions. The audience will be expected to ask clarifying questions, such as "I don't understand how you arrived at this conclusion; I seem to get ___" or "That idea seems to contradict ___."

Presenters are to use transparencies to *help with* their presentations, rather than to *be* their presentations. In other words, they will not present only what is on the transparencies, but should plan to explain the problem, using the transparencies to save the trouble of writing as they talk. They can include diagrams, numeric calculations, and whatever else might be helpful.

Also, presenters may need reminding to plan to talk about *all* parts of the write-up, not only about their solutions. (In later POW presentations, students may find certain sections need less attention, especially given only 5 minutes to present.) Finally, encourage them to make any writing on transparencies large enough to be readable.

You might meet briefly with volunteers to address any questions or concerns they have and to give them transparencies and pens for preparing their presentations.

Once presenters are selected and expectations for the presentations have been communicated, tell everyone that the homework is to finish the write-up.

Discussing and Debriefing the Activity

Presentations

Because making presentations is very difficult for some students, these first POW presenters might have a tough time. Because of their willingness to volunteer, they deserve special consideration from the audience and assistance from the teacher.

Briefly remind the audience of these expectations (which you may want to post):

Acknowledge the effort and courage of the presenter, regardless of the quality of the presentation.

Treat each other with respect and listen attentively.

Listen for what the presenter learned, ask questions when you don't understand, and challenge things you think are incorrect. Being respectful does not mean being passive. It is not disrespectful to question, add to, or challenge each other's work if it is done in the proper spirit.

Multiple routes to the solution: Encourage students to ask questions during the presentations. After all the presentations are over, ask if anyone has anything else to add. Be sure students realize that this invitation includes presenting a different method for finding or explaining an answer—they do not have to have a new or a different answer.

More than one solution: If the presentations did not deal with the issue of the POW having more than one answer, bring that up now. Many students may have stopped exploring when they found that 301 fits all the given conditions. An important mathematical question to ask is, Is this problem one of those that has more than one answer? This problem, like many others, does have more than one answer.

It is not necessary at this time that students find the general expression for all possible solutions, but they should recognize the possibility of multiple solutions. The supplemental activity *More Broken Eggs* asks students for the general solution.

Key Questions

How do you know that (301, for example) is a solution? Is not a solution?

Do you suspect there are other solutions? Why?

Supplemental Activity

More Broken Eggs (extension) expands on *The Broken Eggs*, in which students found a possible number of eggs the farmer might have had when her cart turned over. The task now is to look for other solutions, to find and describe a pattern for obtaining all the solutions, and to explain why all the solutions fit that pattern.

Who's Who?

Intent

This activity is included early in the unit to engage students in the important processes of logical reasoning and proof.

Mathematics

This activity presents interlocking sets of conditions. Using these conditions, students must identify "who's who" and are asked to provide a convincing argument—a proof—of their conclusion. Issues of proof arise repeatedly throughout the curriculum and in daily interactions. Most significantly, students are always expected to justify their solutions, to convince others, and to be convinced by others.

The reasoning students will use to analyze the stated conditions, to make conjectures about the solution, to test those conjectures to convince themselves that their solution meets the stated conditions, and then to determine whether their solution is unique—that is, to prove their solution—is at the heart of what it means to do mathematics.

Progression

Students are asked to find a solution to this puzzle and to determine whether that solution is unique. The activity also gives students the chance to use two components of POW write-ups: Process and Solution.

Approximate Time

5 minutes for introduction

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Review what is expected in the activity. Urge students to start taking notes as soon as they begin thinking about the problem and to use those notes in the Process portion of their write-ups. Emphasize that the Solution part of their write-ups must demonstrate how they are certain of their solution and especially how they know that it is the only solution. (Some students might be unfamiliar with the game of Hearts, but they don't have to know anything about cards or this card game to do this activity.)

Discussing and Debriefing the Activity

Some students might approach this activity using logic. For example, because Felicia passes her cards to the ninth grader, she can't be the person who passes to the eleventh grader.

Others might list all possible cases. With only three students, it is not too difficult to list all possible grade levels and all possible arrangements of the students around the table, and then see which one meets all the conditions.

Whatever approach they use, students have the opportunity to engage in clear, logical argument to explain why their solution is unique.

Give students a few minutes in their groups to compare how they answered Question 1. Suggest that they focus not only on the answer, but also on how they worked on the problem—that is, on the Process component of the write-up.

As you circulate and listen in on the discussions, identify students who approached various parts of the activity in interesting ways, and ask them to present those parts to the class. Try to get several methods presented, both to describe the approaches themselves and to emphasize that there are many possibilities.

When students are convinced that there is a single solution, raise the issue of whether there are other answers, and ask students to explain how they can be sure there is only one. Is this one of those problems that has more than one answer? As with the discussion of solution methods, encourage different approaches.

After several students have offered explanations of why the answer is unique, ask whether students are completely convinced by these arguments. How convinced are you? Use this opportunity to review the word **proof**. Clarify that in this problem, a complete proof involves two aspects:

Showing that the particular seating arrangement satisfies the conditions in the problem; that is, that a solution *exists*

Showing that no other seating arrangement fits those conditions; that is, that the solution is *unique*.

Key Questions

Is this one of those problems that has more than one answer?

How convinced are you?

Supplemental Activities

The Number Magician (reinforcement) asks students to determine the original number that produces one particular answer and to analyze the method used to determine the original number so quickly.

Whose Dog Is That? (extension) is a logic puzzle much like *Who's Who?* Students are given several interlocking conditions and must use logical reasoning to determine a set of conclusions.

The Standard POW Write-Up

Intent

This reference page introduces students to Problems of the Week and to the standard POW write-up. Students will also refer to this reference page throughout the year to aid with their POW write-ups.

Mathematics

Communicating about mathematical thinking is an important part of doing mathematics. This reference page is designed to support students' written communication about their findings when exploring large mathematical problems. Stating the problem, discussing one's methods, and concluding succinctly and with justification, so that a reader will understand what has been written, should be the goal of every student writer. By suggesting extensions to the problem, students will be saying that the mathematics has not been fully explored, given the time constraints. Their self-assessment will be an evaluation of the effort and quality of their work, what they take pride in, and what they wish they could have done better.

Progression

Initially, this reference page will support students as they assemble a paper that communicates the work they did and what they learned. As they become more comfortable with the process of writing about mathematics, the page should be returned to and discussed occasionally. It will prompt students to think more deeply and to share ideas with other students about what each section of the POW write-up means, to evaluate each others' papers, and to focus on improving portions of their own work.

Approximate Time

20 minutes for discussion

Inside Out

Intent

This activity introduces students to a powerful representation of functions, In-Out tables, that will be used repeatedly throughout the IMP curriculum. The In-Out machine metaphor is used to introduce In-Out tables and the terms *input* and *output*. In their exploration, students experiment, make conjectures, and work toward getting clear verbal statements of the rules.

Mathematics

A powerful way to think about a **function** is as a machine. The thing put in is called the *input* (the *In*, for short); the thing that comes out is called the *output* (the *Out*). An *In-Out table* is one representation of a function. Other representations include graphs and symbolic rules. (Students will study these representations in *The Overland Trail*.) We often use the word *function* in such phrases as "the *Out* is a function of the *In*," which means that the *Out* value depends on, or is determined by, the *In* value.

Progression

This activity works well as homework. Students are introduced to the idea of an In-Out machine and learn to keep track of what it can do by using an In-Out table. They attempt to decode some In-Out tables and then create two of their own.

Approximate Time

15 minutes for introduction

20 minutes for activity (at home or in class)

45 minutes for discussion

Classroom Organization

Whole-class introduction, followed by individuals, then small groups, concluding with whole-class discussion

Doing the Activity

Introduce the idea of an *In-Out machine.* Many students may have been exposed to this method of representing functions, but others may not have been. Explain that something, often a number, is put into the machine. The machine does something to that object, and something comes out. The thing put in is called the *input* (the *In*, for short); the thing that comes out is the *output* (the *Out*). With your first example or two, you might show inputs as if they were actually being put into a machine and outputs as if they were coming out of the machine.

A nice way to start is to have students try to figure out what the machine is doing without any initial information. For example, you can ask, What happens if I put in the number 5? Have students make guesses, though they will probably recognize that they can't possibly know for sure. Offer clues, such as "too high" or "too low," until someone gets the answer you have in mind. You may want to have several options in mind at first so that it takes more than one guess.

Then ask for another number to use as the input and have students guess the corresponding output. Continue in this way, as students gradually get additional information about your mystery machine. At some point, ask, **How might you keep a record of the information?**

When most students seem to have figured out the rule, demonstrate how to organize the information using an In-Out table. For example, you might have a table like this for an "add 4 machine."

In	Out
5	9
3	7
0	4
9	?

Can you figure out a rule for this table? Ask students to tell you a rule in words. Work toward getting them to express rules completely, such as "The *Out* is 4 more than the *In*" or "You get the *Out* by adding 4 to the *In*" rather than just "Add 4." Write the rule next to the table.

Offer another example of an In-Out machine, and develop another table of student guesses. Name the input and the output parts of the table, using the metaphor of the machine, and name the table an In-Out table.

Next, give students an In-Out table, and ask them to guess the rule for the In-Out machine that the table is associated with. Again, write the rule as, "The *Out* is . . ."

Discussing and Debriefing the Activity

Ask students, in their groups, to share the rules they found for the In-Out tables in Questions 1 to 5. Also ask each group to prepare a transparency of one of the tables. Explain to the class that as they prepare to present, you would like them to focus more on their thinking process for identifying the rule than on the missing numbers. Also, presenters should write each rule as a complete sentence, beginning with "The *Out* is . . ."

During the presentations, encourage the audience to ask how the presenting group found each particular missing item or rule.

Questions 1 and 2: These two numeric tables represent what students will later classify as linear functions. Some students may have trouble finding the pattern in Question 2. Ask students who have found it to explain, if they can, the process they used to discover a rule to fit the information.

Question 3: The rule generally used for this table is that the *Out* is 1 less than the number of letters in the *In*. Based on this rule, you may want to bring out that there are many possible choices for the missing inputs, but only one choice for each missing output.

Question 4: This is probably the most challenging In-Out table in the activity, as it does not fit any standard idea of what constitutes mathematics, and because there is no simple algorithm for finding a relationship between the inputs and the outputs. As there is no explicit numeric information in the inputs, the first stage in thinking about this problem is to identify something in the pictures that can be associated with numbers. Students may have a variety of ideas about how to do this and will then need to find a rule that connects the numeric information in the pictures to the numbers as outputs.

If everyone is stuck on this problem, ask what changes from picture to picture and use this information to build a new table. For example, students might focus on the number of eyes, in which case they can see the table in Question 4 as equivalent to this table.

In	Out
1	3
3	11
4	15
?	7

Question 5: One rule that works is that the *Out* is the second vowel of the *In*. But students may have other ideas. Some may have decided, based on the first three rows, that the *Out* is the first vowel of the *In*. If so, use this opportunity to remind students to check their rules against all the given information.

Another possibility that may arise is that the *Out* alternates between the fourth and third letter of the *In*. That is, I is the fourth letter of *division*, E is the third letter of *ever*, O is the fourth letter of *opportunity*, and so on. If this or a similar suggestion arises, bring out that in this pattern, the *Out* depends on the sequence in which the input values occur rather than only on the value of the *In*. If the order of pairs shown in the table is changed, this pattern will no longer exist.

Make sure to discuss the "can't be done" entry for Question 5. Ask students what sense they made of it. They might respond with statements such as these.

To get "can't be done" as the *Out*, you need to put in a word with only one vowel.

You can only use words with at least two vowels for this table, so words with only one vowel can't be done.

Before introducing the term **function**, it is important to identify the distinction between functions and arbitrary tables of data. To illustrate, ask students what they think about a table like this one.

In	Out
3	5
2	8
3	7

Bring out that there is something unusual here, as there are two different outputs for the same input. In terms of the metaphor of an In-Out machine, you might identify this as a "broken machine."

With this as background, introduce *function* as the formal mathematical term that roughly corresponds to the idea of a "working" In-Out table. You might also use the phrase *function machine* as another term for an In-Out machine. The key idea is that a function must be consistent. That is, it must give the same output every time a particular input is used.

A related idea is that the output should not depend on where a given input is listed in a table. So a rule such as "The Out alternates between the fourth and third letter of the In" (see the earlier discussion of Question 5) does not describe a function.

Also include a case in which different inputs have the same output, such as Question 5. In other words, bring out that functions can't produce different outputs for the same input, but they are allowed to produce the same outputs from different inputs.

Explain that the concept of a function is one of the major unifying ideas of mathematics and that students will be working with functions throughout their mathematics program. You might mention that rule, table, and function are often used almost interchangeably in informal mathematical work, even though the terms technically have different meanings.

Introduce the term **domain** for the set of things that are allowable as inputs for a given In-Out table. Ask, **What things are allowable as inputs for each table in last night's homework?** You can bring out that in Questions 1 and 2, the *In* must be a number, while in Questions 3 and 5, it must be a word (or perhaps any sequence of letters). In Question 4, the *In* should probably be a picture similar to those shown.

In their work with In-Out tables, students have used such rules as "The *Out* is twice the *In*" or "You get the *Out* by adding 5 to the *In*." In the context of a specific example, you can bring out that a table will generally show only an incomplete picture of a function. It may have enough information to strongly suggest how the rule works, though students will already have seen that this is open to interpretation. But even if we settle on one specific rule, we can't tell from the table exactly what the domain is. Again, we can make an assumption about this, but usually it is only a guess.

Most often, the domain is an infinite set, and thus the function consists of an infinite number of In-Out pairs. Students should recognize that the table can only display a few of these pairs. You might indicate that because of this, we say that the table represents the function, but that technically the function is more than what is shown in the table.

Introduce the term **range** for the set of things that can be outputs for a given In-Out table. This set depends on what the domain is. For instance, for the doubling rule, if the domain is restricted to the whole numbers, then the range consists of the even whole numbers; but if the domain also includes positive fractions, then the range includes all whole numbers as well as all positive fractions.

Question 6: Have students exchange their In-Out tables and look for rules for the tables their fellow group members created. Each group can copy onto a sheet of poster paper two or three favorites from among the tables they created. They should make their tables big enough so that the entire class can read them when the poster is on the wall.

When groups have displayed their posters, they should attempt to find rules for the tables posted by other groups.

Ask the class whether there are specific examples they want to discuss or with which they had difficulty. You can have the group that created the problem or students from other groups offer hints on how to find a rule.

Key Questions

Did you see a method for finding the missing input?

What makes this problem difficult?

What do you think about this table?

Supplemental Activity

A Fractional Life (reinforcement) is part of *The Greek Anthology*, a group of problems collected by ancient Greek mathematicians.

Calculator Exploration

Intent

Graphing calculators will be part of students' tools for doing mathematics throughout the IMP program. This activity will give some students their first chance to learn how these calculators work. It also offers students an opportunity to make a short presentation of something they have discovered.

Mathematics

Given the versatility and power of the graphing calculator, and the wide variety of prior experiences students are likely to bring to this open-ended activity, students' explorations will probably range widely. However, there are some important mathematical issues students will encounter:

- How to handle order of operations on a calculator

  Calculators will evaluate such expressions as 17 − 6 ÷ 3 + 4 x 9^2, without the need for parentheses, by doing the exponent first, then multiplication and division, and finally addition and subtraction.

- How to use the built-in mathematical functions

  To find the square root of 3 you must access the square root function before the number 3, but to find 5! you must access the factorial function after the number 5.

  Sin(30) will not be 0.5 unless the mode is set to degrees.

- The graphing capabilities of the calculator

Progression

Students will explore their calculators in pairs and then share discoveries with the class.

Approximate Time

20 minutes for activity

20 minutes for discussion

Classroom Organization

Pairs, followed by whole-class discussion

Materials

Calculator guidebooks or manuals

TI Calculator Basics (optional), *IMP Year 1: Calculator Notes for the TI-83/84 Family of Calculators*

Overhead calculator

Doing the Activity

Tell students that they will be using a graphing calculator or handheld very often in their math class—so often that it will become a tool with which they think and explore, rather than simply calculate.

Have students read the activity on their own, and then highlight that they are to work with a partner to learn whatever they can about the calculator and, later, to demonstrate something they learned.

Give students time to work on this open-ended activity on their own, free from intervention. Through this experience, they may come to understand that they can learn about calculators, by trial and error, which will help them feel confident with these tools in the future.

Students should focus on simply learning how their calculator works. If you notice a pair fretting about not learning something in particular, encourage them with a reminder that the goal is to explore the calculator until they discover something new. You might also suggest that they explore a button that looks interesting to them. Or ask them to think of something they frequently do in math class and see if they can figure out how to do it on this calculator.

Encourage students to be thoughtful in their preparations to present. Assure them that their presentations can be simple, such as, "If you press this key, such-and-such happens." They don't necessarily have to learn how to accomplish something useful to make the information worth reporting.

Students may want to use manuals to learn how to do a specific activity or to find out what kinds of things the calculator can do. As you circulate, suggest to individual students that they prepare presentations on particular topics.

Discussing and Debriefing the Activity

Give pairs a short time to organize what they learned and to prepare their presentations. Pairs should probably be ready with several ideas to present, so they will have something available if another group presents one of their ideas.

Have pairs make their presentations in an appropriate sequence. For instance, schedule presentations that focus on more elementary aspects of calculator use before those on more advanced or obscure topics.

Students should present using an overhead calculator or appropriate software.

Pulling Out Rules

Intent

This activity gives students more opportunities to find and express rules for In-Out tables, both in words and symbolically, and to use an In-Out table as a problem-solving tool.

Mathematics

This activity raises several mathematical issues.

Finding rules for In-Out tables.

Developing symbol sense. For example, "The *Out* is 2 times the *In*, then add 3," Out = 2 In + 3, Out = 2 x + 3, and $y = 2x + 3$ are equivalent and increasingly abstract ways to use symbols to summarize the rule for Question 1a.

Confronting the idea that the number of data points in a table affects the number of rules that will explain all the data.

Using a function as a mathematical model of a quantitative situation, and then using the model to solve a problem related to that situation.

Introducing the terms **variable**, algebraic expression, **coefficient**, and constant **term.**

Progression

Students are first asked to find rules for In-Out tables that each contain four pairs of numbers. Then they are asked to generate many possible rules that fit tables with only one or two pairs of numbers. Finally, they are presented with a problem for which an In-Out table is a helpful solution tool.

Approximate Time

25 minutes for activity

20 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Tell students that in this activity they will look for rules for more In-Out tables.

Discussing and Debriefing the Activity

Students should begin by asking questions of and comparing results with their group members. During this time, ask some groups to prepare presentations for one of the tables.

During presentations, encourage students to talk about the strategies they used to find their rules. Ensure that rules are presented as complete sentences, and record these sentences on the board.

Now that students have worked with a variety of In-Out tables, it may be valuable to let them share techniques that they have found for finding rules. As they do, help them to clarify their own thinking and encourage them to challenge each other to explain their ideas clearly.

Make sure a variety of rules for each table are shared. In Question 2, with only one data point to explain, students can find an infinite number of possible rules. If an In-Out table contains two points, as in Question 3, there are still infinitely many possible rules, but there is now only one linear rule that will work.

Question 4a asks students to create an In-Out table to analyze a real-life situation.

Number of volunteers	**Number of bags of weeds pulled**
1	2
2	5
3	8

Some students will answer Question 4b by extending the table using the pattern "Add 3 bags for each additional person" until they reach an output that is large enough. Some might find and employ a rule that fits the table *Number of bags* = 3 • (*Number of volunteers*) + 1.

Using either approach, students will discover that 30 does not appear as an *Out* if they stick to whole numbers for the *In*. **So what should the supervisor do?** To address this question, students must make sense of their work so far in light of the problem context. Their ideas could include these.

- The supervisor should "play it safe" by getting 11 volunteers, taking into account that the job might actually involve more than 30 bags of weeds and that some volunteers might work faster than others.
- The supervisor should get 10 volunteers and either make them work extra hard or join in the work as needed to finish the job.

There is no right answer; each of these suggestions (and other possible ideas) makes sense in this context.

Begin moving students from rules expressed in words to rules expressed symbolically. The discussion may be richer if this is a two-operation rule, such as "Get the Out by tripling the In and then adding 1."

Have students find the Out that goes with each of several In values for their rules. For the rule just stated, the table might look something like this.

In	Out
1	4
3	10
6	19
5	16

Help students record the rules as *algebraic expressions*. Students already know symbols for the numbers and operations involved, and most will have used symbols to replace unknown quantities. Remind them how much they know already about writing algebraic expressions.

With a few volunteered responses, students should arrive at an algebraic expression that all agree finishes the sentence. Record the expression in the In-Out table.

In	Out
1	4
3	10
6	19
5	16
t	$3 \cdot t + 1$

Below are a few additional ideas about algebraic notation and terminology that you can either elicit from students or simply present, using the context of the table just discussed.

- It is conventional to abbreviate $3 \cdot t + 1$ as $3t + 1$.
- It is acceptable to write $t \cdot 3$, unusual to write $t3$, and most common to write $3t$. Emphasize that omitting the multiplication sign is simply a convention of notation—that is, an agreement among mathematicians to write things a certain way. There is nothing inherently wrong about using a multiplication sign between a number and a variable or about placing the variable in front of the coefficient—it's just not usually done that way.
- The letter t is a **variable**. Rather than formally defining the term, you might just say that a variable is a letter that is being used to represent a general case.
- $3t + 1$ is an *algebraic expression*. A number used to multiply a variable, such as 3 in the expression $3t + 1$, is a **coefficient**.
- A number by itself that is added or subtracted in such an expression, such as 1 in the expression $3t + 1$, is a *constant term*.

After introducing this terminology and notation, you might ask students to express some of the other rules they found for the tables in Questions 2 and 3 in algebraic form. Using a variety of letters for the inputs emphasizes that the particular letter chosen has no significance.

Key Questions

What rules have we found for this table?

Are these rules really different? That is, would they lead to different tables? Or are they different ways of stating the same rule?

What should the supervisor do?

Lonesome Llama

Intent

This activity is designed to promote student cooperation and communication about mathematics by making the process of working together on a mathematical task an explicit learning focus.

Mathematics

Students will, within certain constraints, be trying to identify the unique card in a stack of 46 cards. The characteristics that distinguish the cards are mathematical—such as the number, type, and orientation of geometric figures—so students will be communicating about mathematics. The mathematical goals of this activity are for students to develop ways to describe the distinctive features within a set of diagrams that are largely alike and to develop a procedure for sorting the diagrams by those features.

Progression

After students, working in small groups, have found the singleton card or made sufficient progress, a whole-class discussion can focus on how they worked and what they discovered.

Approximate Time

30 minutes

Classroom Organization

Groups, followed by whole-class discussion

Materials

Lonesome Llama blackline master (46 cards for each group)

Doing the Activity

Understanding a bit about group dynamics can make a group a better team and enable students to get more out of the experience. The main purpose of *Lonesome Llama* is to get students to look at group processes and roles while they are engaged in problem solving. Everyone must participate in order to complete the task successfully.

Before passing out the sets of cards, have students read through the entire activity, and take some time to review the rules. Emphasize that students don't get to look at each other's cards until the activity is completed (that would make the activity way too easy!) and that what students say about how they work with each other is as important as what they learn about the cards.

Then hand out one set of cards, face down, to each group, and ask a student to deal out the cards approximately equally among the group members. (Because there are 46 cards per set, students in a given group won't all get the same number of cards.) Each student can then look at his or her own cards. Although students will have read the rules, you will likely need to review the rules one at a time, with students looking at their own cards, to ensure that everyone understands them.

Circulate as students work, ensuring that they follow the rules and observing how they are working together.

You might instruct the members of groups that finish early to respond in writing, privately, to this prompt: **What makes a group work well?** When they finish writing, you might have them begin the activity *Role Reflections*.

Discussing and Debriefing the Activity

Ask students for comments about the activity:

How did you know when you were done? How confident were you in knowing you had solved the problem? Why were you so confident?

What mathematics was involved in this activity? What else was mathematical about the ways your group worked?

This last question can be an opportunity to mention that mathematics involves not only knowing terms and facts, being able to use them efficiently and accurately, and solving problems; but also being able to reason, communicate, and share ideas with others so that you can do things as a team.

Ask for volunteers to share their ideas about the prompt, **What makes a group work well?** Students may choose to read what they wrote or may prefer to talk about their thoughts.

You may want to work together to create a poster entitled "Characteristics of a Well-Functioning Group." Such resources, which can be developed throughout the unit and the entire year, are useful for asking students to reflect on how their groups are working or to consider what role they can take to ensure that their group functions well. You might begin by asking students to privately list things they would see happening in a well-functioning group. Then have students volunteer ideas, while you record them on the board or chart paper, until their lists are depleted.

Key Questions

How did you know when you were done?

How confident were you in knowing you had solved the problem? Why were you so confident?

What mathematics was involved in this activity?

What else was mathematical about the ways your group worked?

Supplemental Activity

Counting Llama Houses (extension) asks students to identify the ways in which the houses in *Lonesome Llama* differed and then to determine how many different houses could have been created using these variations.

Role Reflections

Intent

This activity draws upon the *Lonesome Llama* experience in such a way that students recognize the various roles that members must assume in order for a group to function well.

Mathematics

IMP activities are designed to encourage a high degree of interaction, involving collaborative problem solving as well as engaging discussions of ideas, arguments, and presentations. For such a learner-centered environment to be effective, care must be taken to help students work in groups effectively. *Role Reflections* helps students become aware of these norms, or roles, for working in groups and why they are important. See "Assigning and Using Roles in Cooperative Groups" in the Overview to the Interactive Mathematics Program for more information.

Progression

Upon completion of *Lonesome Llama* (or as a filler while groups finish *Lonesome Llama*), students should privately reflect on the activity to identify actions taken by a student engaged in the various roles. A class discussion of responses helps to highlight the value of the variety of contributions that class members made to the successful completion of the activity.

Approximate Time

15 minutes

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

When the students in a group have determined that they are confident they know the *Lonesome Llama*, provide the following instructions. Ask each group member to think back to the activity and remember the various actions taken by their group members that helped them complete the task. Instruct students to organize their recollections using the roles listed in *Role Reflections*. This work should be done individually and privately.

Once *every* member of a group has completed identifying *one* person for each role, have the group members share what they recorded and what moment during the *Lonesome Llama* activity prompted them to identify each person.

Discussing and Debriefing the Activity

Once every group has spent at least some individual time identifying the moments that a member took on a particular role, bring the class together for a discussion.

Ask for volunteers to share actions taken by fellow group members that moved them forward in their success to complete the *Lonesome Llama* activity. Ensure that both *task roles* and *socio-emotional roles* are shared.

Key Questions

Which role might someone in your group have performed that would have helped the group be successful with the activity?

Were their any roles that your group spent too much time attending to?

Communicating About Mathematics

Intent

The activities in *Communicating About Mathematics* build on the strands begun in *The Importance of Patterns,* while focusing on the written and oral communication of students' mathematical ideas.

Mathematics

One of the underlying principles of IMP is that doing mathematics involves offering ideas for analysis and critique as well as analyzing and critiquing the ideas offered by others. The process of convincing others of the validity of solutions and strategies, and of understanding and analyzing the solutions and strategies of others, is what mathematicians do when they present and publish proofs of theorems, solutions to problems, and new techniques for solving problems.

Progression

In *Communicating About Mathematics,* students will present their solutions to the first POW and begin work on the second one. They will explore a model for integer arithmetic and then write a primer on this model for others. This work will extend their understanding of arithmetic with integers and order of operations, building blocks of algebraic thinking. Students will also solve several problems that require careful reading of a mathematical situation and the creation and communication of solutions to an open-ended task. They will use a powerful problem-solving strategy called *working backward.*

Marcella's Bagels

1-2-3-4 Puzzle

Uncertain Answers

Extended Bagels

The Chefs' Hot and Cold Cubes

Do It the Chefs' Way

POW 2: Checkerboard Squares

You're the Chef

Marcella's Bagels

Intent

This activity engages students in a problem that requires close reading. Students also examine a POW-style write-up for this problem, which will support their write-ups for POW 1: *The Broken Eggs.*

Mathematics

Marcella's Bagels gives students an opportunity to use a variety of problem-solving strategies. They might guess at the original number of bagels, examine what happens when they work through the steps in the problem, and then revise their initial guesses accordingly.

The problem also lends itself to the powerful strategy of working backward. Thinking of the story as a movie, students can begin with the number of bagels Marcella has at the end and "run the movie backward," undoing each action she took and arriving at the number of bagels she had at the start. At each step, Marcella gives away half of her bagels plus 2, so in reverse she would add 2 and then double the total.

Progression

Students work backward or use other methods to analyze a complex word problem. They also examine a POW-style write-up for this problem.

Approximate Time

30 minutes

Classroom Organization

Groups

Materials

About 100 beans, counters, or similar items per group

Doing the Activity

You might introduce *Marcella's Bagels* by asking for volunteers to silently enact the events, with or without props, as you narrate the story. Afterward, ask students to restate the problem.

As you direct students to work in their groups on the activity, encourage them to use the materials available to help them think through the problem. If students express that they are beyond using objects like beans or counters, assure them that doing mathematics involves using whatever it takes—pencil and paper, calculators and computers, models and manipulatives—to understand a situation or an idea.

Discussing and Debriefing the Activity

Discuss the various methods students used to solve the problem.

How did you find the answer? How do you know your answer is correct?

The two most likely approaches will be (1) guessing the starting amount and then running the problem forward to see if the guess leads to the correct ending amount and (2) working backward.

Next, students will focus on how to communicate how they solved a problem like this and how to show and explain their solutions. Tell students that this is what they are asked to do in their write-ups for *POW 1:The Broken Eggs*.

Have students turn to the *POW-Style Write-up of Marcella's Bagels* at the end of the unit, which uses *Marcella's Bagels* to illustrate the POW write-up components. Ask them to read the example write-up on their own. When they have finished reading, ask, **What did you notice that is helpful in this write-up? What is missing? What isn't needed?**

Draw students' attention to the ways the writer used the components in the POW write-up. Ask, **How does the process used here differ from the solution?** In particular, bring out the observation that the write-up describes how the writer *thought about* the problem; it doesn't merely present the answer.

Key Questions

How did you find the answer?

How do you know your answer is correct?

What is helpful in this write-up?

What is missing?

What isn't needed?

How does the process used here differ from the solution?

Supplemental Activity

It's All Gone (reinforcement) is a variation on *Marcella's Bagels*, in which a man goes from store to store getting and spending money and winds up with no money in the end. Students are asked to determine how much money he had when he started.

1-2-3-4 Puzzle

Intent

1-2-3-4 Puzzle and its companion activity, *Uncertain Answers*, help students gain insight into the need for rules for order of operations and provide additional experience with the algebraic logic of graphing calculators.

Mathematics

Order of operations is a set of conventions that facilitate mathematical communication. By convention, arithmetic problems are worked out according to the following precedence rules:

- Simplify expressions within parentheses before combining them with expressions outside the parentheses.
- Within parentheses (or where no parentheses exist), do operations in this order: (1) Apply exponents to their bases. (2) Multiply and divide as the operations appear from left to right. (Neither operation has precedence over the other.) (3) Add and subtract as the operations appear from left to right. (Neither operation has precedence over the other.)

These precedence rules have been established to remove the ambiguity from the meaning of such written expressions as $3 \cdot 7 + 2^2$, which might otherwise be evaluated by multiplying 3 by 7, adding 2, and then squaring the result to obtain 529. Using the precedence rules above, the value of this expression is 25, because $2^2 = 4$, $3 \cdot 7 = 21$, and $21 + 4 = 25$.

Progression

This open-ended exploration highlights the importance of order-of-operations rules for communicating mathematically and gives students an opportunity to explore order of operations on the graphing calculator. The activity is also an ideal time to establish the conventional order of operations.

Approximate Time

30 minutes

Classroom Organization

Groups

Doing the Activity

As you begin planning for this activity, keep in mind these three components, which work well together: this activity, the companion activity *Uncertain Answers* (in which students examine order of operations on the graphing calculator), and a brief lecture on order of operations. One sequence is to have students begin the exploration of *1-2-3-4 Puzzle*, conduct the lecture, and assign *Uncertain Answers* for homework.

Many students enjoy this challenging puzzle. There are lots of ways to use 1, 2, 3, and 4 to generate each answer from 1 to 25. For example, $1 + 2 + 3 + 4 = 10$ and $3 + 2 \cdot 4 - 1 = 10$. It isn't necessary that all students find an expression for every number from 1 to 25; rather, just ask them to find as many 1-2-3-4 expressions as they can for 1 to 25. You might hang a poster in the room, with the numbers from 1 to 25 on it, and invite students to add new 1-2-3-4 expressions to it at any time.

Some key points to consider when orchestrating this activity:

- You might need to clarify the meaning of **factorial** and its position within the rules for order of operations. It has priority over the other operations, including exponentiation. For instance, $2 \cdot 3!$ means $2 \cdot (3!)$ not $(2 \cdot 3)!$ Using the ^ notation for exponents, 2^3! is interpreted as 2^(3!) not (2^3)!.
- If any of the following graphing calculator basics were not discussed during *Calculator Exploration,* this activity may present opportunities to raise them. (It is not a requirement that all students know all these techniques at the conclusion of this activity.)

  Editing an expression that has been entered incorrectly (rather than starting over), including use of the INSERT key

  Entering an exponent using the ^ key

  Using parentheses and recognizing that braces (the { and } keys) and brackets (the [and] keys) do not work like parentheses

  Copying the last entry

  Using the answer to the last calculation as part of a new calculation
- Some students may have learned the acronym PEMDAS as a way to remember the order of operations. Unfortunately, this memory device reinforces the common misconception that multiplication is performed before division, and addition before subtraction. Reiterate that within each pair of operations, the operations are performed from left to right. For instance, in the expression $12 \div 6 \cdot 2$, the division is performed first.
- There is nothing wrong with inserting parentheses that aren't strictly required. People often do this in order to avoid any chance of confusion. For example, one might write the expression $5 \cdot 3 + 5 \cdot 7$ as $(5 \cdot 3) + (5 \cdot 7)$. Not only is the latter expression harder to misinterpret, it's also easier to see the intent at a glance.

This is an easy activity to engage students in. Begin by simply asking someone to volunteer a numeric expression using each of the numbers 1, 2, 3, and 4 and any operations they would like. Record their suggestion, and ask the class to calculate the result. Here, or whenever the possibility for multiple interpretations of an expression arises, is a good time to begin discussion of order of operations and the use of parentheses.

Ask for two or three more expressions, again instructing the class to calculate the results, and then wonder aloud, **Do you think we could create an expression for every number from 1 to 25?**

In their groups, students can productively explore for 15 or more minutes easily. At some stage—possibly after a break to introduce order of operations more formally and introduce the activity *Uncertain Answers*—gather the class to review the activity instructions. Reading the instructions will give students more ideas about the operations they can use. Many students won't have thought to use a square root, and few will be familiar with factorials.

Discussing and Debriefing the Activity

Students will be interested in the discussion of this activity in order to see expressions for numbers they haven't yet figured out. Have volunteers share expressions for answers that other students haven't found.

During the discussion, opportunities will arise to clarify order-of-operations rules. As they present themselves, ask the volunteer or the class to help rewrite the solution in the conventional form.

You might ask questions like the following to encourage volunteers to also talk about *how* they found their 1-2-3-4 expressions.

What methods did you use to find your expressions?

Did you proceed in numeric order or did you jump around?

Did you get an expression for one number by adjusting the expression for another?

Did you use any patterns that you saw in the expressions?

Order of Operations

Though many students have been exposed to **order-of-operations** rules, we treat the topic here as if some have not.

To introduce the topic, you might write arithmetic expressions involving several operations, such as these, on the board, and ask students to work on their own to find the value of each expression.

$4 + 5 \cdot 3 + 1$	$3 + 4^2$
$10 + 2 - 4 + 3$	$2 + 4 \cdot 3^2$
$2 + 3(5 + 4)$	$12 \div 4 - 3$

Some students may remember and apply the order-of-operations rules to get the correct answers, while others may never have learned the rules or may have forgotten them. Ask students to share their results for one or two of the expressions, and go through the details of their computations to demonstrate how the expressions can be interpreted in more than one way. Then point out that it would create great difficulties if more than one answer were correct. Mathematicians, scientists, and everyone who deals with numbers must communicate in writing, so there is a need for a set of rules that will govern how to interpret any apparent ambiguity in a problem.

Tell students that, by convention, arithmetic problems are worked out according to these rules:

- Simplify expressions within parentheses before combining them with expressions outside the parentheses.
- Within parentheses (or where no parentheses exist), do operations in this order:

  Apply exponents to their bases.

  Multiply and divide as the operations appear from left to right. (Neither operation has precedence over the other.)

  Add and subtract as the operations appear from left to right. (Neither operation has precedence over the other.)

Post the rules so that you and students can refer to them when needed. You might use a shortened version, such as the one that appears in *Uncertain Answers.*

Key Questions

Do you think we could create an expression for every number from 1 to 25?

What methods did you use to find these expressions?

Did you proceed in numeric order or did you jump around?

Did you get an expression for one number by adjusting the expression for another?

Did you use any patterns that you saw in the expressions?

Uncertain Answers

Intent

This activity gives students opportunities to gain insight into the need for rules for order of operations and helps to establish the conventional rules. This activity also gives them additional experience with the algebraic logic of graphing calculators.

Mathematics

This activity reinforces order of operations as students fix equations by inserting parentheses so that the resulting statements are correct.

Progression

This activity complements students' work in *1-2-3-4 Puzzle*.

Approximate Time

20 minutes for activity (at home or in class)

15 minutes for small-group discussion

Classroom Organization

Groups

Doing the Activity

This activity will require little or no introduction.

Discussing and Debriefing the Activity

You can give groups a few minutes to share their work on the assignment. Students should be able to resolve each other's difficulties within this group discussion.

If you see common errors as you circulate among groups, you may want to draw the class together for clarification, perhaps calling on individual students to explain a given idea. Clear up any conflicts by having students go through the problem one small step at a time.

Supplemental Activity

1-2-3-4 Varieties (reinforcement) adds a rule to those that students used in *1-2-3-4 Puzzle:* now the digits must appear in numeric order. In addition to finding expressions for the first 25 whole numbers, students are asked to find the greatest possible answer given these rules and to make up their own variations to the original activity.

Extended Bagels

Intent

In this activity, students explore what it means to extend a problem, which can help one gain a better understanding of the structure of a problem.

Mathematics

This activity extends *Marcella's Bagels* by posing the question of how altering the final number of bagels would change the initial number of bagels. This is a "functions" question: the starting number is a function of the ending number. Students are asked to find that functional relationship by trying several ending values, organizing their findings in an In-Out table, and then searching for a rule.

Progression

Students use backward reasoning or other methods to further investigate a complex word problem.

Approximate Time

15 minutes

Classroom Organization

Groups, followed by whole-class discussion

Materials

About 100 beans, counters, or similar items per group

Doing the Activity

Introduce the question that frames this extension to the original *Marcella's Bagels* problem: **How does the solution to *Marcella's Bagels* depend on the number of bagels Marcella has when she gets home?**

Monitor group interaction. Encourage students to share ideas and to make sure everyone has an opportunity to contribute his or her ideas.

Discussing and Debriefing the Activity

Have a group or student volunteers share the ideas they pursued and what they learned about the problem. An In-Out table with all their data might look like this.

Number of bagels when Marcella gets home	Number of bagels Marcella started with
0	28
1	36
2	44
3	52
4	60
5	68

It is not crucial that students develop a rule to describe this relationship. If they haven't found a rule, you might post the In-Out table and invite students to continue to think about a rule and bring their ideas to you when they have time.

Conclude this activity with some discussion of the idea that each Problem of the Week (POW) requires students to write out, and sometimes explore, an extension to the original problem. Mention that mathematics is at least as much about creating interesting questions as it is about answering them.

Key Question

How does the solution to *Marcella's Bagels* depend on the number of bagels Marcella has when she gets home?

The Chefs' Hot and Cold Cubes

Intent

The "hot and cold cubes" sequence of activities offer a model for the operations of integer arithmetic, key tools in high school mathematics. This activity reviews some basics about negative numbers and reaffirms some conventions before students begin to make sense of the "hot and cold cubes" model for integer arithmetic.

Mathematics

The IMP program assumes that most students have had prior exposure to negative numbers and have been taught—but may not remember or understand—basic rules for arithmetic with integers. In this set of activities, students are introduced to a model that embodies these rules and serves as a metaphor for thinking about integer arithmetic. Rather than simply reviewing the rules for such arithmetic, this model provides a frame of reference for the rules and will allow students, if necessary, to reconstruct the rules for themselves in the future.

The basic operations for **natural numbers** are defined, at least intuitively, in terms of putting sets of objects together and taking objects away from a set, but this definition doesn't make sense for negative numbers. In moving from **whole numbers** to **integers**, numbers are no longer simply a magnitude, but also a direction. Treating an integer as merely an opposite of a whole number, as do such commonly memorized rules as "subtracting a negative is the same as adding a positive," does not encourage a more powerful understanding of integer. The hot and cold cubes model emphasizes both the magnitude and the direction of an integer and encourages awareness of the meaning of the operation involved.

Progression

The activity begins with discussion of the need to justify solutions when doing integer arithmetic and is followed by a brief review of notation, language, and conventions. Students are then introduced to the model and, in their groups, perform some integer arithmetic to help make sense of the model. They are asked to translate the chefs' moves (using hot and cold cubes to change the temperature in the cauldron) into integer arithmetic and to translate integer arithmetic into chefs' moves.

Approximate Time

35 minutes

Classroom Organization

Groups

Materials

Manipulatives, such as cubes or tiles, in two colors

Doing the Activity

Many students have probably been exposed to the basics of computing with negative numbers and may be reluctant to learn another way of thinking about the process. The hot and cold cubes model, however, will help them to understand *why* the rules work. It offers a frame of reference for the rules and will allow students to reconstruct the rules for themselves in the future.

Two quick questions can provide information on students' prior knowledge.

What is the answer to $(^{-}3)(^{-}5)$? How do you know your answer is right?

What is the answer to $^{-}3 + {}^{-}5$? How do you know your answer is right?

You will likely receive a variety of solutions; record them all on the board. Remind students that they should be able to state why their answers are correct. Some students may be able to apply the rules to find the correct answers, but many will have trouble explaining why the product of two negative numbers is positive while the sum of two negative numbers is negative. The intent of this introductory challenge is to convince students that they still have something to learn about working with negative numbers and that it might be worthwhile to have an approach that doesn't rely on memorizing rules.

Review the notation and terminology of positive and negative numbers. These activities use the "raised sign" notation, such as $^{+}5$ and $^{-}7$. These should be read as "positive five" and "negative seven" not "plus five" or "minus seven." Using clearly defined terminology helps to distinguish between positive and negative *numbers* and the *operations* of addition and subtraction.

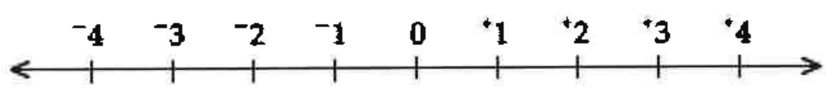
Once these activities are completed, the student book reverts to the standard notation conventions: positive numbers typically include no sign, and negative numbers are denoted with the same symbol as subtraction. Tell students that this convention is typical in most contexts.

Also review the following terminology and notation with students.

- The *sign* of a number indicates whether it is positive or negative. Zero is considered neither positive nor negative.
- The numbers in a pair such as $^{+}3$ and $^{-}3$ are sometimes called *opposites*. That is, $^{-}3$ is the opposite of $^{+}3$, and $^{+}3$ is the opposite of $^{-}3$.
- The word **integer** refers to a number that is zero, a natural number, or the opposite of a natural number. (The **natural numbers** are positive whole numbers: 1, 2, 3, 4 and so on.) Thus the set of integers is

$$\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, \ldots\}$$

- The *number line* is a way to picture both positive and negative numbers. By convention, positive numbers are on the right and negative numbers are on the left; numbers are considered to get larger as one moves to the right on the number line. Thus, for example, $^{+}5 > {}^{-}8$ and $^{-}7 < {}^{-}3$.



Have students read the introduction to *The Chefs' Hot and Cold Cubes* and the first five paragraphs of The Story.

Introduce students to the manipulatives—such as two colors of cubes or tiles—for representing hot and cold cubes. Ask groups to use their manipulatives to create several cauldrons, each representing a temperature of 0°, to introduce the idea that a hot cube and a cold cube "cancel out" one another.

Have groups read the next paragraph (beginning "For each hot cube . . .") and then create cauldrons for other specific temperatures, such as $^{+}5°$ or $^{-}3°$. The idea is for students to get a sense of the cancellation mechanism and to see that a given temperature can be represented in many ways.

After this introduction, let students read the rest of The Story individually and then work in their groups on the questions.

As groups work, you may want to emphasize that the equations and arithmetic expressions focus on the *change* in temperature and not on the temperature itself.

Some students may be confused by the fact that the same notation is used in different ways. For instance, $^{+}5$ can mean "add five bunches of a certain number of hot or cold cubes" (as in $^{+}5 \cdot {}^{+}20 = {}^{+}100$) or "a bunch containing five hot cubes." If this comes up, acknowledge that this part of the model is something they may have to pay extra attention to. It is similar to the dual meaning in multiplication of whole numbers, in which $5 \cdot 3$ can mean "5 groups with 3 objects in each group" or "3 groups with 5 objects in each group," with 5 representing either the number of groups or the size of each group.

Discussing and Debriefing the Activity

This activity will likely not require a formal debriefing.

Key Questions

What is the answer to ($^{-}3$)($^{-}5$)? How do you know your answer is right?

How might you represent the situation with objects?

Supplemental Activity

Positive and Negative Ideas (extension) extends the work with hot and cold cubes and asks students to consider other ways they might model integer arithmetic.

Do It the Chefs' Way

Intent

This activity gives students more experience with the model of hot and cold cubes for integer arithmetic.

Mathematics

Students use the hot and cold cubes model to understand arithmetic with integers. This is also a good time to introduce the concept of **absolute value** and to explore patterns in operations with integers.

Progression

Students have worked in their groups to make sense of the "hot and cold cubes" model. Now they will spend some individual time practicing with and confirming their understanding of the model. After comparing their work with one another and discussing questions that arise, students review a few more basic ideas related to operations with integers.

Approximate Time

20 minutes for activity

30 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Materials

Manipulatives, such as cubes or tiles, in two colors

Doing the Activity

To introduce the activity, emphasize that students are to explain each expression in terms of the "hot and cold cubes" model.

Discussing and Debriefing the Activity

Have a volunteer from each group explain one of the answers. Insist on the use of the "hot and cold cubes" model, even if the student prefers to quote arithmetic rules.

For example, for Question 4, a student should say something like, "The term ${}^{-}4$ (read "negative four") means the chefs put in four cold cubes, which lowered the temperature 4°. The $-{}^{+}6$ (read "subtract positive six") means they took out six hot cubes, which lowered the temperature 6°. Altogether the temperature went down 10°. In terms of the model, you would write ${}^{-}4 - {}^{+}6 = {}^{-}10$."

Some students may comment that taking out hot cubes has the same effect as putting in cold cubes. Such alternative explanations of the expressions, in terms of the model, should be encouraged.

In a problem like $^{+}5 \cdot {}^{-}2$ (Question 3), students may say that the answer is negative "because a positive times a negative makes a negative." Encourage a reinterpretation of such an expression in terms of the model. For example, they might say, "This is as if the chefs were putting in bunches of cold cubes, which lowers the temperature."

If some students continue to resist learning the model, insisting that they can get the answers more easily from rules, mention that part of learning mathematics is being able to explain it and that being able to explain simple situations like this is good practice for explaining more complex problems later. Point out that students do not have to use the model for every computation they do now or in the future, but they should be prepared to justify their work in terms of the model when asked to do so.

Absolute Value

Working with the hot and cold cubes model is an ideal opportunity to introduce the term **absolute value**. Tell students that the absolute value of an integer is the number of cubes it represents. Help them to see that any integer except zero is a combination of a sign and an absolute value.

Also introduce the notation for absolute value through examples, such as $|5| = 5$, $|{}^{-}7| = 7$, and $|{}^{-}0| = 0$.

Ask students, **What's the difference between the operation of subtraction and the negative sign?**

If students develop their own general rules about the relationship between sign and operation, such as "adding a negative gives the same result as subtracting a positive," that's fine. However, tell them that familiarity with the hot and cold cubes model will give them something to fall back on if they happen to forget the rule.

At this point you can announce that you (and their books) will generally omit the raised plus sign prefix for positive numbers and will write the negative sign the same way as a subtraction sign.

Explain that in order to avoid seeming to write two arithmetic operation symbols next to each other, it is common to insert parentheses. For example, instead of $10 + -7$ or $8 - -4$, we might write $10 + (-7)$ or $8 - (-4)$. Also, for $5 \cdot -3$, we might write $5 \cdot (-3)$, but we often omit the multiplication sign when there are parentheses to indicate multiplication: $5(-3)$. Students are probably familiar with the use of parentheses for multiplication in such expressions as $5(2 + 7)$, but may not have seen it used in conjunction with a symbol immediately inside the parentheses that could be interpreted as an arithmetic operation, such as $5(-2 + 7)$.

Patterns in Integer Arithmetic

At this point, you may want to present the following pattern approach to operations with integers. Begin by writing this sequence of addition equations.

$7 + 3 = 10$

$7 + 2 = 9$

$7 + 1 = 8$

$7 + 0 = 7$

$7 + (-1) = ?$

Ask students to look for a pattern and use it to explain what number belongs in place of the question mark. **What should come next in this sequence?** Presumably they will see that the sequence of answers suggests that $7 + (-1)$ should equal 6.

Continue with $7 + (-2) = ?$ and similar problems. Ask students what is happening. They should see that as negative numbers of greater magnitude are added, the resulting sum gets smaller.

You may want to continue through $7 + (-7)$ and on into examples that give a negative sum, such as $7 + (-8)$.

After the pattern has been described, ask the class whether this pattern gives the same answers as the hot and cold cubes model. Students should be able to explain how to get the same results from the model.

Present the next series of equations, which relate to subtraction, and ask, **What should come next in this sequence?**

$7 - 5 = 2$

$7 - 6 = 1$

$7 - 7 = 0$

$7 - 8 = ?$

From the continuation of this pattern, students should see that if a greater number is subtracted from a lesser number, the result is a negative number. Some may also notice that the result is the opposite of the result when the two numbers are reversed.

Continue the pattern above in the opposite direction, subtracting smaller and smaller numbers from 7. **What should come next in this sequence?**

$7 - 5 = 2$

$7 - 4 = 3$

$7 - 3 = 4$

$7 - 2 = 5$

$7 - 1 = 6$

$7 - 0 = 7$

$7 - (-1) = ?$

Students should notice that as the number being subtracted grows smaller, the result increases. Some may recognize that subtracting a negative number gives the same result as adding the corresponding positive number, so that there is a related addition equation for each subtraction equation. For example, the subtraction equation $7 - (-5) = 12$ relates to the addition equation $7 + 5 = 12$.

Finally, have students look for patterns in the products of integers, as in this sequence.

$6 \cdot 3 = 18$

$6 \cdot 2 = 12$

$6 \cdot 1 = 6$

$6 \cdot 0 = 0$

$6 \cdot (-1) = ?$

Students should observe that as the second factor decreases by 1, the products decrease by 6.

Key Questions

What's the difference between the operation of subtraction and the negative sign?

What should come next in this sequence?

How does this pattern relate to the hot and cold cubes model?

POW 2: Checkerboard Squares

Intent

This is the second POW of the course. The primary purpose of this, as for all POWs, is to give students the opportunity to solve a significant problem outside of class, to generalize their solutions, and to prepare a formal written account of their work.

Mathematics

This POW will draw on and strengthen students' ability to visualize geometrically, to collect and organize a complex set of information, and to generalize their solutions. Students are asked to generalize their methods for counting the number of squares of different sizes on an 8-by-8 checkerboard to produce a method for counting the squares on a board with dimensions n by n.

This activity is connected to an important mathematical theme of this unit: patterns. In counting all the squares of different sizes on a checkerboard, students will have to be systematic to ensure they have accounted for all the squares. To do so, they will have to recognize patterns in the locations of squares of various sizes.

Progression

This POW is posed toward the end of *Communicating About Mathematics*, and students will work on it into *Investigations*. Unlike *The Broken Eggs*, the student book includes no follow-up activities in support of the various components of students' write-ups of this POW. Students will be introduced to summation notation in another activity, *Add It Up*, as they are working outside of class on this POW. This notation can then be brought into *Checkerboard Squares* as appropriate.

Approximate Time

10 minutes for introduction

10 minutes for discussion

1 to 3 hours for activity (at home)

20 minutes for presentations

Classroom Organization

Whole-class introduction, concluding with presentations and class discussion

Doing the Activity

The first part of every POW write-up is the student's statement of the problem. In addition to helping students learn how to write a problem statement, having students work on and share their problem statements soon after the POW is assigned will help clarify for many students what the problem is.

Announce when the write-up is due. Solicit presenters immediately, or nearer the due date, reminding the presenters of basic expectations and providing them with transparencies and pens.

Discussing and Debriefing the Activity

Discussion of this POW can possibly extend over two days.

Before students turn in their write-ups, you might offer them an opportunity to review the work of other students. This is their second POW, so they will have formed some idea of what is expected, but seeing each other's work may be of great value.

As they read other students' work, you might have students focus on what makes a good paper, what makes an adequate paper, and what makes a poor paper. After the sharing of POW write-ups is complete, you might want to ask students to do focused free-writing on this topic: **What makes a good POW write-up?** (see "Focused Free-Writing" in the Overview to the Interactive Mathematics Program). After they have written for about five minutes, let students share their ideas. They can read aloud from their written work or simply discuss what they wrote about.

Have the assigned students give their presentations, limiting each to about five minutes. Encourage presenters to speak about their investigation process at least as much as they speak about their findings. When findings overlap, presenters may wish to emphasize slight nuances they saw, questions they explored, and the like.

In the discussion that grows out of the presentations, focus on the patterns that students have discovered. Bring out that finding patterns helps us to analyze mathematical situations.

Student interest may offer opportunities to extend the exploration. For example, this POW lends itself to trying to explain *why* the square numbers appear. Students, or you, may raise such questions as these: **Why is the number of squares of each size itself a square number? Why is it the particular square that it is?**

The problem can also be another opportunity to use summation notation. You might inquire, **Do you see a way to express your findings using summation notation?**

Key Questions

Why is the number of squares of each size itself a square number?

Why is it the particular square that it is?

Do you see a way to express your findings using summation notation?

Supplemental Activities

Different Kinds of Checkerboards (extension) is a follow-up to *POW 2: Checkerboard Squares* in which students find the number of squares on nonsquare checkerboards and search for a general rule for checkerboards of dimensions m by n.

Lots of Squares (extension) is a substantial investigation in which students are asked to divide a square into different numbers of smaller squares. The goal is to determine which numbers of smaller squares are impossible and which are possible and to prove their results.

You're the Chef

Intent

This activity concludes work with the "hot and cold cubes" model for integer arithmetic, for now. This activity requires students not only to make meaning of the model but to create a thorough explanation of the model, using examples. Their write-ups of this activity will be part of their *Patterns* portfolios.

Mathematics

Throughout the rest of the program, students are expected to use negative numbers where appropriate.

Progression

Students will use their previous experiences to put the hot and cold cubes model into their own words, including selecting several examples to provide a complete explanation of the model.

Approximate Time

5 minutes for introduction

30 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Have students read the activity. Advise them to be thoughtful about including sufficient examples in their manuals so that the new assistant chef will know how to read, interpret, and write recipes.

Tell students that they will include their manuals in their portfolios, which they will create at the end of this unit.

Discussing and Debriefing the Activity

You might have a few students read their papers to the class. Encourage students to identify parts of the manual they struggled to write. Ask for other students to comment, sharing how they figured out how to write that section or even reading the passage aloud.

Rather than collecting the assignment now, you may want to discuss the ideas and then let students turn in revised versions tomorrow.

Supplemental Activity

Chef Divisions asks students to model division with hot and cold cubes.

Investigations

Intent

Investigations contains several rich mathematical explorations that offer students opportunities to employ the tools and techniques they developed in *The Importance of Patterns* and *Communicating About Mathematics.*

Mathematics

The two groups of investigations in *Investigations* draw on important ideas from number theory and plane geometry. In the first set, students will propose, test, and justify conjectures (and find counterexamples) about sums of whole numbers. They will be introduced to some formal notation for expressing conjectures and will continue to explore the notions of existence and uniqueness of solutions. In the second set of activities, students will apply their developing skill at using In-Out tables to find patterns in the relationship between the sides and angle measures of polygons.

Progression

Investigation of Consecutive Sums

- Consecutive Sums
- Add It Up
- Group Reflection
- That's Odd!

Investigation of Polygon Angles

- Pattern Block Investigations
- Degree Discovery
- Polygon Angles
- An Angular Summary

Consecutive Sums

Intent

The core for a series of activities that open *Investigations, Consecutive Sums* poses an open-ended situation in which students are encouraged to make and test conjectures, construct proofs, and find counterexamples. This activity helps to establish a classroom environment of student-student interaction during the exploration of a challenging mathematical investigation.

Mathematics

Students examine complex, open-ended mathematical questions, develop and test ideas, write proofs using logical reasoning and algebraic notation, and disprove **conjectures** using **counterexamples**.

Students are also introduced to summation notation. Consecutive sums are defined as sums of consecutive natural numbers, such as $6 + 7 + 8 + 9 = 30$, $35 + 36 = 71$, *and* $1 + 2 + 4 + \cdots + 10 = 55$. *The third example can be written, using summation notation, as* $\sum_{i=1}^{10} i = 55$.

Progression

This activity requires a brief introduction, followed by a significant amount of time working in small groups to investigate, prepare posters, and present findings to the class. Split the activity over at least two days, assigning the individual activity *Add It Up* in the break. In *Group Reflection*, students reflect on the nature of the activity and the ways in which they worked together during the investigation.

Approximate Time

85 minutes

Classroom Organization

Groups

Doing the Activity

Students' task is to explore patterns in consecutive sums and to create a poster summarizing their work, including descriptions of confirmed, disproved, and still-open conjectures.

Once you ensure that students understand what consecutive sums are and what they are being asked to do, the best way for students to begin is to "try stuff." For the purpose of this activity, a **conjecture** might be defined as a "guess based on some evidence."

To introduce the activity, you might spark student interest by offering a few examples of consecutive sums and posing a challenging question, such as one of those given below.

This activity will be explored over at least two days. The end of the first day's work is a good time to refocus groups on the products they are to create: posters that summarize their results and contain summary statements of the patterns they have observed. Remind students that they will classify patterns as conjectures, certainties (statements that are always true), and false conjectures.

Encourage groups to record clearly worded summary statements about what they think the pattern is. Offer an example of a clear summary statement, such as "Every number can be written as a consecutive sum." Tell students that while this statement may or may not be true, it is the type of statement you are looking for.

The following are some of the questions that groups might investigate.

What numbers can be written as consecutive sums?

What numbers can be written as more than one consecutive sum?

Are there patterns to the answers to consecutive sums that are two terms long (such as 4 + 5), three terms long, or four terms long?

If groups have gathered some information but are not seeing any patterns, suggest that they try to reorganize the information in a way that might make patterns more visible.

As the exploration draws to a close, circulate to help groups focus on their summary statements. Following are some possible summary statements.

- Every odd number greater than 1 can be written as a consecutive sum of two terms. (This particular statement is the subject of the activity *That's Odd!*) Because only positive whole numbers are permitted in the activity, 1 itself cannot be written as a consecutive sum.
- The numbers 1, 2, 4, 8, 16, . . . (powers of 2) cannot be written as consecutive sums.
- The numbers that cannot be written as consecutive sums are all even. (This statement is incorrect, because 1 is odd but cannot be written as a consecutive sum. It can be written as 0 + 1, but the activity allows only positive terms, not 0.)
- Every third number—that is, every multiple of 3—except 3 itself can be written as a consecutive sum of three terms. (The number 3 is 0 + 1 + 2, but again, 0 is not permitted.)

Discussing and Debriefing the Activity

Once groups have displayed their posters, review and discuss this collection of conjectures and summary statements. Ask a member of each group to state one of the patterns that the group found that hasn't yet been mentioned. Continue until no group has summary statements that haven't already been mentioned.

It may work best to have all the statements read before getting into discussion of or challenges to any of them. When ready, invite students to comment on the summary statements of other groups. They may have facts that contradict a given statement, or they may simply question whether a given generalization is valid.

Introduce the word **counterexample** in the context of these summary statements by asking whether there are any cases in which a generalization doesn't hold. (If no one offers one, suggest one yourself.) For example, the summary statement "If a number can be written in three or more ways as a consecutive sum, then it must be odd" is false, and 30 is a counterexample. Although 30 fits the condition that "it can be written in three or more ways as a consecutive sum," it doesn't have the property "it must be odd."

On the basis of this discussion, the class may eliminate or confirm some of the summary statements, while others will remain conjectures. For example, the statement "Powers of 2 cannot be written as consecutive sums of positive whole numbers," though a true statement, will probably remain unproven at this time.

Key Questions

What numbers can be written as consecutive sums?

What numbers can be written as more than one consecutive sum?

If a number can be written as a consecutive sum, is that consecutive sum unique?

What numbers are not answers to some consecutive sum? Are there patterns in these numbers?

Are there patterns to the answers to consecutive sums that are two terms long (such as 4 + 5), three terms long, or four terms long?

Supplemental Activities

Three in a Row (extension) offers students an opportunity to explore sums of three consecutive numbers as well as sums of other lengths.

Any Old Sum (extension) asks students to examine sums that are not consecutive.

Add It Up

Intent

This activity introduces the mathematical symbol for summation notation. Students begin to understand the utility of this notation by working with both numeric and geometric examples.

Mathematics

One of the challenges of secondary mathematics teaching is helping students to understand the notational systems used to express complex ideas in a compact form. This activity introduces one such system, summation notation, and offers students opportunities to start to make sense of it.

Progression

This activity serves as a useful way to break up student work on *Consecutive Sums.* Students may elect to utilize summation notation in their posters for *Consecutive Sums* and in *POW 2: Checkerboard Squares.*

Approximate Time

10 minutes for introduction

20 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Introduce the activity with a multiterm example of a consecutive sum, such as $3 + 4 + 5 + 6 + 7 + 8 + 9$. Demonstrate that there is a shorthand way for writing such sums: $\sum_{i=3}^{9} i$. Explain that this symbol is an uppercase letter in the Greek alphabet, called *sigma*, and that the expression is read, "The summation, from *i* equals 3 to 9, of *i*." Invite students to articulate the connection between the shorthand and the full expression.

Explain that the letter *i* is called a *dummy variable* and that any letter would work. The expression $\sum_{t=3}^{9} t$ means exactly the same thing as $\sum_{i=3}^{9} i$.

Use a more complex example to illustrate in detail how this notation works. For example, ask students what they think this expression means.

$$\sum_{w=3}^{7}\left(w^2+2\right)$$

How can you "act out" the process described by this summation expression? Help students act out the process.

First, w is 3, so the first term is $3^2 + 2$.

Then, w is 4, so the next term is $4^2 + 2$.

Then, w is 5, so the next term is $5^2 + 2$.

Then, w is 6, so the next term is $6^2 + 2$.

Finally, w is 7, so the next term is $7^2 + 2$.

Since the symbol $\sum$ indicates summation, these terms must be added together. In other words, the notation represents the expression.

$$(3^2 + 2) + (4^2 + 2) + (5^2 + 2) + (6^2 + 2) + (7^2 + 2)$$

Point out that although this example does not give a consecutive sum, the values for w are a sequence of consecutive numbers.

The mechanics of summation notation are summarized in the student activity. Students will work with this notation in geometric as well as in purely numeric contexts. Don't get bogged down on mastery of the notation; it is intended only as a tool to help students express their ideas.

You may want to introduce the use of ellipsis notation, such as writing $1 + 2 + \cdots + 100$ for the sum of the whole numbers from 1 to 100

Discussing and Debriefing the Activity

Give students an opportunity to share responses and ask questions of one another.

For Question 2, students will likely see the picture in terms of the sum $1 + 2 + 3 + 4$ and produce an expression like $\sum_{i=1}^{4} i$.

The expressions for Question 3 can be written in various ways. Question 3c is especially likely to lead to different answers, such as $\sum_{t=2}^{6}(3t+2)$ and $\sum_{j=3}^{7}(3j-1)$. You can leave this question open if students cannot find a way to write the expression using summation notation.

For Question 4, the diagram suggests the idea of a sum of squares and can be expressed as $\sum_{n=1}^{5} n^2$.

Students' facility with summation notation will increase as they find situations where it is useful.

Key Question

How can you "act out" the process described by this summation expression?

Group Reflection

Intent

This activity is an opportunity for students to thoughtfully consider how they participate in groups.

Mathematics

In an open-ended exploration such as *Consecutive Sums,* productive group work is quite helpful. In this activity, students reflect on their experiences as members of a collaborative problem-solving group.

Progression

Students reflect on their participation in the open-ended activity *Consecutive Sums* to identify norms for productive interaction, to assess their level of participation, and to consider ways that they can draw others into the group.

Approximate Time

10 minutes for introduction

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Whole class, then individuals, followed by whole-class discussion

Doing the Activity

Once students have completed *Consecutive Sums,* you might introduce this activity by asking them to think about this question: **What makes a group work well?**

Have students do focused free-writing on this topic. (See "Focused Free-Writing" in the Overview to the Interactive Mathematics Program.) Briefly explain the expectations of focused free-writing, if you haven't already. There are several key points to make.

- Their focused free-writing will not be collected (though students may have the opportunity to read some of it aloud or just share their ideas).
- Students should write, write, write—a sort of stream of consciousness.
- Students should try to stick with the assigned focus.
- Explain that it is better to write things like "I can't think of what to say" than to stop writing completely.

After a few minutes, have volunteers share their ideas, either reading what they wrote or just talking about their thoughts. If students are reluctant to share their ideas, you might prompt some conversation by asking questions such as the following.

Describe a time when you thought your group was working especially well together, when you were all achieving more than you could alone.

Describe a time when your group might have been able to be more productive. What was going on?

What strategies have you noticed you or other group members using that brought the group back together and that helped you be more successful?

Tell students that group work often does not come naturally and may not be a great experience if group members aren't thoughtful about how they are working and interacting. Learning to work well in groups is not necessarily an easy thing. However, as long as students get involved in their groups and respect the other members, they should see improvement.

Tell students that in this activity, they will think more specifically about the actions people take in groups.

Discussing and Debriefing the Activity

Let students share what they want about this activity. Such open discussion is important for developing a classroom climate in which students are willing to share their ideas and opinions.

To break the ice, you may want to talk about your own experiences working in groups. This is also an appropriate time to return to (and possibly edit) the posted classroom norms, group roles, or the Characteristics of a Well-Functioning Group poster from *Lonesome Llama*.

You may wish to collect and read these assignments, not for grading, but to gain insight into the dynamics of your classroom and the attitudes of your students.

Key Questions

Describe a time when you thought your group was working especially well together, when you were all achieving more than you could alone.

Describe a time when your group might have been able to be more productive. What was going on?

What strategies have you noticed you or other group members using that brought the group back together and that helped you be more successful?

Supplemental Activity

Getting Involved (reinforcement) asks students to reflect on a situation in which one person in a group is not contributing.

That's Odd!

Intent

The purpose of this activity is not for students to learn a proof that odd numbers greater than 1 can be written as a sum of two consecutive numbers. Rather, it is intended to help students do the following:

- Begin to learn what a proof is
- Learn to distinguish between specific examples and a general argument
- Gain experience in communicating complex, abstract ideas
- Become familiar with a more precise way of thinking than they may have encountered before

Mathematics

This activity challenges students to evaluate whether a conjecture is true or false. The conjecture in question is "If an odd number is greater than 1, then it can be written as the sum of two consecutive numbers." Students are asked to find a **counterexample** if they think the statement is false or to devise a set of instructions for writing any odd number greater than 1 as a sum of two consecutive numbers if they think the statement is true.

Progression

This activity should follow the conclusion of *Consecutive Sums*. Students work alone to consider a conjecture likely to have been made during that activity. Follow-up discussion will introduce the concept of a **proof**.

Approximate Time

15 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Introduce the activity by asking, **Is the conjecture in the activity true? How confident are you about your answer?**

Clarify the instructions for students so that they understand what is expected of them.

Discussing and Debriefing the Activity

Begin the discussion by asking the class again whether they think the conjecture is true. Then ask how confident they are that it is true for *every* odd number greater than 1. Most students will likely be fairly sure that it is always true, but encourage skeptics to voice their opinions.

Ask for volunteers to share any instructions they developed for writing an odd number as a sum of two consecutive numbers, and have them illustrate their methods using specific examples. If the class is at a loss about how to do this, you might ask a series of questions, such as, **How would you write 397 as the sum of two consecutive numbers? How would you write 4913 as the sum of two consecutive numbers? How would you write 157,681 as the sum of two consecutive numbers?**

Encourage students to explain how to find the pair of consecutive integers in each case, as this is key to developing a general argument.

There are several ways to describe the general process; elicit as many as possible from your students. Here are some commonly suggested procedures.

- Subtract 1 from the odd number to get an even number. Divide this even number by 2. That quotient and the next number are the desired consecutive numbers.
- Divide the odd number by 2, getting "something and a half." The whole numbers just above and below this mixed number are the desired consecutive numbers.
- Add 1 to the odd number to get an even number. Divide this even number by 2. That quotient and the previous number are the desired consecutive numbers.

Careful examination of any of these methods will show that they don't work if the initial odd number is 1, because one of the numbers in the consecutive sum will be 0 rather than a positive whole number as required.

Whichever methods students suggest, ask them to explain how they know that a given method works. For example, for the first procedure above, you might ask how students know that subtracting 1 from an odd number gives an even number. The best response to this question would refer to a definition of the term *odd*. That is, students should recognize that, ultimately, they can't say anything for sure about odd numbers unless they begin with a clear definition. Similarly, ask how students know that dividing an even number by 2 gives a whole-number result. Again, encourage them to see that the answer to this challenge depends on having a precise definition of the term *even*. It is not necessary to go into formalities about the meaning of the terms *odd* and *even*. What is important is recognizing the value of having a precise definition if one is to give a complete proof.

Use the discussion to help bring out the difference between a collection of examples of a phenomenon and a legitimate general proof. A proof does not need to use algebraic symbols. For example, when appropriate and precise definitions are given for *odd* and *even*, the arguments above constitute completely legitimate proofs that every odd number can be written as a consecutive sum with two terms. Help students to see that these arguments are better than only giving a few examples such as $23 = 11 + 12$ and $47 = 23 + 24$.

Each procedure listed above demonstrates that every odd number is expressible as a consecutive sum of two terms by showing *how to do it,* that is, how to find the two terms. Such how-to arguments are considered legitimate proofs and are known as *constructive proofs*.

Algebraic symbols do sometimes help students understand a situation, and your students may be able to express their arguments symbolically. For example, if you suggest using n for the number obtained after subtracting 1 and dividing by 2, students can probably write the next number as $n + 1$.

You might extend this problem by encouraging students to express each method using a general equation. For instance, the first method listed above can be expressed by the equation $N = \frac{N-1}{2} + \left(\frac{N-1}{2} + 1\right)$. Students can be asked to explain why, if N is odd and greater than 1, both terms must be positive integers.

Key Questions

Is the conjecture in the activity true? How confident are you about your answer?

How would you write 397 as the sum of two consecutive numbers?

How would you write 4913 as the sum of two consecutive numbers?

How would you write 157,681 as the sum of two consecutive numbers?

Supplemental Activities

The General Theory of Consecutive Sums (extension), the final activity in this group, asks students to explore consecutive sums of integers.

Infinite Proof (extension) asks students to prove that the square of every odd number is odd and that every prime number greater than 10 must have 1, 3, 7, or 9 as its units digit.

Pattern Block Investigations

Intent

This activity introduces students to pattern blocks, determining angle measures, and learning to use a protractor. Some students will already know how to measure angles with a protractor, and some will benefit from a refresher.

Mathematics

Pattern blocks are polygons that share side and angle relationships. By fitting and stacking these blocks, students can observe many geometric relationships inherent in these special manipulatives.

The equilateral triangle, the square, both parallelograms, and the hexagon have the same side lengths. The trapezoid has three sides of that length and one side twice that length.

Two triangles cover the blue parallelogram, three triangles cover the trapezoid, and six triangles cover the hexagon.

Three blue parallelograms, or two trapezoids, cover the hexagon.

The large angles in the blue parallelogram are twice the size of that figure's small angles.

The small angles in the tan parallelogram are half the size of the small angles in the blue parallelogram.

Students will also deduce the sizes of the interior angles of these polygons by examining the relationships among the polygons. Finally, they will use these known angle measures to figure out how to measure angles using a protractor.

The division of a complete turn into 360 equal parts is quite ancient and is often attributed to the Babylonians, whose number system was based on 60 and for whom the number 360 played an important role.

Progression

During the next sequence of activities, students will learn about the concept of angle. *Pattern Block Investigations* introduces students to pattern blocks, a manipulative tool that they will use in the development of this concept. In Part I of this activity, students create pattern block designs and focus their attention on the point formed by the blocks' vertices. In Part II, based on the idea that a full turn is 360 degrees, students deduce the sizes of all angles of all the blocks. Finally, in Part III they trace the blocks, extend their sides, and then use their deductions of angle sizes to learn to measure angles using a protractor.

Approximate Time

40 minutes

Classroom Organization

Groups

Materials

Pattern blocks

Overhead pattern blocks (optional)

Doing the Activity

When students encounter a new manipulative, they often need time to explore its properties and possibilities. Begin by providing groups with a large set of pattern blocks and encouraging a few minutes of exploration. As students explore, review the names for the various blocks: *triangle, hexagon, parallelogram* (or *diamond*), *square*, and *trapezoid*. You can refer to the two different parallelograms by shape (*wide* and *thin*) or by color (*blue* and *tan*). Also introduce the general term **polygon** as well as the term **quadrilateral** for any four-sided polygon.

Part I: Pattern Block Designs

After the free play, refocus groups on Part I of the activity, creating a group design. As groups work, if they aren't already considering the two questions in Part I, pose these to them.

Groups will likely discover that four squares fit together, three hexagons fit together, and six triangles fit together. Whatever cases they do find, you can point out that these blocks at least *appear* to fit together, but that students can't be sure yet whether they actually fit together perfectly or just come very close. This uncertainty will help foreshadow proving the angle sum formula in *Polygon Angles*.

After approximately 15 minutes of exploration, get students' attention for a brief lecture on **angle**. Angles can be thought of in different ways, and today's activity looks at them from two perspectives. One perspective is dynamic, in which angle is thought of as a turn. The other is static, in which angle is thought of as a geometric figure. For most students, the dynamic concept of an angle as a turn is an easier place to start.

Begin by demonstrating a complete turn. Stand facing the class, make a complete turn, and ask, **How far have I turned?** You might mention the fact that you have not traveled any distance and therefore the traditional measures of length are inappropriate for measuring a turn. Some students may say that you have turned "one complete turn." Others familiar with degree measurement may say that you have turned 360 degrees. Explain that both answers are correct and that a **degree** is the name for a turn that is $\frac{1}{360}$ of a complete turn. Use the symbol for degrees, writing 360° for the complete turn.

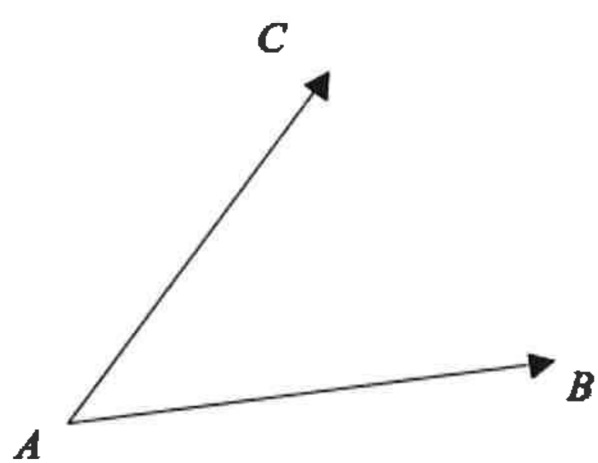
Ask students to demonstrate some other turns. For example, ask everyone to stand and do a half turn. Ask, **How many degrees are in that turn?** Then have students perform some other fractions of a turn.

Also go from degrees to turns. For example, ask students to turn 120°, and ask the class to describe what fraction of a whole turn that is. Most students need to develop a physical feeling for the turning concept, and this approach lets everyone get involved physically and mentally.

Emphasize that, when in doubt, students can return to the fact that a complete turn is 360°. They can always use this frame of reference to go from a fraction of a turn to degrees, and vice versa.

Ask if anyone knows a special name for a quarter turn. Introduce the term **right angle** and have students figure out how many degrees it must be. Also mention that an angle between 0° and 90° is called an **acute angle** and that an angle between 90° and 180° is called an **obtuse angle.**

Another important way to think about an angle is as a geometric figure (or part of one). To introduce this idea, show students a diagram such as that below, and ask, **Where is the angle in this diagram?**



As needed, explain that in order to think of this diagram as showing an angle in the sense of a turn, students can imagine standing at *A* and facing *B*, and then imagine turning to face *C* (while continuing to stand at *A*).

Extend the lengths of the sides of the angle and ask how this changes the angle itself. Many students confuse side lengths with the size of an angle, so it is important to bring out early and often that the angle itself remains unchanged.

Tell students that point *A* is called the **vertex** of the angle and that the rays from *A* through *B* and from *A* through *C* are called the *sides* of the angle. Also introduce the notation $\angle BAC$, read as "angle *BAC*." Mention that if there is no chance for confusion, such an angle can be simply referred to as $\angle A$.

Make it clear that whether we start facing *B* or facing *C*, we generally assume that we turn "the short way." Thus, if we start at *A*, facing *B*, we would turn counterclockwise to face *C*, rather than make almost a whole turn clockwise.

Part II: Pattern Block Angles

For Part II, students will need to be familiar with the concept of an angle in a polygon. You can introduce this concept by drawing any polygon. You may need to begin with the terms *side* and *vertex* as applied to a polygon and introduce the plural *vertices* as well.

Then ask students to identify the angles in the polygon. Explain, if needed, that an angle in a polygon is an angle formed where two sides meet at a vertex. Thus, a polygon has the same number of angles as vertices (which is also the same as the number of sides). Use the special case of a square or rectangle to illustrate this fact, and ask students to find the sizes of the figure's angles. They should be able to connect this idea with the earlier discussion and see that each angle is a quarter turn, or 90°.

Have groups now turn their attention to Part II. Explain that they are to determine the measure of the angles in degrees using only the blocks themselves. Remind them to consider what they learned about fitting blocks together to make complete turns.

As students complete Part II, encourage them to continue into Part III.

Once most groups have worked through at least a few of the pattern block angles, bring the class together for a brief discussion on methods and findings. If you have overhead pattern blocks, they will be useful here.

Most of the explanations should be straightforward, such as "I could fit six triangles together at a single point, so each angle is a sixth of a turn, which is 60°." Students will need to do something subtler to find the large angle of the thin parallelogram, such as fit it together with the right angle from a square and an angle from the hexagon. You might again make note that the methods being used to determine these angle measurements are based on the assumption that the blocks fit together perfectly.

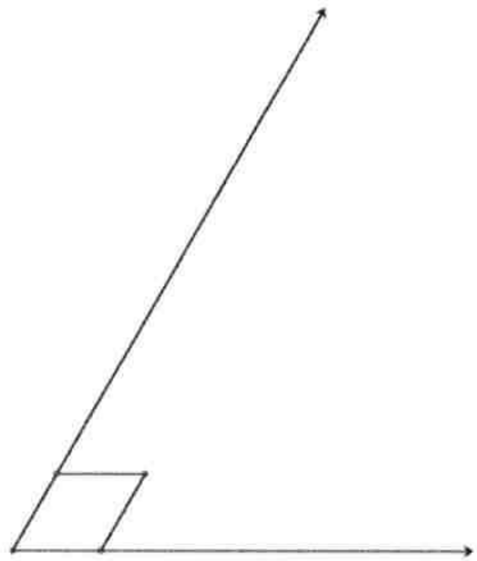
Part III: Pattern Block Angles with a Protractor

The protractor is a difficult tool for many students to learn to use. This activity is intended to help students develop a meaningful understanding of and experience with angles so that they can use the protractor to measure angles and get the same results. The activity assumes students will use the known angle measures of the pattern blocks to learn to read that measure on the protractor.

Students will have to learn how to align the vertex of the angle, as well as each side of the angle, with the protractor, and then how to read the measurement. Each protractor works slightly differently, so they will need to spend some time exploring with their own protractors.

Help students begin by suggesting that the protractor is a tool to measure angles. Remind them that they now know the measures of several angles—namely the angles of all the pattern blocks—and explain that they can use that knowledge to investigate how to use the protractor to obtain the same measurements.

Because pattern blocks are rather small, suggest that students trace one angle of a block onto paper and then extend its sides. In the example below (the blue rhombus), the angle is known to measure 60°. Let students learn how to use their protractors to arrive at this measure.



Discussing and Debriefing the Activity

You might conclude the activity with some students demonstrating how to measure angles using a protractor. You can provide more experiences for them to practice this skill, although many opportunities come in the next activities.

Key Questions

How far have I turned?

How many degrees are in that turn?

Supplemental Activities

A Protracted Engagement (reinforcement) is an open-ended activity in which students are asked to decode a message created using angles of different sizes to correspond to different letters of the alphabet, and then to code a message of their own. In the process, they gain additional experience measuring angles with protractors.

From Another Angle (extension) extends students' work with pattern blocks.

Degree Discovery

Intent

In this activity, students explore conjectures about the sum of the angles in triangles and quadrilaterals and gain further practice in the use of protractors.

Mathematics

The central mathematical idea underlying the next three activities—*Degree Discovery, Polygon Angles,* and *An Angular Summary*—is that there is a functional relationship between the number of sides of a polygon and the sum of the measures of its interior angles: sum of angle measures is equal to 180 degrees multiplied by (*number of sides* – 2).

Students get a good deal of practice with measuring angles in *Degree Discovery*. In fact, upon completing this activity, they should have measured at least 20 angles. The design of the activity provides students with feedback on correct protractor use—students tend to check their readings when patterns aren't emerging or when one polygon stands out differently from the others.

Progression

Students will draw several triangles, measure and sum the angles in them, and then do the same for quadrilaterals. Their observations are noted in class and initiate a sequence of activities in which students derive and prove the angle sum formula for polygons. *Degree Discovery* works particularly well as a homework assignment and sets up class work on *Polygon Angles*. *An Angular Summary* serves as a wrap-up assignment for this part of the course.

Approximate Time

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

To transition, mention that in *Pattern Block Investigations*, students found the measures of the interior angles of the special polygons represented by the six pattern blocks. Now they will be asked to make conjectures about the sum of the angle measures of any triangle and then of any quadrilateral.

Point out that students are to draw a variety of triangles. Ask the class to come to an agreement on how many each person should draw (we suggest three at least).

Discussing and Debriefing the Activity

Give students a short time to share results and ask questions within their groups. Then invite some students to share their observations about triangles. Since they are using approximate measurements, their angle sums may not be exactly 180°, but they should see that regardless of the shape of a triangle, the angle measures always seem to add up to about 180°.

This might lead to the conjecture that the angle sum for any triangle is exactly 180°, perhaps using the analysis in *Pattern Block Investigations* for the triangle pattern block as support. But this is a strong statement, one that cannot be proved by measuring, as no measurement is ever exact. What if the real answer were 181° or 179.5°? And what if the result is different for some triangles? This activity is designed to raise, rather than settle, these questions.

A thoughtful argument could arise from the green pattern block triangle. Students should have noted that six of these blocks seem to fit together around a single point, so the angles are apparently 60° each. However, this too is not a conclusive argument, as students have no way yet to be sure that the blocks fit together perfectly.

Use the word **conjecture** to describe the hypothesis that the angle sum for every triangle is 180°.

In the discussion of the next activity, *Polygon Angles*, students will be told that the angle sum for triangles is always 180° and that they will see a proof for this later in the year (in the unit *Shadows*). For now, leave the issue unresolved, so that students are not yet certain whether their conjecture is true.

For Question 3, let students share their conclusions about angle sums for quadrilaterals. They will probably see that the sum always appears to be approximately 360°. Bring out that this observation, like the one for triangles, is only a conjecture (at least for now), since the measurements are only approximations. Have students discuss how their work on *Pattern Block Investigations* relates to this conjecture. Their results for the four quadrilaterals should confirm the conjecture.

Some students might offer that if the angle sum for triangles is 180 degrees, then it makes sense for the sum for a quadrilateral to be 360 degrees, because a quadrilateral can be seen as two non-overlapping triangles. This connection will be explored more deeply in the next activity.

Key Questions

What if your protractor measurements are not exact?

Why might measurement results vary for some triangles?

Polygon Angles

Intent

Building on previous work with patterns, In-Out tables, and functions, this activity asks students to generalize their observations about the relationship between the sum of interior angles and the number of sides of a polygon.

Mathematics

In this activity, students generalize the results from triangles and quadrilaterals to all polygons. This mathematical investigation is a valuable opportunity for them to learn about what doing mathematics is and to see themselves doing mathematics.

Progression

Students work in groups on a rather open task to explore and record what they notice about the sums of polygon angles. If they record their observations about different types of polygons in an In-Out table comparing number of angles to angle sum, they may observe another pattern.

Approximate Time

30 minutes

Classroom Organization

Groups

Doing the Activity

This activity provides another opportunity for students to engage in a fairly open, unstructured exploration allowing them to approach the problem, structure their own time, and organize their data in their own way.

This activity immediately follows the observations made about angle sums for triangles and quadrilaterals. Tell students that now they will explore polygons with more than four sides.

As groups explore, you will have an opportunity to observe who may be having trouble with the protractor. Encourage group members to help each other use the tool properly.

While groups work, encourage them to gather data, make observations, and look for patterns. Ask, **Think about different ways to organize your data to see if there might be patterns in your findings.** By drawing polygons and measuring and adding their angles, students can build an In-Out table of conjectures, like the one below. **What do you notice about your table?**

Number of Sides	Angle Sum
3	180
4	360
5	540
6	720
7	900

Ask questions to help students progress from simple pattern identification to rule building.

Given these results, what might be a conjecture for the angle sum for a 10-sided polygon? A 12-sided polygon? A 100-sided polygon? An *n*-sided polygon?

Is there a general formula for connecting the *In* to the *Out* in this table?

Students may recognize that all angle sums are a multiple of 180 degrees, but what multiple? This generalization—if n is the number of sides, then $(n - 2)180$ is the angle sum—is the key underlying functional relationship. It is likely that several groups will notice the generalization, but may not have symbolic notation for the rule; it is not an easy step to recognize that $(n - 2)$ can be written to represent "two less than the number of sides." You might encourage students to write their rules as sentences.

The important challenge is the proof that this relationship must always hold, even beyond the data in the table. For groups that have achieved some confidence with this pattern, begin by reminding them it is only a pattern in the shapes they have seen—are they certain the pattern continues?—and then challenge them to prove the relationship they have conjectured.

Why must your rule be true for all polygons?

A slightly simpler question, to get a group started, is, **Why should the triangle sum for quadrilaterals be exactly twice that for triangles?**

Discussing and Debriefing the Activity

Bring the class together when you think that groups have made good progress exploring and observing patterns and a whole-class discussion can help them to move forward.

You may want to begin by asking students to review what they saw yesterday about angle sums for triangles and quadrilaterals. Then let volunteers state their conclusions about angle sums for polygons with more sides. Although their measurements will again be approximate, they will probably come up with conjectures that can be entered into an In-Out table. Take this table as far as students' results lead, and then ask, **Did anyone come up with a general formula expressing the angle sum as a function of the number of sides?**

If you get a clear statement of the generalization, try to determine whether the class sees where the formula came from. If not, you can build up to the formula by asking students to guess what the angle sum would be for polygons with a specific number of sides not covered yet, based on information in the table.

For example, if the table goes up to a 7-sided polygon, ask students to use the data to formulate a conjecture for 10-sided polygons. **What do you think is the sum of the angles in a 10-sided polygon?** They should probably be able to extend the table by adding 180° three times to get additional rows. By now, they will probably have recognized that the *Out* values all seem to be multiples of 180°.

You can follow up with a large numerical case, such as a 100-sided polygon. **What should you multiply 180° by to get the sum of the angles in a 100-sided polygon?** Students should be able to confirm that the necessary factor seems to be found by subtracting 2 from the number of sides.

Add a row to the table to show this formula.

Number of sides	Angle sum
3	180°
4	360°
5	540°
6	720°
7	900°
8	1080°
9	1260°
10	1440°
100	17640°
n	$(n - 2)180°$

Ask if anyone can explain why the angle sum for quadrilaterals should be exactly twice that for triangles. They should be able to see that a diagonal can be constructed to split a quadrilateral into two triangles; this works even for concave quadrilaterals. Without getting into a lot of detail, use that fact to conclude that the angle sum for a quadrilateral is the sum of the angle sums for its two triangles. Emphasize that this argument does not prove that the angle sum for a triangle is 180° or even that every triangle has the same angle sum. It *does* prove that if every triangle has an angle sum of 180°, then every quadrilateral has an angle sum of 360°.

Finally, ask how the argument for quadrilaterals might be used to explain the formula for the general polygon. Here are two approaches students might use.

They may see—using more examples, if needed—that a polygon with *n* sides can be divided, using diagonals, into (*n* – 2) triangles. This is easy to see for convex polygons (all angles less than 180°), but can also be done for concave polygons, with slightly more effort.

They may see that a single diagonal can be used to divide an *n*-sided polygon into a triangle and an (*n* – 1)-sided polygon. This explains why each side added to the polygon increases the angle sum by 180°. (This approach is another example of recursive reasoning; see the discussion of *Diagonally Speaking*.)

Key Questions

Think about different ways to organize your data to see whether there might be patterns in your findings.

What do you notice about your table?

What would be your conjecture for the angle sum for a 10-sided polygon? A 12-sided polygon? A 100-sided polygon?

Is there a general formula connecting the In to the Out in this table?

All angle sums are a multiple of 180 degrees, but what multiple?

What do you think is the sum of the angles in a 10-sided polygon?

What should you multiply 180° by to get the sum of the angles in a 100-sided polygon?

Why must your rule be true for all polygons?

Why should the triangle sum for quadrilaterals be exactly twice that for triangles?

Supplemental Activity

A Proof Gone Bad (reinforcement) asks students to explain the contradictions in another student's proof.

An Angular Summary

Intent

In this activity, students reflect on and apply their knowledge of the relationship of sides and angles in polygons. This activity emphasizes the important mathematical relationships they have worked on recently, including the unproven fact that the sum of the angles in a triangle is 180° and, based on this conjecture, the proven polygon angle sum formula.

Mathematics

This activity draws upon the polygon angle sum formula to introduce the concept of a **regular polygon**, a polygon in which all angles have the same measure and all sides are the same length. Students draw and measure angles using a protractor one more time.

Progression

This activity serves as a wrap-up for the angle and polygon investigation sequence. After recalling and writing about what they know about polygon angles, students solve the angle measures of a regular pentagon and regular octagon and then draw these polygons.

Approximate Time

20 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, followed by small groups

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

You may want to have one or two volunteers read or present their work on Question 1 and then allow the class to add to or correct the material presented.

Be sure to get some explanations for the angle sum formula. For example, these might refer to the idea of dividing a polygon into triangles or to the numerical pattern found in *Polygon Angles.*

For Question 2, ask students to explain their work. Use the opportunity to briefly emphasize the definition and basic properties of a **regular polygon**. Then ask what difficulties students had in drawing the polygons.

In Question 3, students are asked once more to use protractors to draw polygons, reinforcing their grasp of the pattern developed in the previous activities. If they correctly determine the size of each angle in a regular polygon, and measure correctly, their figures should close—that is, the last side of each figure should meet the first side at the final vertex.

Putting It Together

Intent

As the name suggests, *Putting It Together* encourages students to bring to bear all of the mathematical tools and techniques they have been developing throughout the unit, and their developing identity as a learning community, on a group of summary activities.

Mathematics

In these activities, students will use In-Out tables to solve several problems. One activity will introduce two famous, mathematically important patterns that will surface again later in IMP. Others will combine algebraic thinking, and in particular the concept of function, with ideas from geometry. Finally, students will review and summarize their work over the entire unit in a portfolio.

Progression

Squares and Scoops

Another In-Outer

Diagonally Speaking

The Garden Border

Border Varieties

Patterns Portfolio

Squares and Scoops

Intent

This activity challenges students to draw on their work with patterns and their explorations in *Consecutive Sums* to identify general rules for two important patterns.

Mathematics

Question 1 involves consecutive sums starting with 1. Students investigate the relationship between the height of a stack of squares and the number of squares in the stack. The stack of squares is arranged in a triangular pattern, with each row of squares one unit longer than the one above. A 1-high stack contains 1 square, a 2-high stack contains $1 + 2 = 3$ squares, and a 3-high stack contains $1 + 2 + 3 = 6$ squares. The numbers 1, 3, 6, and so on are called the *triangular numbers*. In general, an n-high stack will contain $1 + 2 + 3 + \cdots + n$ squares.

Question 2 involves an analogous idea for consecutive products starting with 1. It poses a combinatorial question: How many ways are there to arrange n scoops of ice cream on a cone? There is 1 way to arrange 1 scoop and 2 ways to arrange 2 scoops. However, there are 6 ways to arrange 3 scoops. To make the problem easier to think about, imagine that each scoop is a different flavor. For 4 scoops, once the first flavor is chosen, we know there are 6 ways to arrange the rest, and with 4 ways to choose the first flavor, there are $4(6) = (3)(2) = 24$ arrangements altogether. In general, there are $n(n - 1)\ (n - 2)\cdots(2)(1) = n!$ (read "n factorial") ways to arrange n flavors.

Progression

Each question presents the first few rows of an In-Out table. Students are asked to predict the values in subsequent rows and then to generalize the patterns they used to make their predictions.

The activity is particularly appropriate for small-group exploration. The activity *Diagonally Speaking* follows a similar numeric approach and then challenges students to identify *why* the rule they discover must always hold.

Approximate Time

20 minutes

Classroom Organization

Groups

Doing the Activity

Tell students that they will now explore two very important number patterns—patterns that they will see repeatedly, and in surprising places, in their future mathematics work.

In Question 1, students might see a vertical recursive pattern, in which a value in the second column is found from the previous value, and a "zigzag" addition pattern.

Encourage groups to write a general rule for the patterns they find, but allow that they do not necessarily have to be written with symbols alone. Encourage use of words and sentences as well. Some students may recognize that the number of squares in an n-high stack is equal to the sum of the numbers 1 to n. If so, you might remind them of summation notation, which was introduced during *Consecutive Sums.*

None of these patterns is optimal when searching for, say, row 40, or for row n. In these cases, a rule that relates the *In* value to the *Out* value is best.

For Question 2, students might notice an analogous zigzag pattern, in this case a multiplicative one. If groups focus on the recursive pattern and you decide to challenge them to identify a functional pattern, you might turn their attention to Question 2c. In their solutions, even if they begin with their answer for ten scoops, they will probably say something like, "Multiply this by 11, then by 12, then by 13, and so on, all the way up to 100."

If students recognize some connections, you might remind them of **factorial** notation, mentioned briefly in *1-2-3-4 Puzzle.* The notation for products is analogous to summation notation, using the uppercase Greek letter pi (Π) in place of sigma for sums. For example,

$$\prod_{t=3}^{7} t$$

means 3 • 4 • 5 • 6 • 7.

Discussing and Debriefing the Activity

There is no need for a formal debriefing of this activity. You might invite posting of solutions or some class discussion if students wish to examine other groups' work.

Depending on students' interest, you might identify the common name for the set of numbers in the *Out* column of Question 1: the *triangular numbers*.

Also of note, the patterns in these In-Out tables would be much harder to find if the entries were not arranged sequentially. You might mention that while this is a good principle for analyzing information, the entries of an In-Out table do not, in general, have to be arranged in any particular order.

Key Questions

Is 1.5 an appropriate input for either of these tables?

What do you call the set of possible inputs for an In-Out table?

What other examples have you seen in which only certain inputs were allowed?

Supplemental Activity

From One to N (extension) asks students to find a simple expression in terms of *n* that allows one to find a sum without repeated addition. If students find such an expression, they look for a proof that their answer is correct.

Another In-Outer

Intent

In this activity, students practice integer arithmetic and finding and using rules for In-Out tables. They also return to the focus on language and symbolic notation begun in the earlier activities *Inside Out* and *Pulling Out Rules*.

Mathematics

The six questions in this activity give students additional opportunities to express the relationships between the *In* and the *Out* in an In-Out table representation of a function. Students write algebraic equations for expressing the *Out* as a function of the *In* and use their rules to find both the *Out* given the *In* and the *In* given the *Out*.

Asking students, in effect, to find both y given x and x given y emphasizes the "doing and undoing" aspect of algebraic thinking. The values in these tables also offer students the chance to use their knowledge of integer arithmetic, stressed earlier in the unit in the "hot and cold cubes" activities.

Progression

This activity is particularly appropriate for students to begin as a homework assignment.

Approximate Time

20 minutes for activity (at home or in class)

30 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Tell students that in this activity, they will look for patterns in more In-Out tables. Some of the tables are tricky and will draw upon their creativity. Students will be asked to write some of their rules as algebraic equations.

Discussing and Debriefing the Activity

Have students convene in groups to briefly share findings and ask each other questions. You might then have the members of various groups begin the discussion by sharing what they noticed in one of the In-Out tables.

Use Questions 2, 3, and 6 to review the usage of the term **function**. Ask volunteers to talk about how they translated their rules from verbal form into algebraic equations.

Questions 1, 4, and 5 are nonnumeric logic puzzles. The most engaging question in the activity is probably Question 5 (the one with the funny faces). If students are stuck, you might suggest making an In-Out table in which the number of eyes and the number of hairs are both *Ins*.

In		Out
Number of eyes	**Number of hairs**	
2	2	6
2	3	11
3	4	19
2	5	?

Finding a rule for this table could be left open. Here are two.

$$\textit{Out} = 3(\text{number of eyes}) + 5(\text{number of hairs}) - 10$$

$$\textit{Out} = \text{number of eyes} + (\text{number of hairs})^2$$

Both formulas fit all three given rows, but the first formula gives 21 for the missing entry, while the second gives 27.

Diagonally Speaking

Intent

This activity revisits and pulls together ideas in the *Patterns* unit. It reinforces In-Out tables and the search for patterns as powerful problem-solving tools. Students use an In-Out table in a geometric context to find a functional relationship and are challenged to prove why their pattern holds.

Mathematics

Students explore geometric ideas in this activity, such as the definition of a polygon diagonal and the connection between the number of sides and the number of vertices of a polygon. They use their developing algebra skills to analyze a geometric situation: the number of **diagonals** in any polygon is a function of the number of sides (or vertices) of the polygon. And finally, returning to the developing notion of proof, students utilize the geometric properties of polygon angles to prove that their numeric pattern must hold for all polygons.

Progression

Students gather and organize data, search for patterns and use them to make predictions, and use the problem context to explain why the patterns they found hold. This activity can be used as a small-group activity or assigned for homework and then discussed in class.

Approximate Time

20 minutes for introduction

20 minutes for activity (at home or in class)

20 minutes for discussion

Classroom Organization

Individuals or small groups, followed by whole-class discussion

Doing the Activity

This activity illustrates a common use of tables in mathematics—to organize information about a complex situation in order to gain insight into the situation itself. The context determines a well-defined, unique function. Part of students' task is to examine how to use that context to justify and explain any pattern or rule they find.

Arouse students' curiosity by posing the simple question that forms the basis for this activity.

Can you predict how many diagonals a polygon has?

Encourage everyone to draw a polygon (suggest from four to seven sides) and count its diagonals. Have a few volunteers report their findings, and then turn students to their groups and encourage them to continue their investigation.

Discussing and Debriefing the Activity

You might begin by drawing a table on the board and calling on someone to fill in one row of the table based on that group's work. Pause after each row and ask students if they agree or if they have any concerns about the numbers just recorded. After several rows have been added, the table will look something like this.

Number of sides	Number of diagonals
3	0
4	2
5	5
6	9
7	14

A *convex polygon* is defined as one in which all diagonals are inside the figure. A *concave polygon* is one in which all diagonals are not inside the figure. Some students might notice that for a concave polygon, one might argue that a segment should not be considered a diagonal if it goes outside the figure. If so, assure them that such a segment does not violate the definition of a *diagonal:* a line segment that connects two vertices of a polygon and is not one of its sides.

Ask for volunteers to describe patterns they found. Encourage them to draw on the board to aid their descriptions. During the discussion, draw out the rule that defines the pattern, and press students to develop a justification for why that rule must always hold. Some examples of patterns, rules, and possible justifications follow.

If students organize the table with increasing inputs, they will more easily notice that the number of diagonals increases by 2, then by 3, then by 4, and so forth, as the number of sides goes up by 1. This vertical or recursive pattern will allow many students to predict that a polygon with 12 sides has 54 diagonals and given that a 20-sided polygon has 170 diagonals, a 21-sided polygon must have 189 diagonals—results they would have great trouble obtaining by drawing the figures and counting diagonals.

The recursive pattern can be summarized in several ways. Two equations that express the relationship are these.

$$Out = \text{previous } Out + (\text{previous } In - 1)$$

$$Out = \text{previous } Out + (In - 2)$$

Some students, drawing on their work in *Consecutive Sums*, might see the following pattern in their tables and predict, correctly, that the number of diagonals in a 12-sided polygon is given by the sum 2 + 3 + ··· + 10.

Number of sides	Number of diagonals
3	0
4	2
5	5 = 2 + 3
6	9 = 2 + 3 +4
7	14 = 2 + 3 + 4 + 5

If so, you might ask, **How can you use summation notation to express this pattern of consecutive sums?** The key is determining how to use the *In*. Each *Out* is the sum of whole numbers from 2 to *In* – 2.

Some students might use their previous experience with In-Out tables to try to find a way to relate each *Out* to its corresponding *In* and notice that the following pattern emerges.

Number of sides	Number of diagonals
3	0 = 3(0)
4	2 = 4(0.5)
5	5 = 5(1)
6	9 = 6(1.5)
7	14 = 7(2)

From this, they might see that the multipliers in parentheses are one-half the quantity (*In* – 3), which leads to this closed-formula rule.

$$Out = \frac{In\left(In - 3\right)}{2}$$

Why does this rule make sense? The figure itself provides a clue. If students focus on a single vertex, they will see that the number of diagonals from that vertex is 3 fewer than the number of sides, because no diagonal is drawn to that vertex or to the two vertices immediately adjacent to it.

Key Questions

Can you predict how many diagonals a polygon has?

Can you use the table to predict the number of diagonals for an 8-sided polygon, without drawing one?

Why must all 5-sided polygons have the same number of diagonals?

Why would a 7-sided polygon have five diagonals more than a 6-sided polygon? Why would a 12-sided polygon have ten diagonals more than an 11-sided polygon?

Why does your rule make sense?

Supplemental Activity

Diagonals Illuminated (extension) is a follow-up activity that draws a distinction between recursive and **closed-formula** rules and asks students to develop a closed-formula rule for the number of diagonals of any polygon and to explain why it makes sense.

The Garden Border

Intent

This activity sets the stage for *Border Varieties,* in which students will gain additional experience using algebraic language and symbols to represent geometric situations. In addition, it strengthens students' understanding of equivalent expressions and their skill in working with the distributive property.

Mathematics

The algebra-geometry connection is again a key mathematical element of this activity and the next. In this activity, students derive general approaches for counting the tiles along the border of a square garden. The number of tiles is a linear function of the size of the garden.

Progression

Students begin this two-activity set by creating as many ways as they can think of to count border tiles for a 10-by-10 garden, without counting one tile at a time. They discuss and compare the variety of counting methods they find, setting the stage for developing a symbolic representation of each method and for recognizing the equivalence of the expressions created.

Approximate Time

25 minutes

Classroom Organization

Individuals or groups, followed by whole-class discussion

Materials

Overhead tiles (optional)

Square tiles

Doing the Activity

You may want to present this activity without referring to the student book. If students open their books to *The Garden Border,* they might notice the page for *Border Varieties,* which includes diagrams that provide possible answers for today's activity, and you want students to come up with these ideas on their own.

Introduce the context of the problem, noting that the garden, including the tiles, is to be 10 feet by 10 feet. Draw or display a picture of the tiles, or ask groups to agree what the tile arrangement must look like and have a volunteer share a sketch.

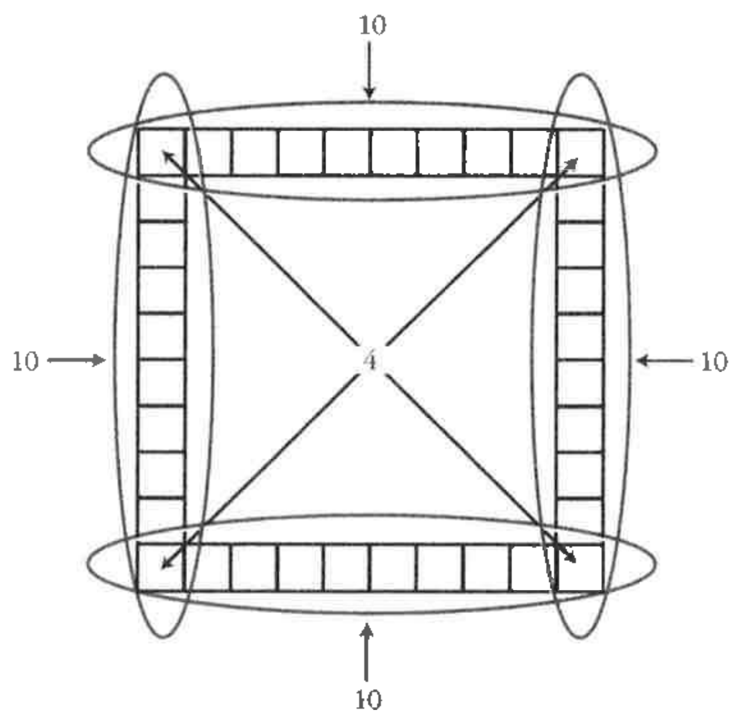
Tell students that their challenge is to figure out how many tiles Leslie needed, without counting the tiles individually, and to write down as many ways as they can for doing this. Then ask that they draw a diagram that indicates how each method works.

Note that you have posed the entire activity verbally. If some students would benefit from written instructions, have them turn to their own books. Or, you could display the instructions for students to refer to.

Discussing and Debriefing the Activity

Ask group representatives to report on one method for calculating the number of border tiles, giving the details of the arithmetic as well as the diagram.

One method is to start with the ten tiles along each edge and subtract 4 to account for the fact that each corner tile is on two edges. For this method, the arithmetic might be 4(10) − 4. Various diagrams can be created to represent this method. The diagram below shows the four 10s along the edges and indicates the four corner tiles that have been counted twice. (A more schematic diagram and five additional approaches appear in *Border Varieties*.)



Once the first presenter from every group has reported, ask for any additional ways not yet mentioned. Collect all the approaches that students have come up with.

Take a few minutes to talk about how some of the strategies collected might be applied to a 5-by-5 garden. For example, applying the method described above would show that such a garden uses $4 \cdot 5 - 4 = 16$ tiles.

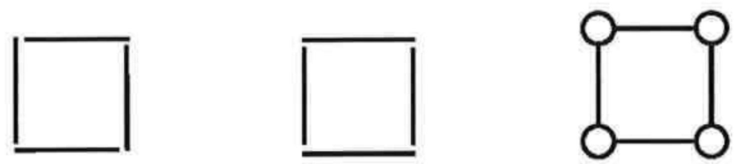
Border Varieties

Intent

This activity gives students additional experience using algebraic language and symbols to represent geometric situations. It also strengthens their understanding of equivalent expressions and skill in working with the distributive property.

Mathematics

In *The Garden Border,* students perceived a geometric context in many ways. The variety of approaches they developed lead to different-looking, but equivalent, methods for counting tiles. Here are three such methods for counting border tiles, along with rules that reflect these three ways of viewing the problem.



$4(s - 1)$ $2(s) + 2(s - 1)$ $4(s - 2) + 4$

These rules are **equivalent** because each gives the same total number of tiles for a given value of s.

Progression

In this follow-up activity to *The Garden Border,* students compare a variety of methods for counting border tiles for a 10-by-10 garden and then review and generalize these methods for gardens of any size. The classroom conversation emphasizes the equivalence of the resulting expressions and the occurrence of the distributive property.

Approximate Time

5 minutes for introduction

20 minutes for activity (in class or at home)

20 to 40 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity asks students to create "formulas" and offers the example $4s - 4$. In the early use of symbolic algebra in IMP, words like *formula* and *rule* are not precisely defined. In the context of this activity, students will be comparing expressions, so an equation is not necessary.

Have students read up to Question 1, and then ask a volunteer to summarize what was stated. Invite others to add points of clarification. Once the important points have been covered, tell students that they will be creating similar formulas for several more methods.

This might be a good assignment to collect to help assess and support students' understanding of using variables to express generalizations.

Discussing and Debriefing the Activity

Ask students to share their answers. They will probably want to use the diagrams in the activity to explain how they found their formulas. The diagrams will help make the generalizations more understandable.

To help clarify the techniques, ask students to test another specific case, such as $s = 100$. **Would this method work for a 100-by-100 square?** For example, for Question 1, the diagram suggests that the border for a 100-by-100 garden would have 100 tiles along the top and along the bottom, leaving 98 for each of the other two sides. Getting students to express this as $2 \cdot 100 + (100 - 2)$ will help elicit the expression $2s + 2(s - 2)$. You might use the phrase "imitating the arithmetic" to describe this technique for developing algebraic formulas to represent situations. Students might also express the method in Question 1 with the formula $s + s + (s - 2) + (s - 2)$.

The various expressions students develop provide an excellent opportunity to review equivalent expressions, the distributive property, and the idea of combining terms.

As much as possible, have students demonstrate equivalence for all the formulas they found for the various diagrams. For example, for Question 1, you might ask a volunteer to explain how to be sure that the expressions $s + s + (s - 2) + (s - 2)$ and $2s + 2(s - 2)$ are equivalent, independent of the problem setting. **How can you be sure these expressions are equivalent?** Students might recognize that $s + s$ is the same as $2s$ and that $(s - 2) + (s - 2)$ is the same as $2(s - 2)$. They might also see why these expressions are equivalent to $4(s - 2) + 4$.

At any time during this discussion, you might use numeric examples to check equivalence or to simply help students see that the operations being carried out in the different orders do yield equal values. Communicate that such examples do not prove equivalence, just help to confirm it.

To wrap up the conversation, remind students of the distributive property and ask them to identify instances in which they determined that two expressions were equivalent and in which they can see the distributive property in action, such as $2(2s - 2) = 4s - 4$. You might ask groups to work on this question for a short period of time and then to share with the class any other instances they found. Record their observations as equalities; for example, $4(s - 1) = 4s - 4$ and $2(s - 1) = 2s - 2$. Students will likely see that the distributive property is just a subset of the bigger idea of equivalent expressions.

A general statement of the distributive property might look like this.

The expressions $N(a + b)$ and $Na + Nb$ are equivalent.

You might want to post this statement, or another that students develop, for students to refer to.

Key Questions

Would your method work for a 100-by-100 square?

How can you be sure these expressions are equivalent?

Supplemental Activities

More About Borders (extension) contains variations on the *Border Varieties* activity.

Programming Borders (extension) asks students to write a program that answers some or all of the questions posed in *More About Borders*.

Patterns Portfolio

Intent

Students review and document their mathematical activity and learning during the course of the unit. Their product is an opportunity for assessing what they have learned and what they believe is important in their learning (see "About Portfolios" in the Overview to the Interactive Mathematics Program).

Mathematics

Students review their work on all the mathematical topics of the unit, with an emphasis on In-Out tables, proof, and integer arithmetic. Other ideas, such as functions and the distributive property, are approached more informally at this stage and are not specifically addressed.

This metacognitive activity is an important mathematical task in itself, in that mathematics learned in school has a quality distinct from the natural mathematical activity that emerges from interaction with real problems, or at least problems that are real to students. The distinction involves learning the agreed-upon conventions, notation, and terminology for referring to particular ideas that the formal study of mathematics requires.

Progression

Students are formally introduced to the notion of a portfolio and the particular expectations for the *Patterns* portfolio. They review their materials and begin to compile their portfolios as they write their cover letters.

Approximate Time

20 minutes for introduction

30 minutes for activity (at home or in class)

Classroom Organization

Whole-class introduction, then individuals

Doing the Activity

Prior to this day, emphasize the need for students to bring all their work from the unit to school. For some students, having to assemble a portfolio can become a reminder for the need to maintain some organization of their materials.

Disorganization can be an early indicator of students who may find less success in school. You might invite students who are particularly disorganized to visit with you privately to help them organize their materials from the unit, possibly into a three-ring notebook, and then draw out particular activities for their portfolios. Such private time will give you an opportunity to make a personal connection with students as well as support them in getting off to a good start in your class.

Before students begin work on their portfolios, you may want to ask what they recall about portfolios from the discussion at the beginning of the unit.

What is the purpose of a portfolio?

What would be good items to include in a portfolio?

After this brief review of portfolios in general, ask students to read *Patterns Portfolio* carefully. You may then wish to lead a general review discussion of the unit before students begin assembling their portfolios, or you may prefer to let students work in groups or on their own.

The primary goal of this first IMP portfolio is that students have some success becoming aware of and completing each of the three phases: Cover Letter, Compiling Papers, and Personal Growth. The writing of cover letters and the selection of portfolio materials are intertwined activities. Students should probably at least begin their general review of the unit before selecting portfolio materials, but they will need to make certain selections in order to complete the cover letter.

You might suggest two possible orders for work on the portfolios. Students could (1) identify papers to include, (2) write their cover letters, and (3) write their personal growth statements. Or they might (1) write their cover letters only through the point of describing the central ideas of the unit, (2) identify papers to include, (3) complete the cover letters by explaining their choices, and (4) write their personal growth statements.

As students review the unit and their work, they should think about both the mathematical content of the unit and the quality of their writing. Suggest that they choose work that conveys the essence of the unit as well as work that illustrates their expertise in solving and writing about problems.

Have students complete the *Patterns* portfolio for homework. Ask that they bring the portfolios back tomorrow with the cover letter as the first item. You might also mention that they will be able to refer to their portfolios when they work on unit assessments.

Discussing and Debriefing the Activity

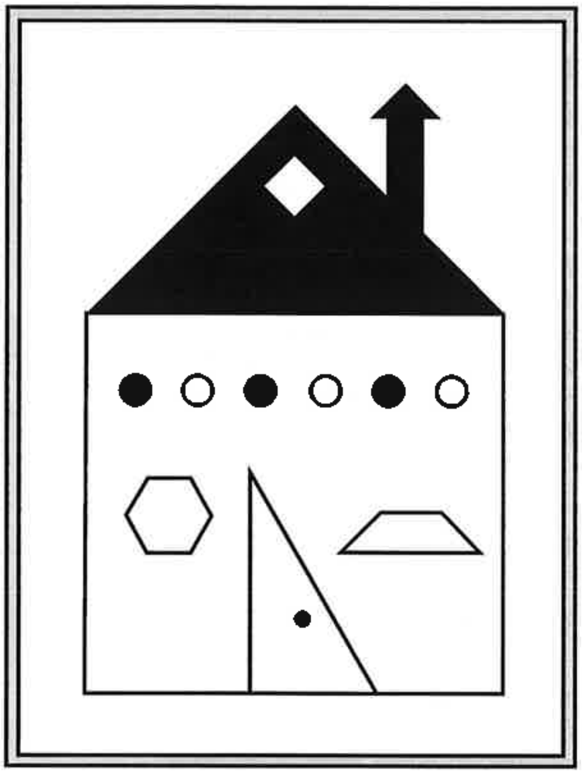
You might have students share their portfolios as part of the closing activity to the unit, the *Unit Reflection,* by reading cover letters and/or personal growth statements as a class or in small groups, sharing portfolios among group members, or having a class conversation about what was learned during the unit.

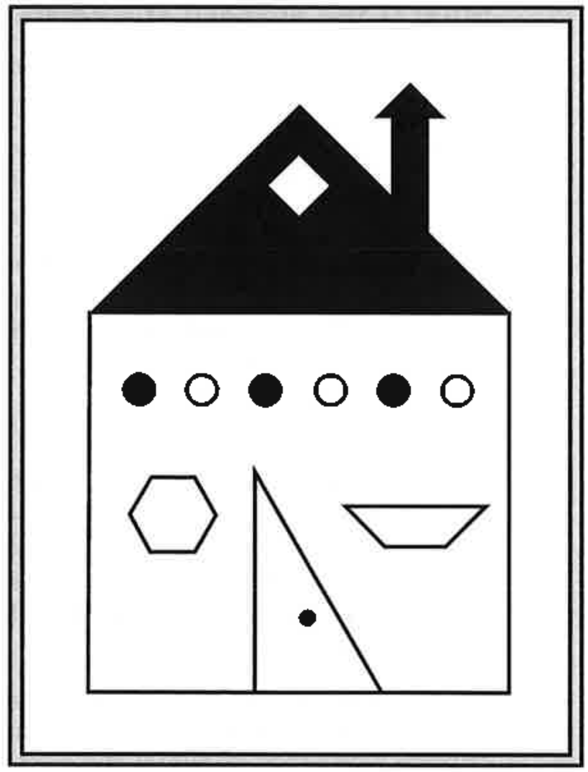
The portfolio is a very personal product for many students, something they may take a great deal of pride in. Public recognition of the thoughtfulness that students invested in creating their portfolios will add to the classroom learning environment and to students' connection to that environment.

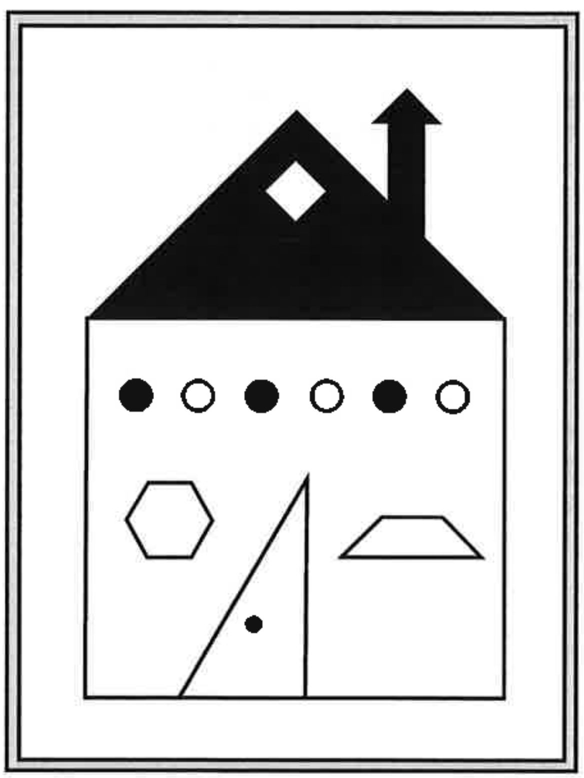
Key Questions

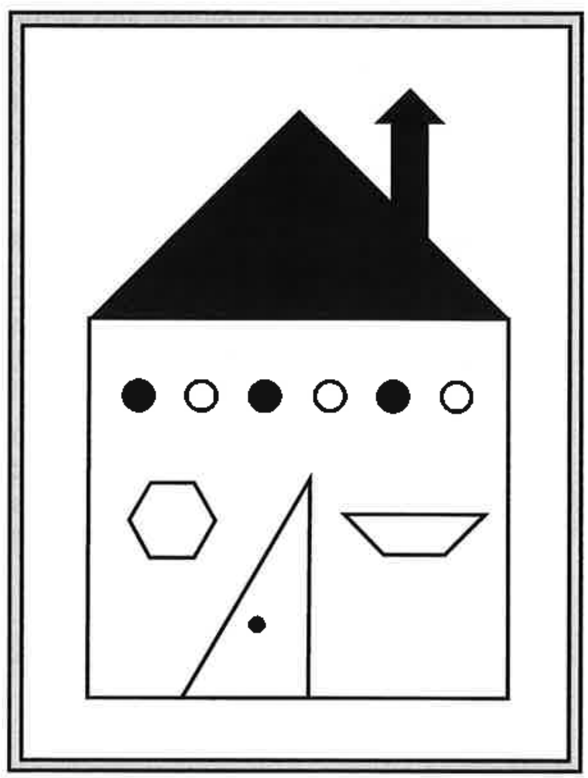
What are portfolios? What should go into them?

Lonesome Llama









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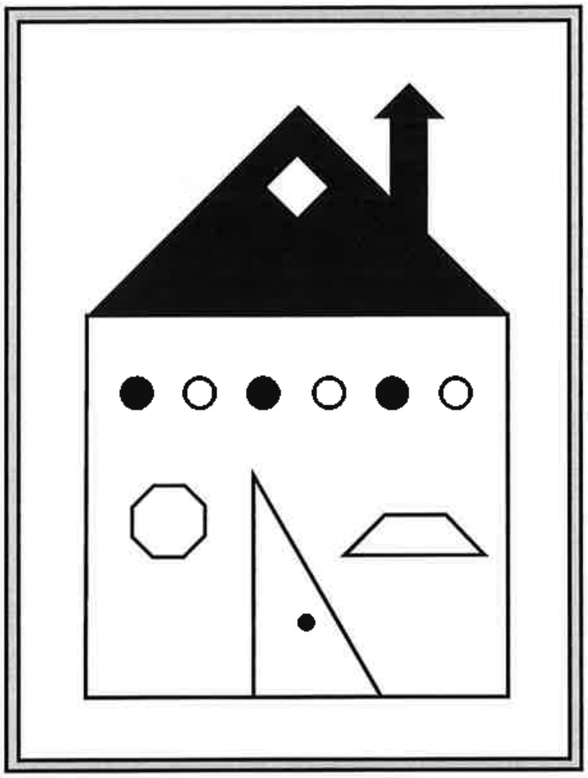

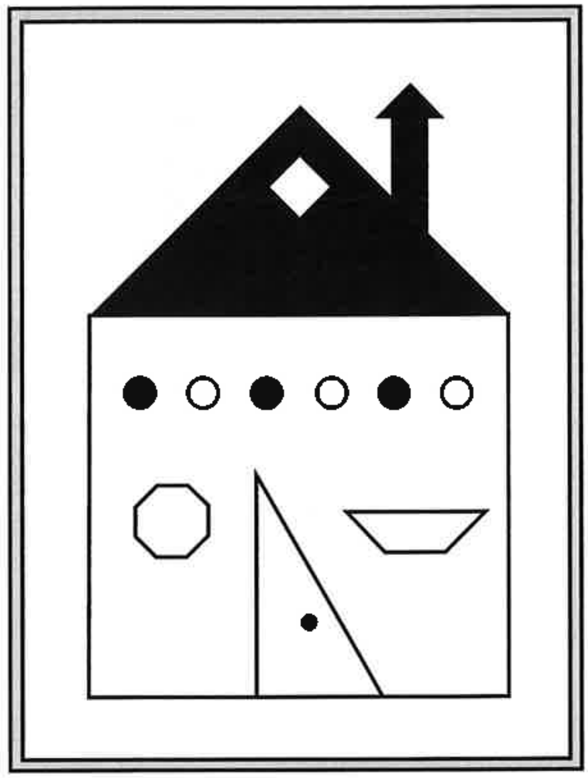

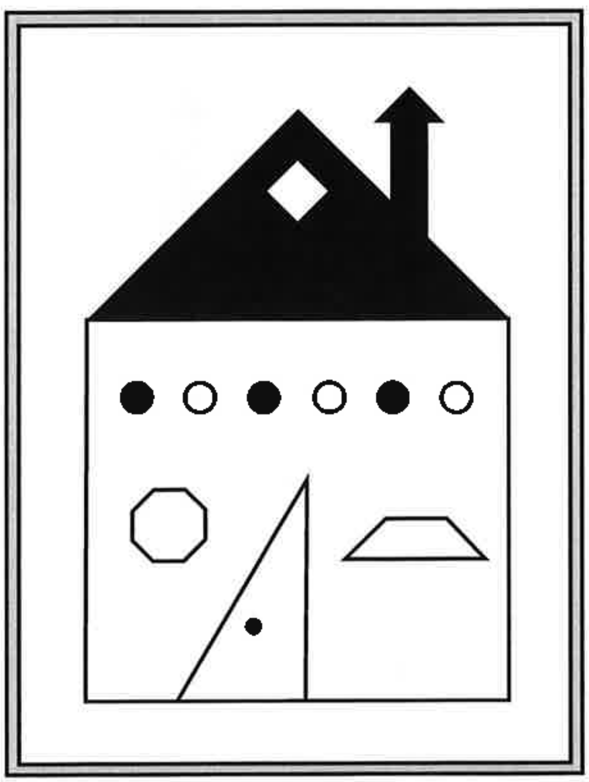

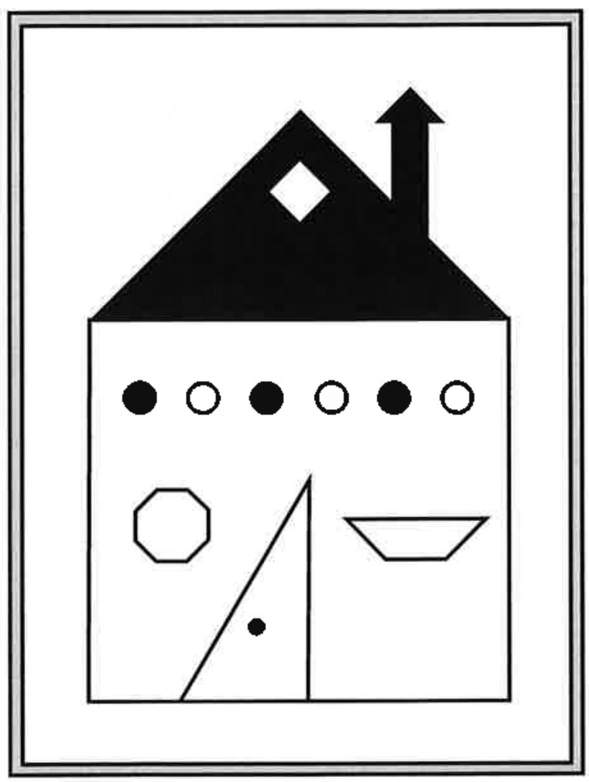

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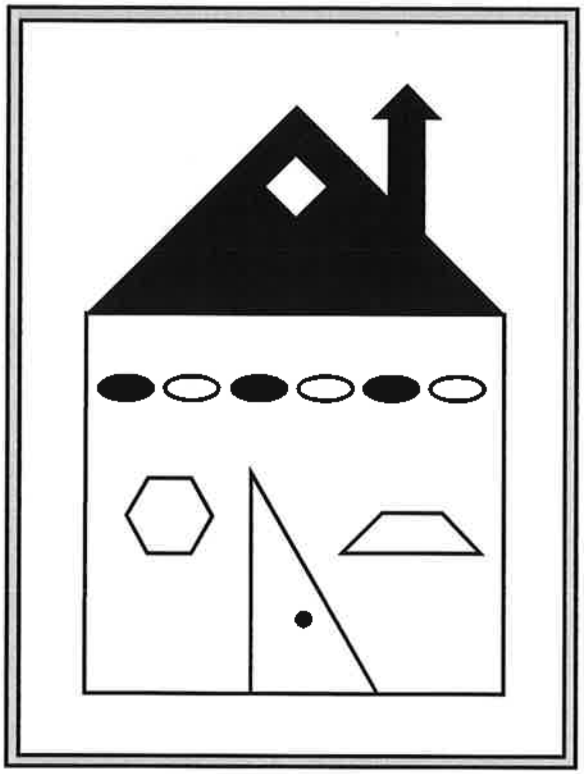

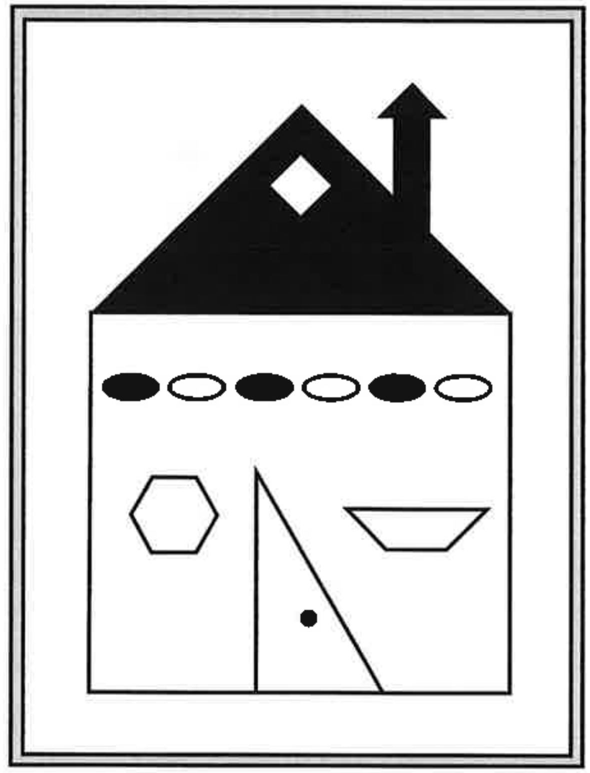

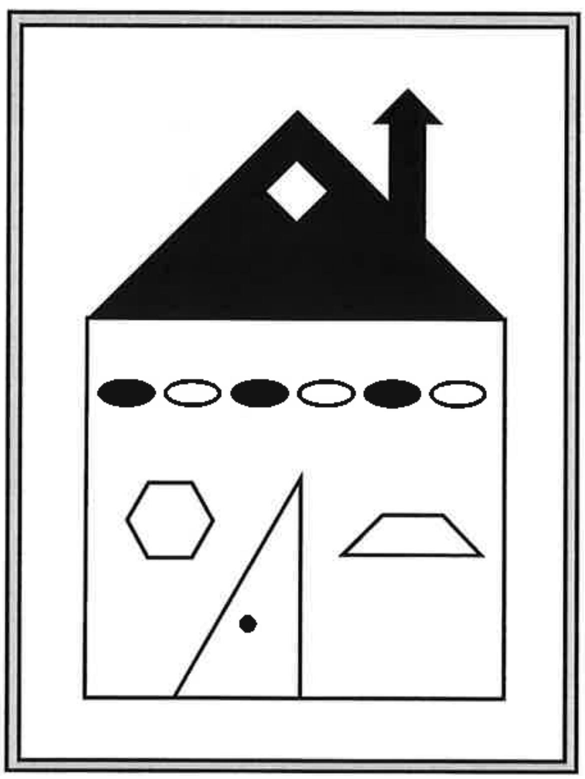

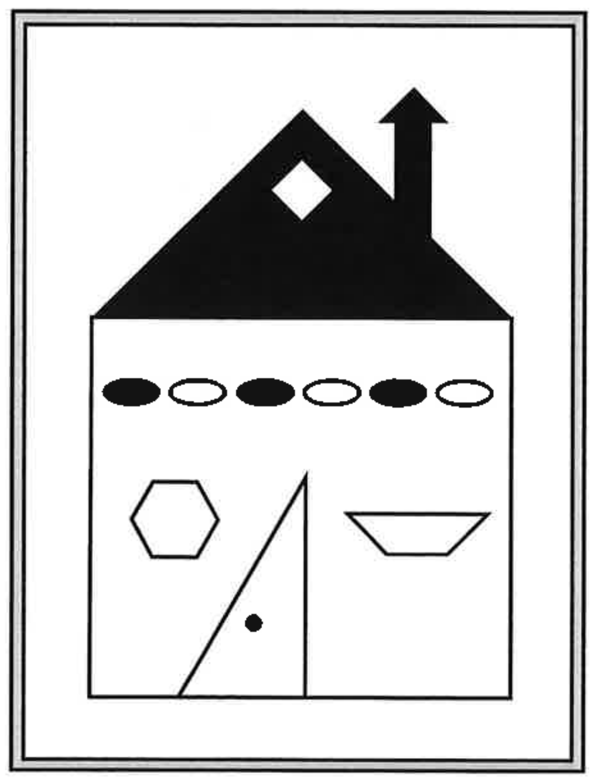

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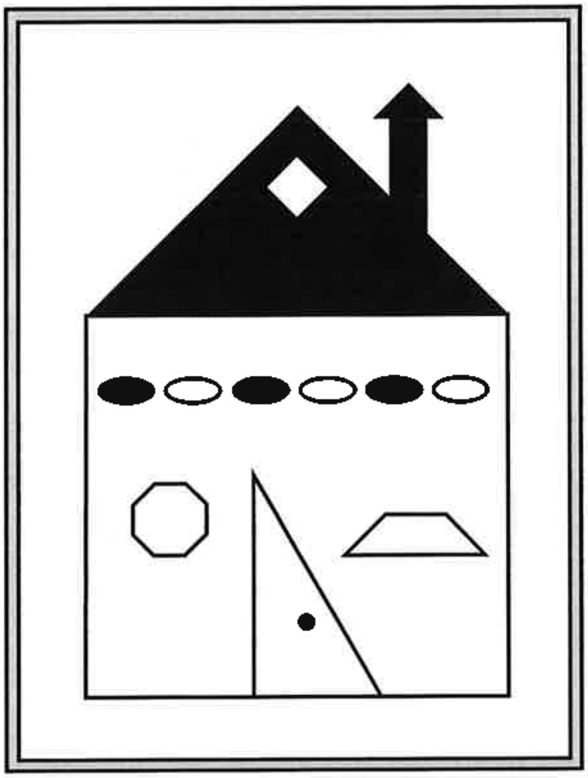

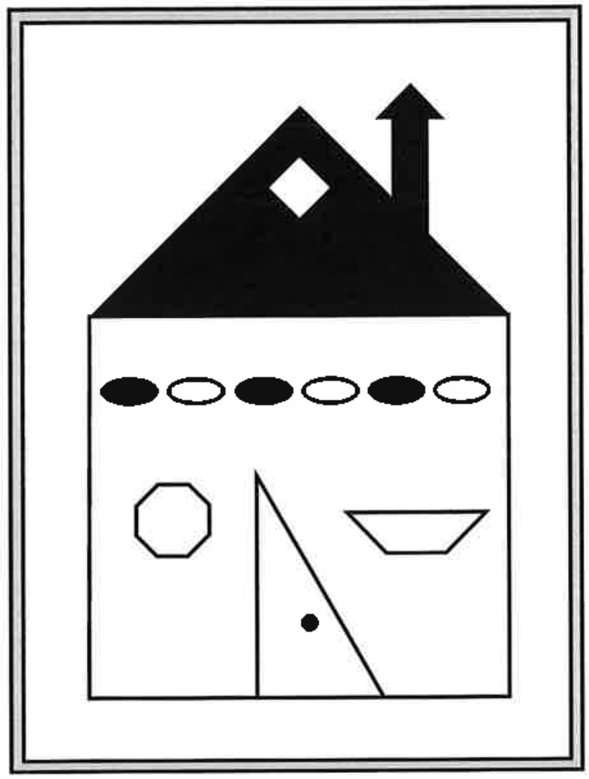

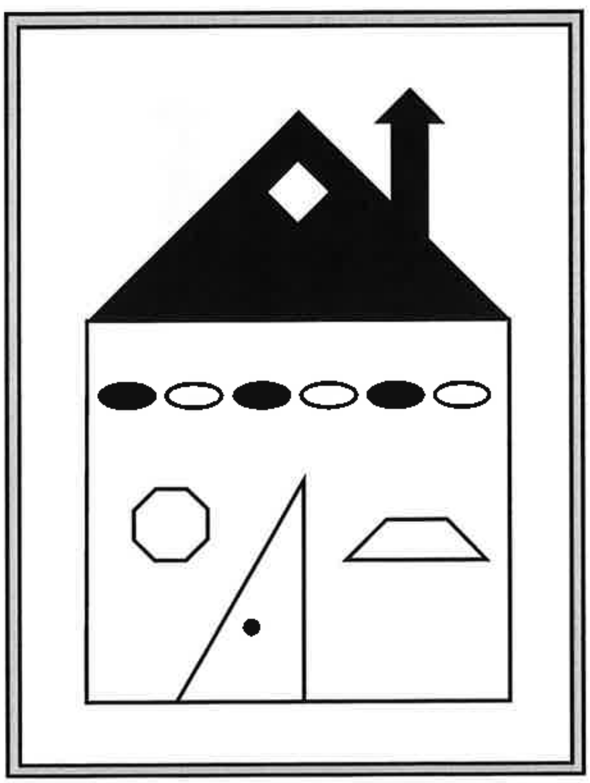

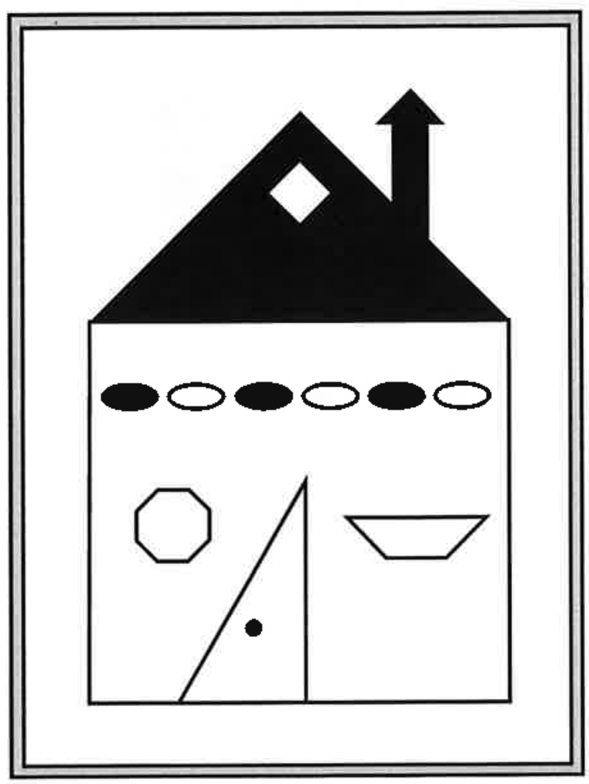

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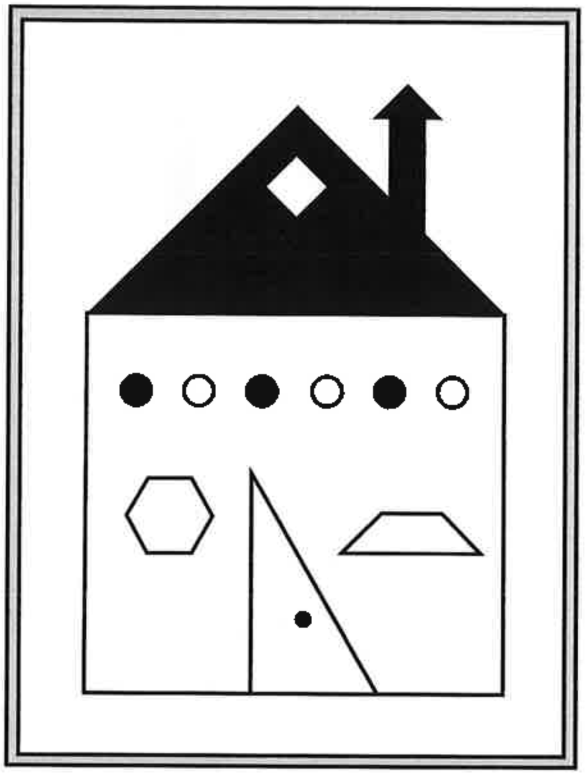

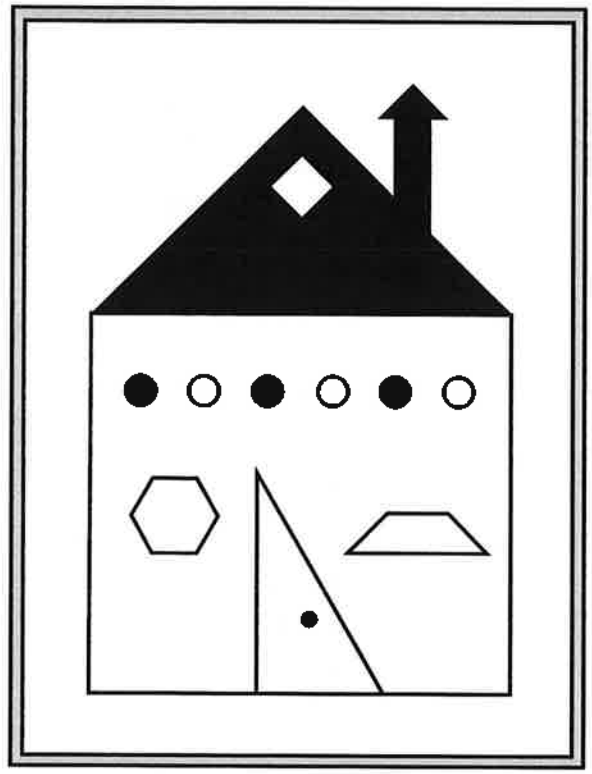

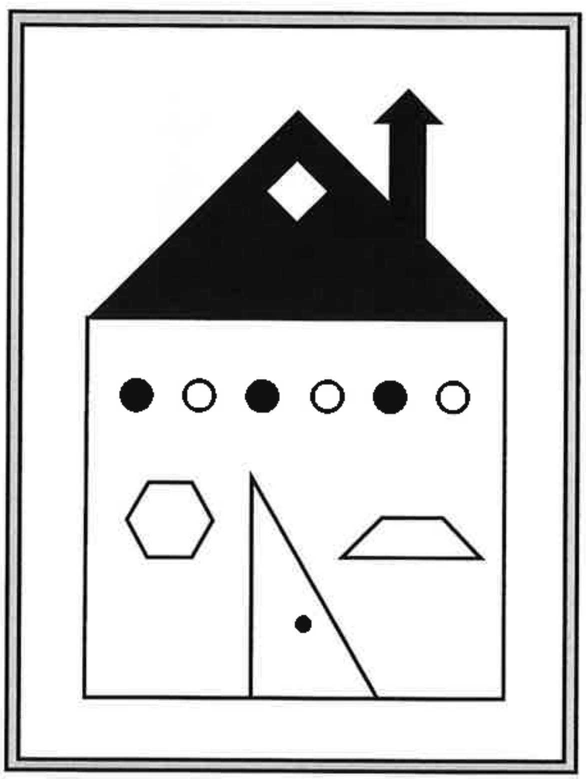

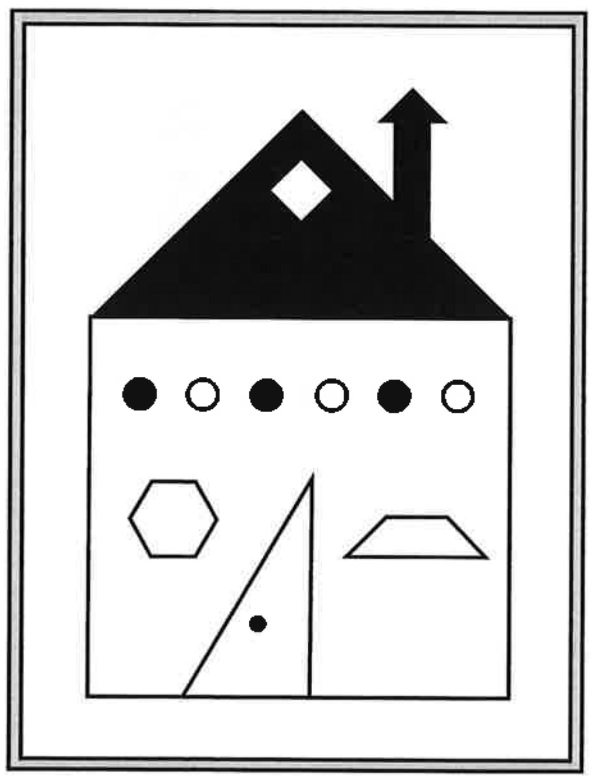

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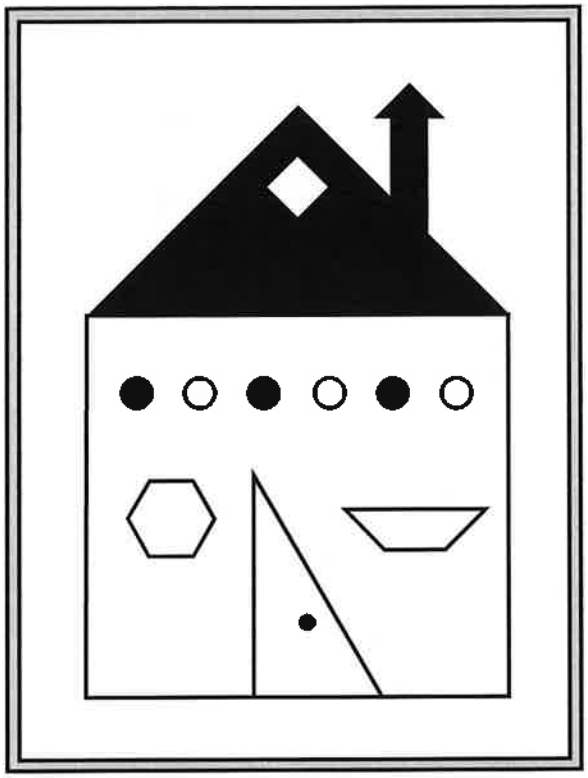

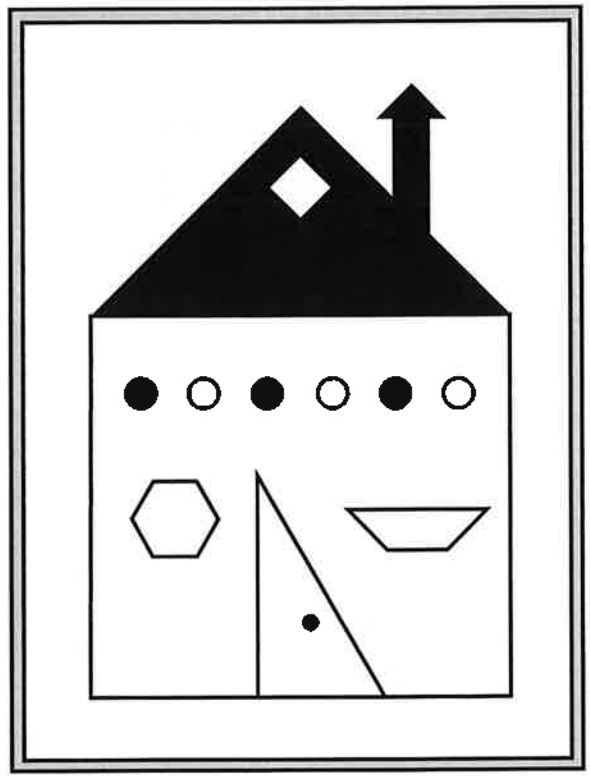

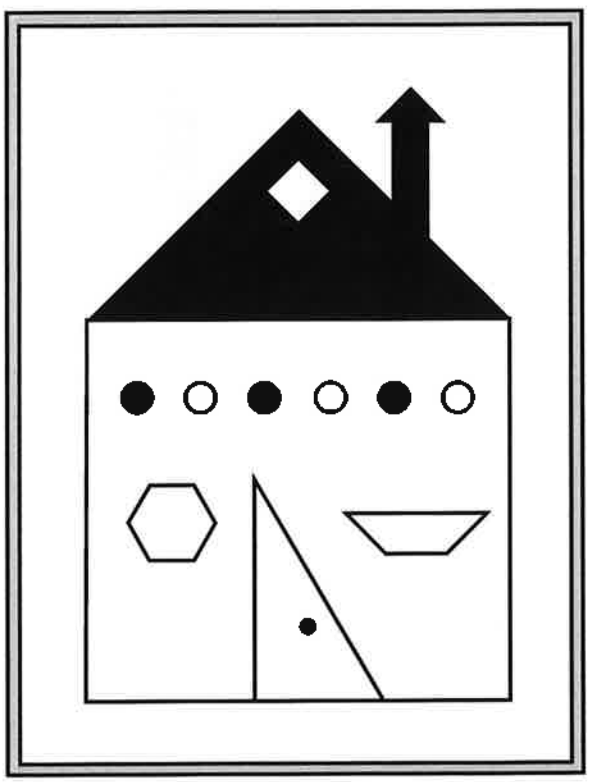

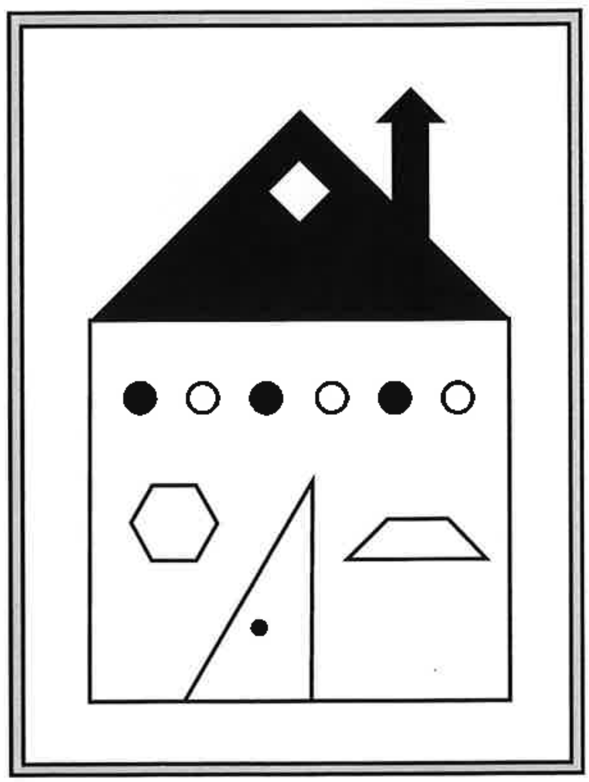

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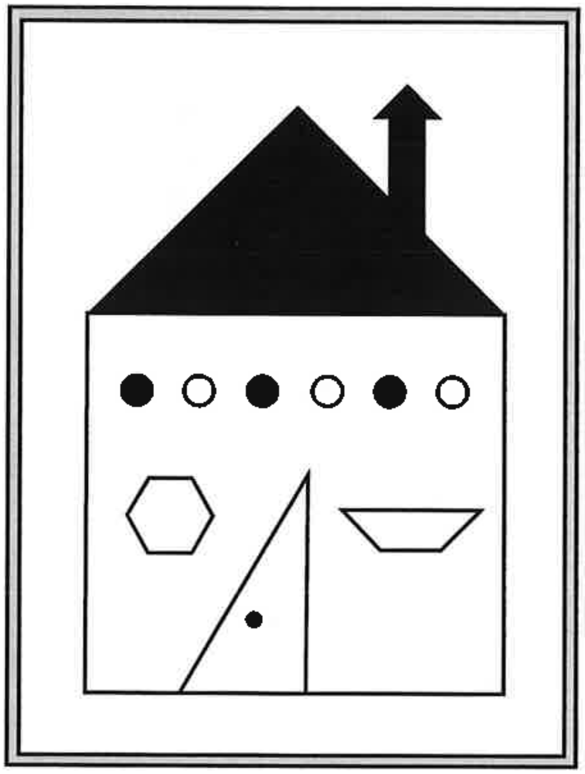

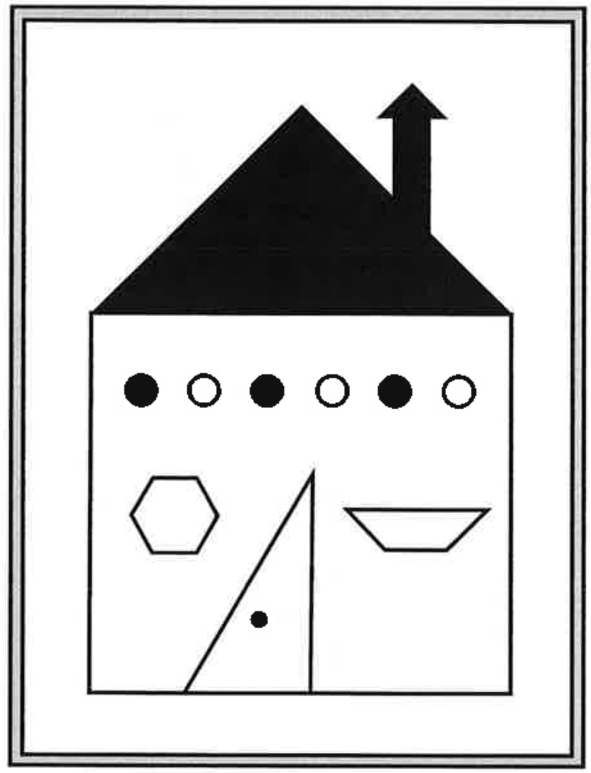

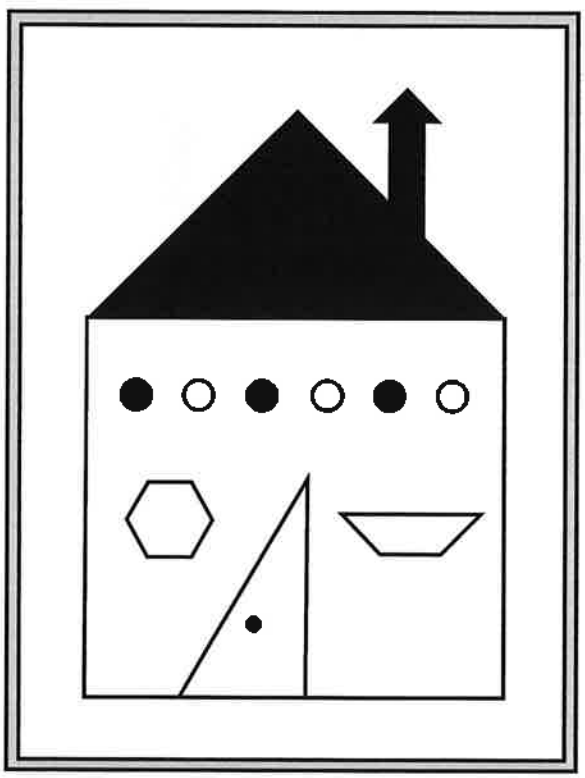

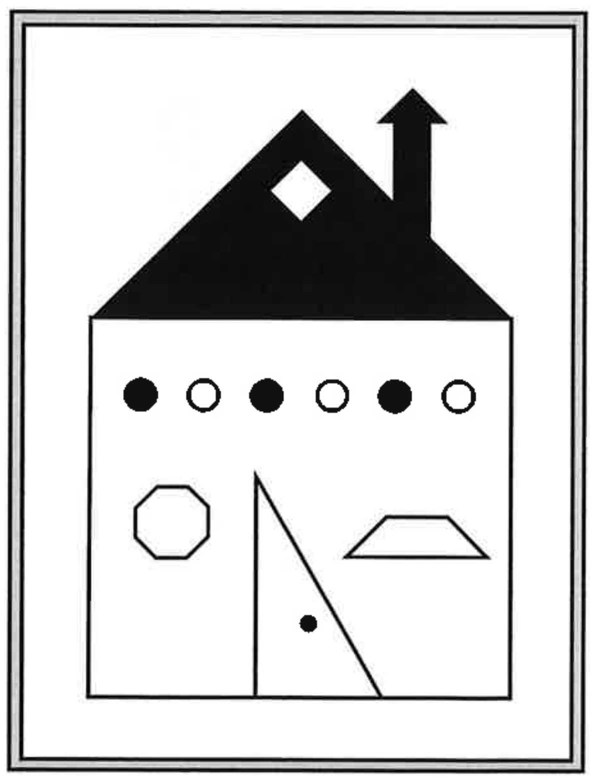

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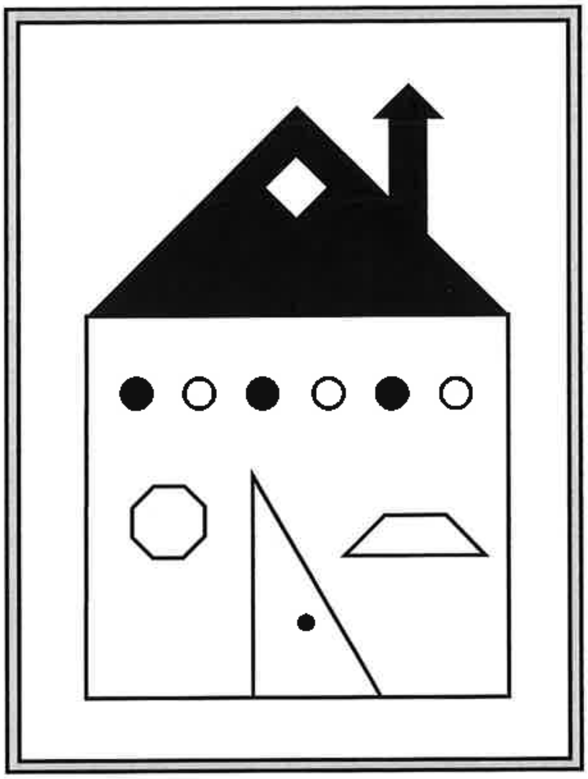

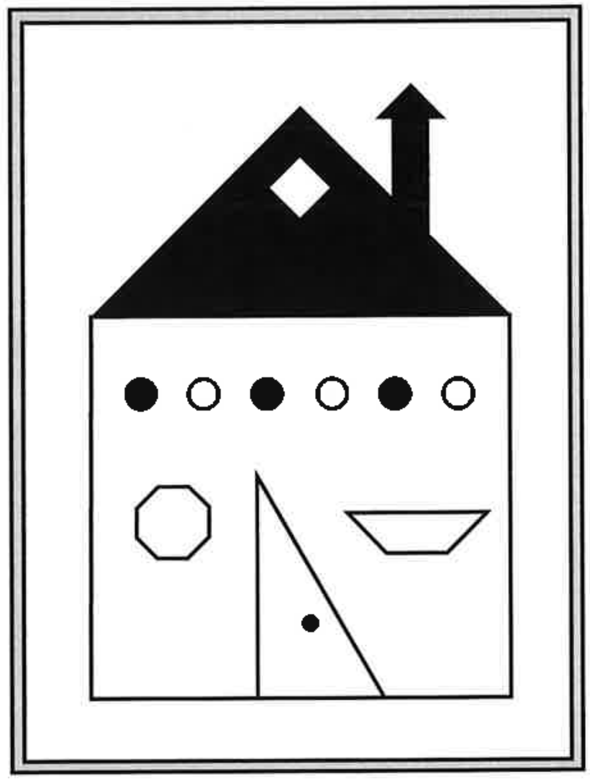

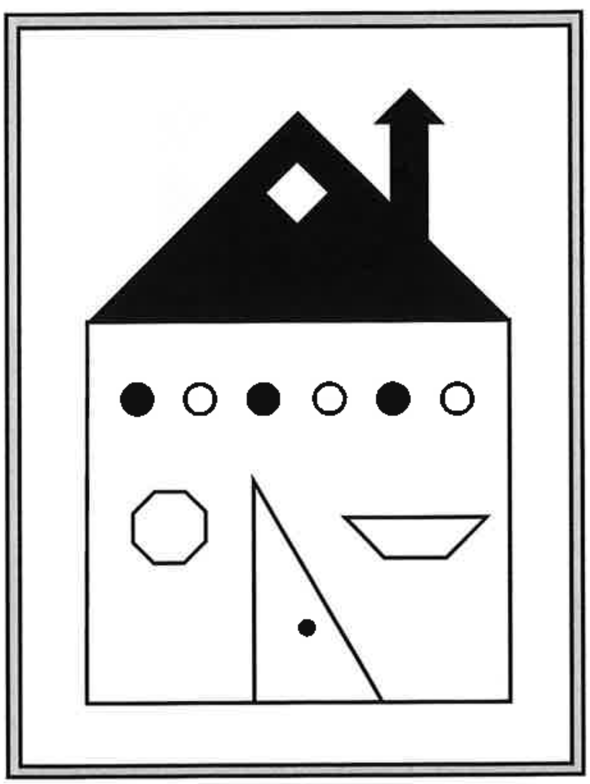

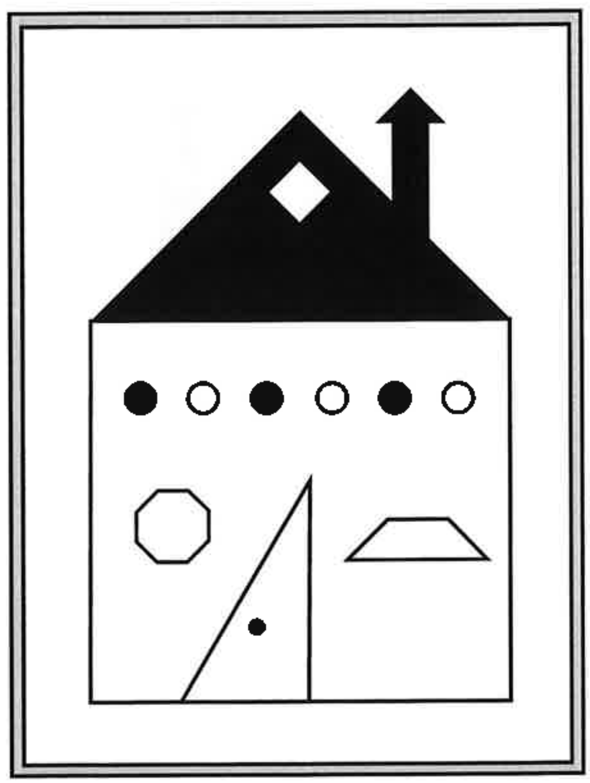

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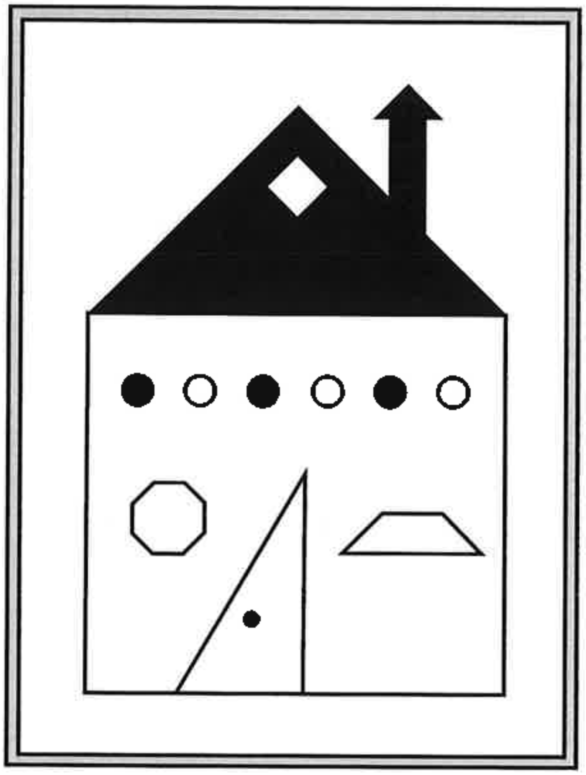

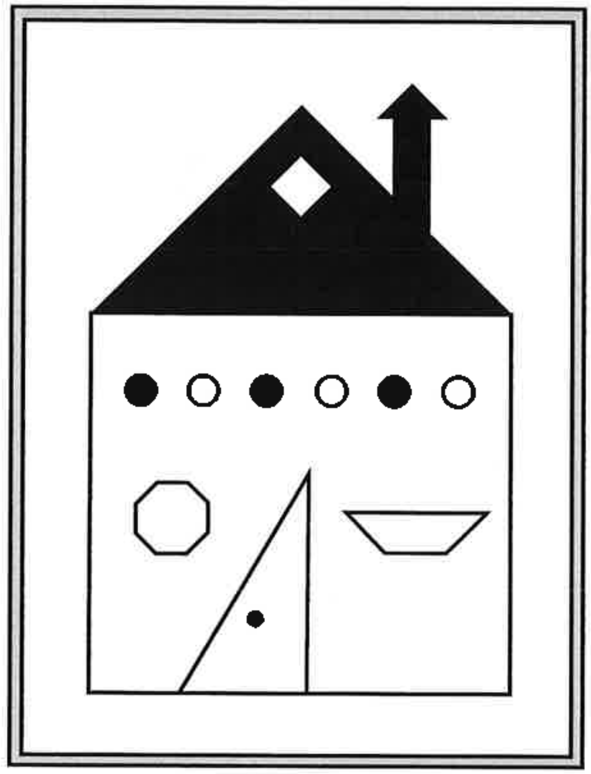

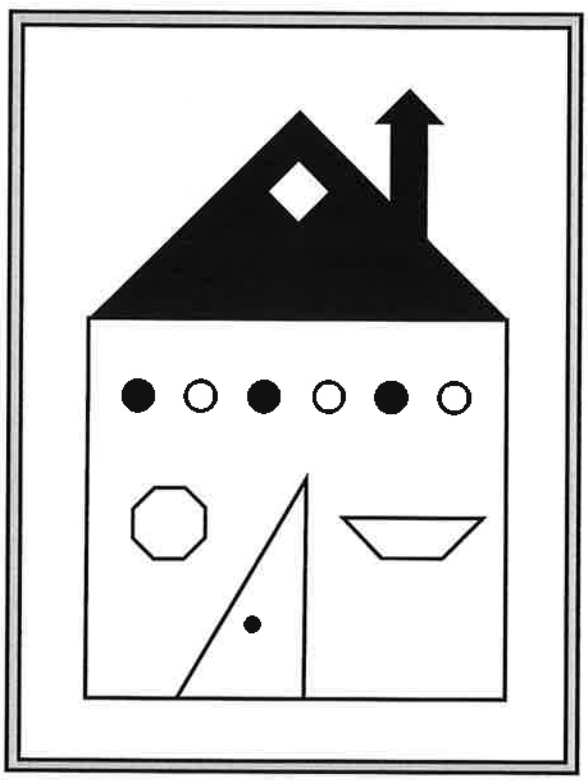

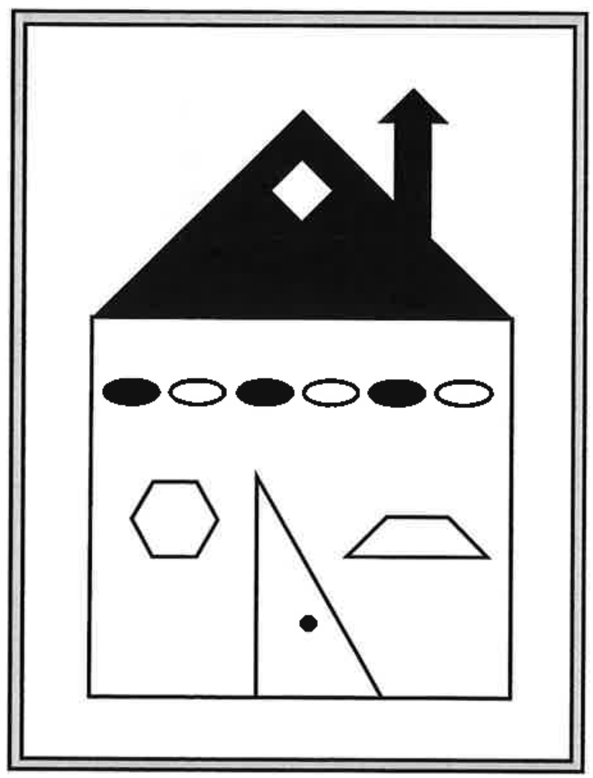

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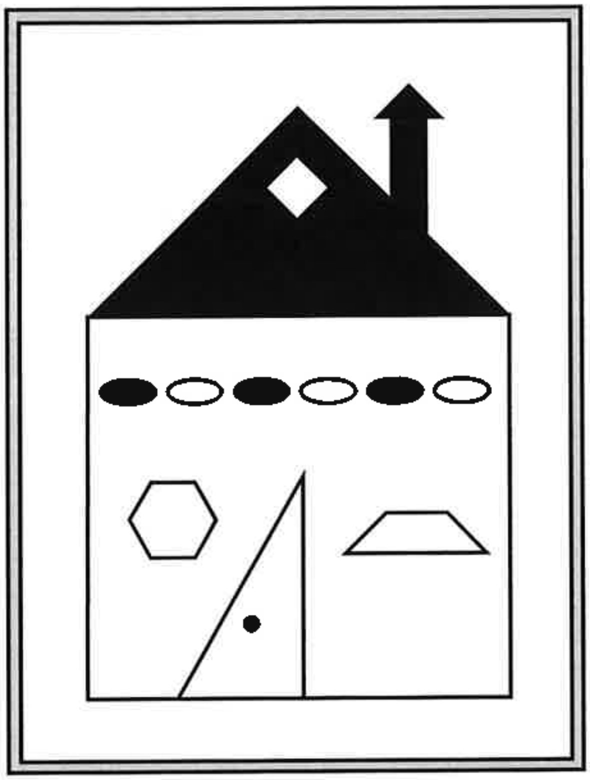

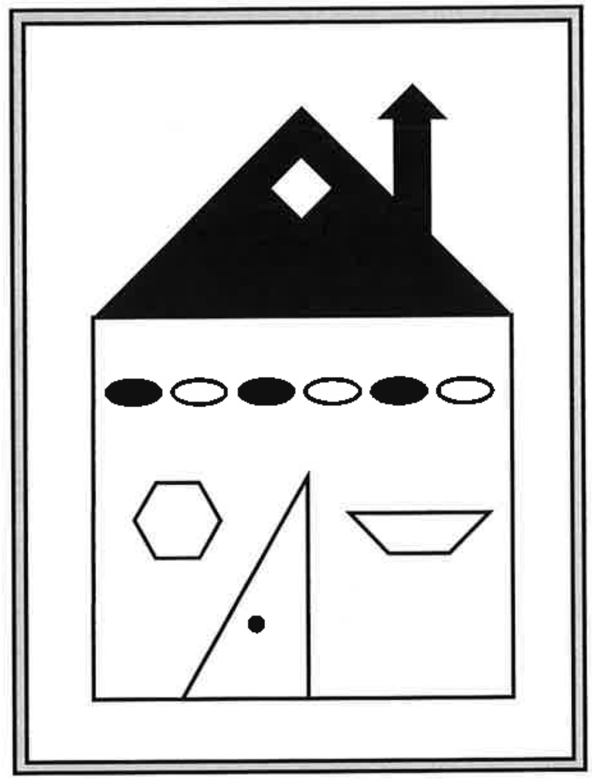

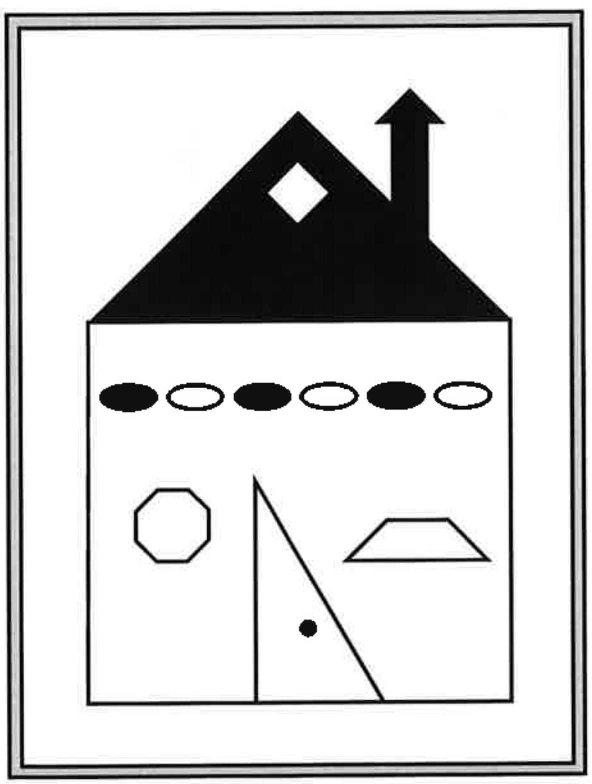

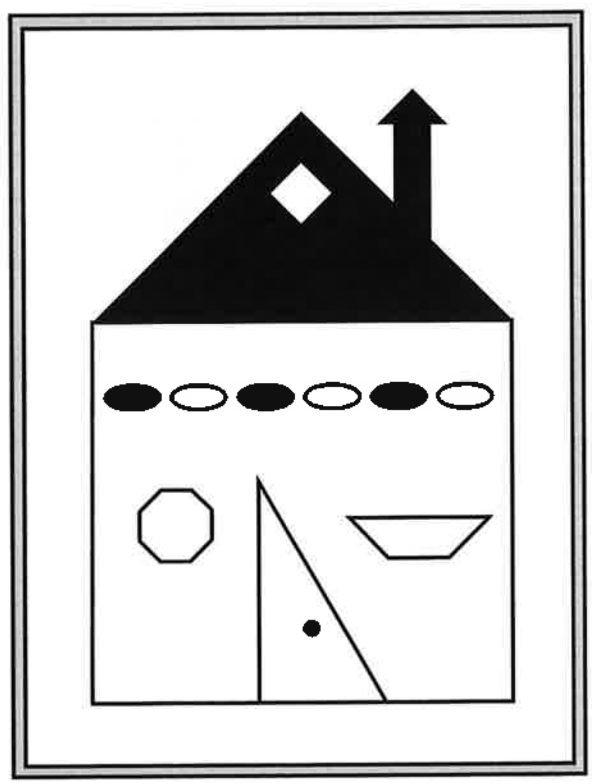

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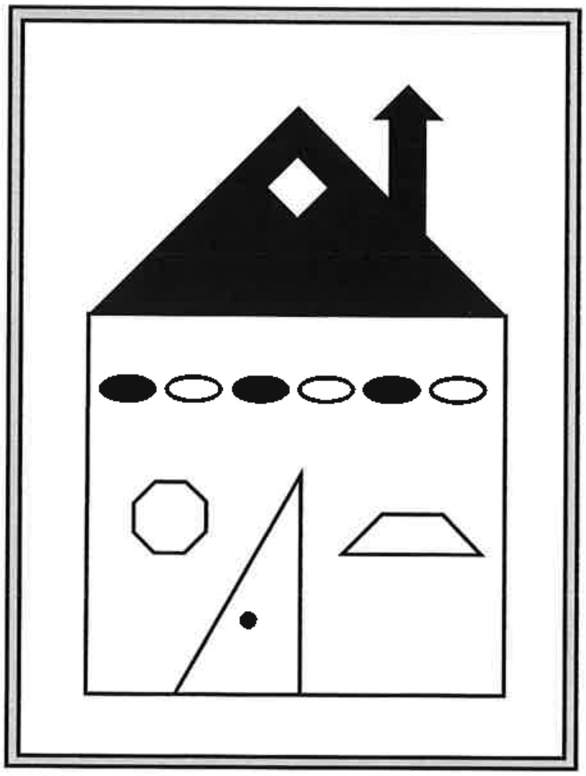

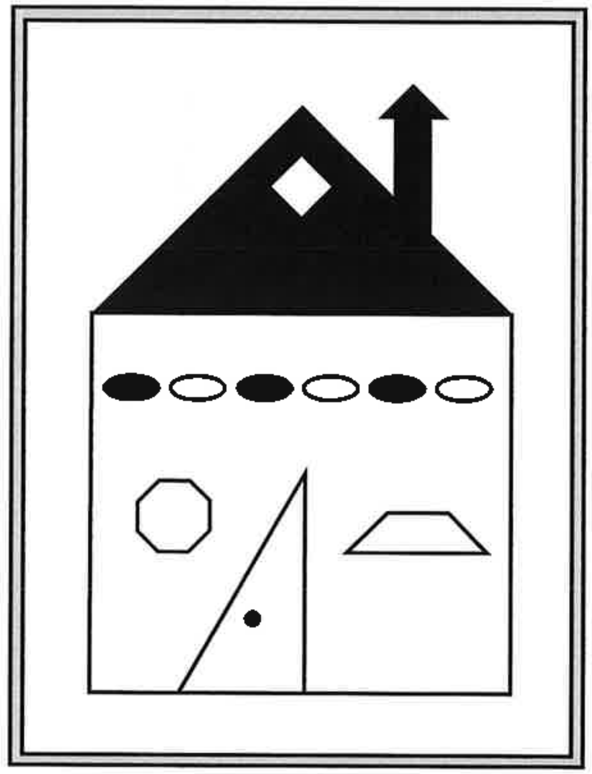

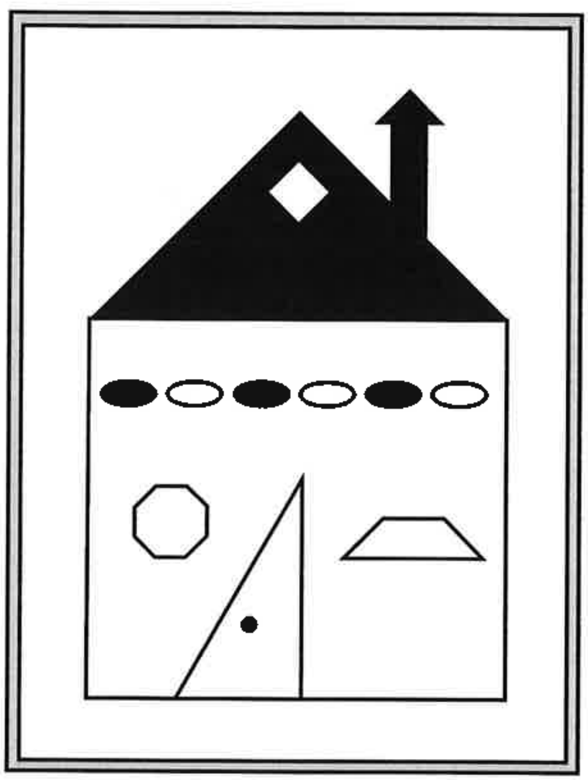

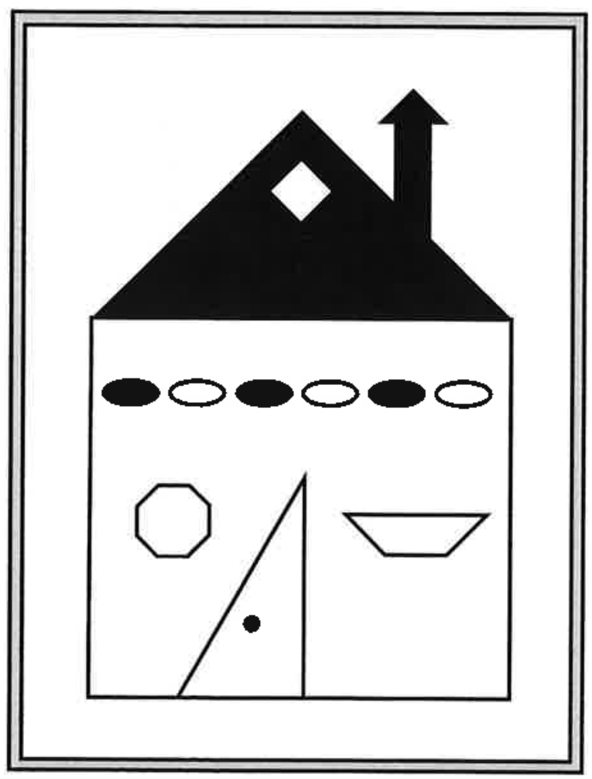

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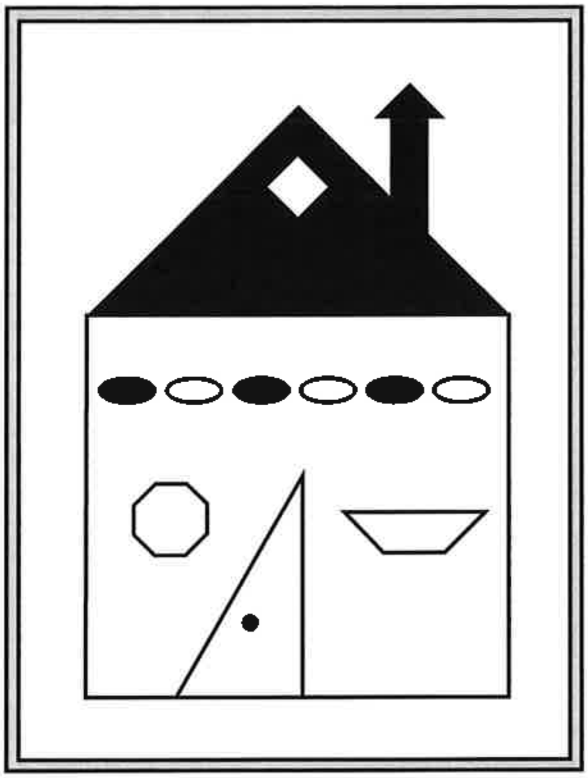
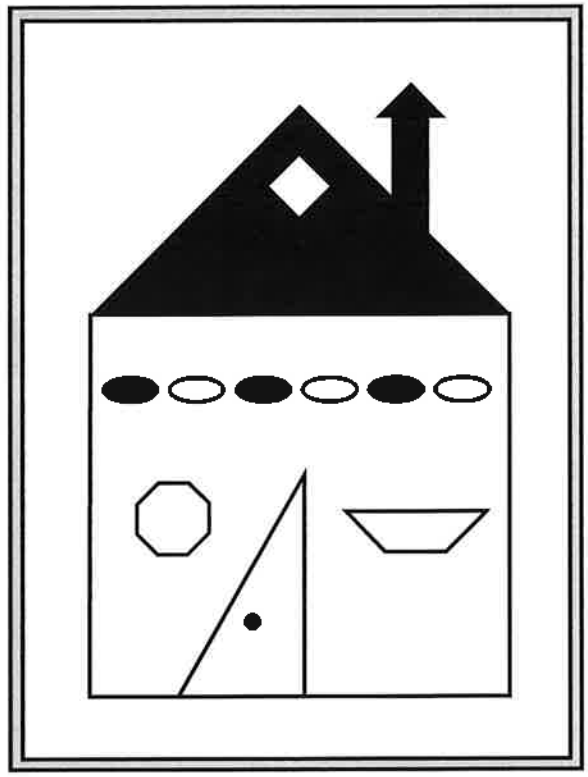




1-Inch Graph Paper

$\frac{1}{4}$-Inch Graph Paper

1-Centimeter Graph Paper

1. Explain each of the following problems in terms of the model of hot and cold cubes. Each explanation should include a statement of how the temperature changes overall.

   a. $4 - (-3)$

   b. $4 \cdot (-5)$

2. **a.** Find the missing entries for the In-Out table.

   b. Describe the function represented by the table both in words and by using an algebraic expression.

In	Out
4	7
8	15
−2	−5
10	?
−5	?
?	31

3. An **isosceles triangle** is a triangle in which at least two of the angles are equal.

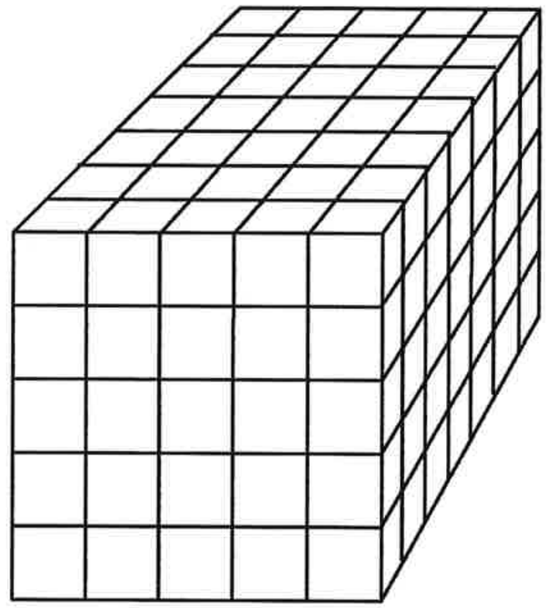
   The figure at right represents a general isosceles triangle in which $\angle B$ and $\angle C$ are equal.

   a. Suppose $\angle B$ is 60°. Find the size of $\angle A$.

   b. Suppose $\angle B$ is 72°. Find the size of $\angle A$.

   c. Make an In-Out table and develop an expression that will tell you the size of $\angle A$ in terms of the size of $\angle B$. That is, $\angle B$ should be the *In* and $\angle A$ should be the *Out*.

This problem is like the border problem, but it involves cubes instead of squares.



Imagine that you have a rectangular solid with dimensions 5 inches by 5 inches by 7 inches. This large rectangular solid is made up of smaller cubes. Each small cube is 1 inch on every edge.

Someone comes along and paints the large rectangular solid on all of its faces, including the bottom. None of the paint leaks to the inside.

How many of the small cubes have paint on them?

How many have only one face painted?

How many have two faces painted?

Your write-up of this problem should have these components:

- *Problem Statement:* Restate the problem in your own words.
- *Process:* Show all the work you did in solving this problem. If you used tables or diagrams, include them.
- *Solution:* Give your solution and justify it so that someone else will be convinced that your solution is correct.
- *Extra credit:* Suppose the dimensions of the rectangular solid are 5 inches by 5 inches by n inches. Find a formula that gives the number of small cubes with only one face painted.

Patterns Guide for the TI-83/84 Family of Calculators

This guide gives suggestions for selected activities of the Year 1 unit *Patterns.* The notes that you download contain specific calculator instructions that you might copy for your students. NOTE: If your students have the TI-Nspire handheld, they can attach the TI-84 Plus Keypad (from Texas Instruments) and use the calculator notes for the TI-83/84.

Students will come in with a wide range of knowledge about graphing calculators and other technologies. The goal is for technology to become a tool that helps students ask and explore interesting questions. Students will have an opportunity to learn new things about graphing calculators during the open-ended *Calculator Exploration*. Until then, keep calculators available and let students discover and teach each other.

For one activity, this guide provides support for required calculator use in the text. For other activities these notes contain ideas about how to use calculators for enrichment or extension of the activities or hints and suggestions for pitfalls to avoid.

POW 1: The Broken Eggs: In their work on the first POW, students may wish to use their calculators to list possibilities for the number of eggs. For example, when they consider the statement "She knows that when she put the eggs in groups of three, there was one egg left over," students might generate the sequence of numbers that match this statement on their calculators: 4, 7, 10, 13, and so on. In calculations that involve performing the same operation (in this case, adding 3) over and over again, the calculator provides a shortcut. This shortcut is described in the *Patterns Notes* in the section "Using the Previous Answer."

Calculator Exploration: In this activity, students explore the capabilities of their graphing calculator without structured directions. Teachers have found that relatively unstructured exploration can help build students' confidence about using technology. Students realize that they can figure out a great deal without asking for directions or help. For this reason, use the notes "TI Calculator Basics" conservatively with students. You might review the material to remind you of the things you will eventually want to bring out in discussions if they are not mentioned by students.

1-2-3-4 Puzzle: Students who use their calculators for this activity will want to find the square root command and the factorial command. They should also know that the calculator uses parentheses exclusively as grouping symbols. (That is, brackets and braces are reserved for other contexts.)

Uncertain Answers: The TI calculator follows the order-of-operations rules. Nevertheless, encourage students to use parentheses in long calculations. Parentheses make calculations easier to follow both on and off the screen.

Extended Bagels: In this activity, students use algebraic expressions to describe the relationship between two columns of a table. You can use algebraic expressions to generate tables like these on the TI calculator. You probably don't want students to work with these tables on a calculator until they are more comfortable with the ideas of tables and algebraic expressions. The instructions in the section "Using Lists to Build a Spreadsheet" describe how to create the table of values containing Marcella's bagel count.

Two calculator features create tables: Table and List. For instructions on using the Table feature, see "Graphing Basics" in *The Overland Trail Notes*. You will notice that the List features allow the calculator to function like a simple spreadsheet.

Do It the Chefs' Way: Students frequently confuse the negative key with the subtraction key. The negative key, which is located beside the decimal key and has a negative in parentheses, changes the sign of the calculation that follows it. The subtraction key, which is located above the addition key, subtracts the next term from the previous term. You cannot begin a calculation with the subtraction key. Use the negative key to represent negative numbers. On the screen, a negative is shorter and raised when compared to subtraction.

Add It Up: It is possible to use [2nd] [**LIST**] to build a sequence. It is also possible to sum this sequence. The steps to do this are fairly complicated, and the notation is different from the traditional sigma (Σ) notation, so avoid showing students the calculator procedure right away. However, if your students become comfortable with the activity and you have some extra time, they will probably enjoy seeing how quickly the calculator can sum complicated sequences.

Programming Borders: The note "Programming Your Calculator for Borders" is designed to assist with this activity. It does not address the general concept of a program, so you will still want to discuss that with the class. Instead, it focuses on the mechanics of creating and entering a simple program on a TI calculator.

The *Patterns Notes* contain a simple program BORDER, or BORDER1. There are two modifications: BORDER2 includes some display; BORDER3 behaves the way the supplemental activity in the textbook implies and is advanced, requiring some commands not mentioned in the notes. To save time and prevent transcription errors, you could load the programs BORDER1.8xp, BORDER2.8xp, and BORDER3.8xp in student calculators.

```
PROGRAM:BORDER
:Input S
:4S–4  K
:Disp K
```

```
PROGRAM:BORDER2
```

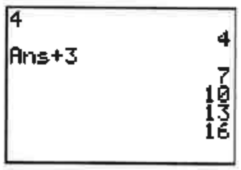
```
:Disp "SIDE LENGTH?"
:Input S
:4S–4→K
:Disp "TILES:",K
```

```
PROGRAM:BORDER3
:ClrHome
:Disp "HOW WIDE IS THE"
:Disp "GARDEN?"
:Input W
:Disp "HOW LONG IS THE"
:Disp "GARDEN?"
:Input L
:Disp "HOW WIDE IS THE"
:Disp "BORDER?"
:Input B
:2*B*(W–2B)+2*B*(L–2B)+4*B*B→T
:(W–2B)*(L–2B)→S
:3*T+0.20*S→C
:ClrHome
:Disp "YOU WILL NEED"
:Output(2,1,T)
:Output(2,6,"TILES AND")
:Output(3,1,"WILL HAVE TO")
:Output(4,1,"COVER")
:Output(4,7,S)
:Output(5,1,"SQUARE FEET WITH")
:Output(6,1,"TOPSOIL.")
:Output(7,1,"THIS WILL COST")
:Fix 2
:Output(8,1,C)
:Float
:Output(8,9,"DOLLARS.")
```

Using the Previous Answer

Sometimes you want to perform a sequence of calculations in which each answer builds on the previous one.

For instance, suppose you want to generate the sequence 4, 7, 10, . . . , in which you add 3 to generate the next term every time. Press [4] [ENTER] to get the first line of this display, and then [+] [3] [ENTER] to get the next line. The calculator assumes you are adding onto the previous answer, in this case, 4.



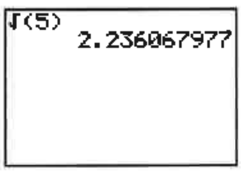
After you perform an operation in this manner, the calculator remembers the operation. Repeat the operation as many times as you wish by successively pressing [ENTER].

You can also use the previous answer in any part of the next calculation by pressing [2nd] [ANS].

TI Calculator Basics

Many of the keys on the TI calculator have two other operations in addition to the primary operation written on the key itself. The second operation appears in small letters above the key on the left. The ALPHA operation appears above the key on the right.

Using the [2nd] Key



The second operation usually performs an operation that is the inverse, or opposite, of the primary operation on the key itself. For example, the [x^2] key has the square-root function as its second operation, because squaring and finding the square root often work as inverses. Notice that the [ON] key has OFF as its second operation.

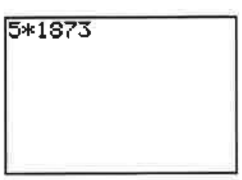
To indicate a second operation, this calculator guide uses the [2nd] key and then shows the operation itself in brackets. For example, [2nd] [√] indicates the square-root function above the [x^2] key.

Now try it yourself. Use the [2nd] [√] key to find the square root of five. The square root function includes an open parenthesis, so you should close the expression within the radical with the close parenthesis. Check your answer with the [x^2] key.

Returning to the Home Screen

There are several different displays, or screens on your TI calculator. The main screen is called the home screen. If you find yourself at another screen, press [2nd] [QUIT] to return to the home screen.

Editing What You Type



To clear your home screen, press [CLEAR].

Enter a calculation like the one shown here, but don't press [ENTER] yet. You can use this calculation to practice your editing.

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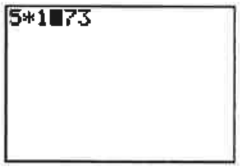
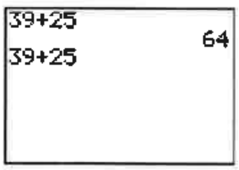


To replace a character or command, use the arrow keys to place the cursor over the character or command you wish to replace. Then type the new character.

To delete an entry, place the cursor over the entry you wish to delete and press [DEL].

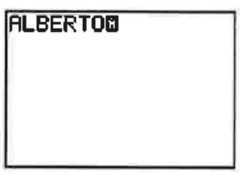
To insert a character or command, place the cursor on the character that will follow your new entry and press [2nd] [INS]. Then type what you want to insert.

Recalling a Previous Entry



To recall a previous entry, press [2nd] [ENTRY]. This will repeat the last entry you typed onto your screen. If you press [2nd] [ENTRY] repeatedly, you will move back through your work one line at a time, so you can actually recover something you did several steps earlier.

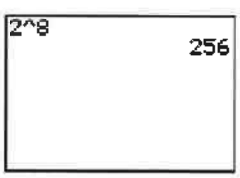
Using [ALPHA] and [A-LOCK]



The [ALPHA] key puts you in ALPHA mode, which allows you to use the secondary operations above the keys on the right. Most of these are letters of the alphabet. For example, pressing [ALPHA][A] produces the letter A on your screen.

The [ALPHA] key only affects the very next key you press. That is, ALPHA mode lasts for only one key stroke. Press [2nd] [A-LOCK] to put your calculator into ALPHA mode so that it stays in this mode. Now write your name on your calculator screen.

Working with Exponents



We often use the ^ symbol to indicate an exponent. The TI calculator uses this symbol, too. For example, you use the keystrokes [2] [^] [8] to calculate 2^8.

Continued on next page

We raise numbers to the power 2 often enough that the calculator provides the x^2 key for this purpose.

Using the MODE Key

Press MODE. This key gives you the option of controlling certain aspects of how the calculator displays and interprets information. (The screen shown here is a TI-84 Plus screen. Other calculators have slightly different mode options, so their screens will look different.)

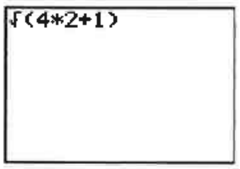
The first two lines of this display control the way numbers are displayed on the home screen. On the first line highlight **Normal** for now. The **Sci** option refers to scientific notation, which you will not use in the IMP curriculum until Year 2. (*Note:* If the calculator displays a very large or very small number, it will use scientific notation no matter what option you highlight in the MODE screen.) The **Eng** option refers to engineering notation, which is a variation of scientific notation. You will not need this option in IMP.

On the second line highlight the **Float** option. When this option is in use, the calculator displays numbers using up to 10 digits, as well as a negative sign and a decimal point (if needed).

The third line refers to units for angle measurement. Highlight **Degree**. You will not need radian measurement of angles in IMP until Year 4.

Some of the other standard mode settings are **Func**, **Connected**, **Sequential**, **REAL**, and **FULL**. Highlight these if you have not already done so.

Finding the Square Root



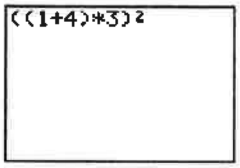
To calculate a square root, press [2nd] [√]. In order to find the square root of a long expression, use parentheses. The square root function includes an open parenthesis, so you should close the expression within the radical with the close parenthesis.

Using the Factorial Command

```
3+4!+1-2
```

To find the factorial symbol, press [MATH], highlight the **PRB** (probability) menu heading, and find ! in the submenu.

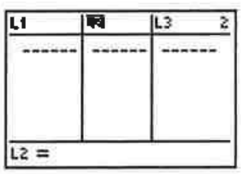
Using Parentheses as Grouping Symbols



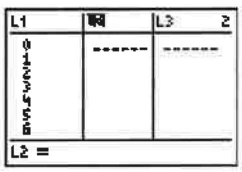
In printed mathematical expressions, you can use parentheses ( ), brackets [ ], and braces { } as grouping symbols. However, your TI calculator only uses parentheses as grouping symbols. Brackets and braces are used for other kinds of mathematical notation.

For example, to evaluate the expression $[(1+4)\cdot 3]^2$ on your calculator, you must replace the brackets with parentheses and enter $((1+4)\cdot 3)^2$.

Using Lists to Build a Spreadsheet

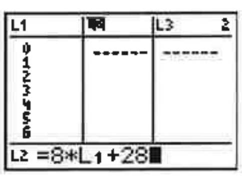


To work with a list, press [STAT]. **EDIT** is already selected, so just press [ENTER]. You should see at least three lists of numbers. For now, you are going to use list **L1** and list **L2**. If your lists already contain values, you will want to clear them first. (If others have been using the calculator, you might check with them before you clear their lists.) The fastest way to clear a list is to put the cursor on the list name (in this example the cursor is on **L2**) and then press [CLEAR] [ENTER].

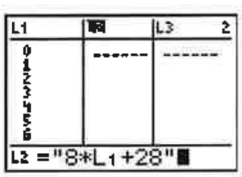


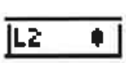
List **L1** will contain the number of bagels Marcella has when she gets home. List **L2** will contain the number of bagels Marcella started with. In the original example Marcella had no bagels when she got home, so the first number in list **L1** will be 0. Move the cursor to the first entry in list **L1** and press [0] [ENTER]. Continue to the numbers 1, 2, 3, and so on until you get at least as far as 10.

Now create list **L2**. You could figure out these entries and enter them one at a time, but instead you will use your understanding of variables and algebraic expressions to make the calculator build this list for you.



You have discovered that in order to generate a number in list **L2**, you take the corresponding number in list **L1**, multiply it by 8, and add 28 to that result. To tell your calculator to do exactly that for the entire list, put your cursor on the label **L2**. Now enter **8*L1+28**. (Find **L1** by pressing [2nd] [L1].) Press [ENTER] when you are done. Use your arrow keys to examine the entries in your completed list 2. If Marcella ends up with 5 bagels, how many did she start with?



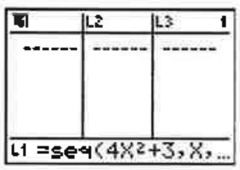


The "lock" symbol.

If you change the entries in list **L1**, you'll see that the entries in list **L2** do not change. That's because your calculator does not remember that list **L2** is calculated by **8*L1+28**. If you want list **L2** to automatically update when you change list **L1**, you can permanently attach the expression to the list using quotation marks, [ALPHA] ["]. Now the list has a "lock" symbol and its entries update when you change entries in list **L1**.

Building a Sequence

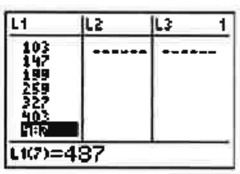
You can use your calculator to create a sequence. Your sequence will appear in a list. These instructions use an example from *Add It Up* and work with the sequence that follows the rule $4t^2 + 3$, starting with $t = 5$. To demonstrate the power of the calculator, let's have the sequence end with $t = 50$. (Later on we will add all the terms of this sequence.)



First, press [STAT] [ENTER] to show your lists. You will build your sequence in list **L1**, so place your cursor on **L1**. Clear this list if needed by pressing [CLEAR] [ENTER]. Put your cursor back on **L1** if necessary, and press [2nd] [LIST]. In the **OPS** menu highlight **seq(**. Press [ENTER] and return to your list.

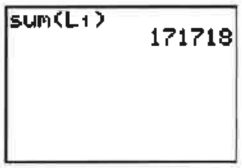
After the **seq(** function enter **4X²+3, X, 5, 50, 1)**.

The information in parentheses gives your calculator the following information in order: the algebraic expression that describes your sequence, the variable in the expression, the lower and upper limits of the variable, and finally the increment for the variable x. It's easier for the calculator to use x as the variable in this case. That's why you're entering x instead of t.



Press [ENTER]. Use the arrow keys to navigate around your list.

Finding the Sum of a Sequence



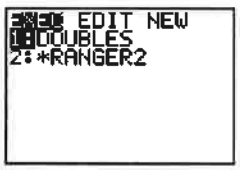
Once you have your sequence in a list, finding the sum is not difficult. Press [2nd] [QUIT] to return to your home screen. Then press [2nd] [LIST], but this time choose the **MATH** menu. Find the **SUM** function and press [ENTER] [2nd] [L1]. Then press [ENTER]. Your calculator will display the sum of all the numbers in your list!

Now use your calculator to sum another sequence of numbers.

Programming Your Calculator for Borders

For the supplemental activity *Programming Borders* you will program your calculator to help the employees at an outdoor supply store. Your program will use information about the dimensions of a rectangular garden and calculate the number of border tiles, amount of topsoil, and cost.

The steps here tell you how to create a simpler program that uses the rule $4s - 4$ to calculate only the number of tiles needed for a one-tile-wide border around a square garden. After writing this simple program, you can embellish for rectangular gardens and thicker borders. (You could load the program **BORDER1.8xp** on your calculator.)



Writing the Program

Press [PRGM] to get a display like the first one shown here. The screen shows a list of preexisting programs on your calculator, if any.

Highlight **NEW**, and press [ENTER].

To name your program, type **BORDER** and press [ENTER]. In this screen, the Alpha-Lock is on by default, so you can just press the necessary letters.

You will actually write your program in the next screen. Certain keys perform differently when you are in the programming editor.

Press [PRGM] to see the program menu. Highlight **I/O**, choose **Input**, and press [ENTER].

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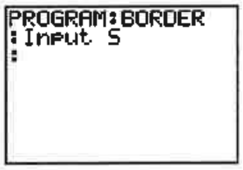
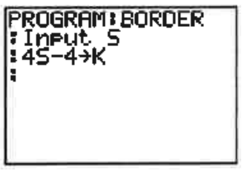
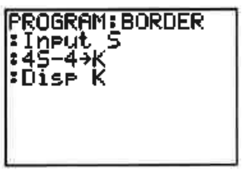


The **Input** instruction will appear on your programming screen. You want the program to store the input value for the variable **S**. Press [ALPHA] [S] to complete the program line as shown here. Press [ENTER] to go on to the next line of the program.

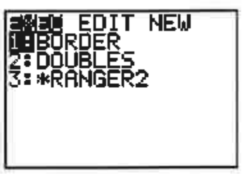


Now enter the instruction **4S−4→K**. The symbol → means "store"; you enter it by pressing the [STO▸] key. This computes the value of the expression **4S−4** for the value of **S** that was previously input, and it stores that result in a variable called **K**. In other words, the value of **K** is the number of tiles needed. Press [ENTER] to go on to the next line of the program.

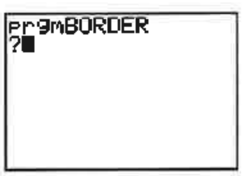


Press [PRGM], highlight **I/O**, choose **Disp** ("display"), and press [ENTER]. Tell the calculator what to display by entering [ALPHA] [K]. This is the end of your program.

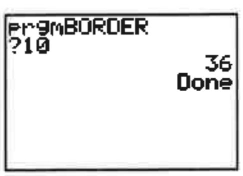
Running the Program



To run your program, press [2nd] [QUIT] to return to your home screen. Then press [PRGM]. Choose **EXEC** ("execute"), highlight the **BORDER** program, and press [ENTER]. **PrgmBORDER** will appear on the home screen.



Press [ENTER] to run the program. The calculator will show **?** and a flashing cursor, indicating that you need to enter a number as an input.



Input any number and press [ENTER]. The calculator will display the output according to the rule $4s - 4$, For example, if you input the number 10, the calculator will compute $4(10) - 4$ and display the result, 36, as shown here. It will also tell you that the program is finished, by displaying **Done**.

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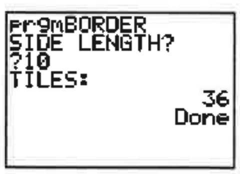
To run your program again, you can press [PRGM] and choose **BORDER** again. However, if you haven't yet performed any other tasks on your calculator, simply press [ENTER], and the program will repeat. Test your program a few times using different input values.

Editing the Program

To change your **BORDER** program, press [PRGM], highlight **EDIT**, and choose **BORDER**. Use the arrow keys to move around within your program and edit, insert, or delete whatever words or commands you wish. To make the program fully functional for an outdoor supply store, you'll ultimately need three input variables and three output (display) variables. This happens in the program **BORDER3.8xp**.

Making the Program User-Friendly



It is always nice to include messages so that a user knows what's happening as the program runs. For example, the simple program can be modified to include input and display messages, as shown here. (See the program **BORDER2.8xp**.)

To do this, go back into program editor mode and insert or edit **Disp** lines. You'll probably want to turn on the Alpha-Lock. Each text message must be in quotation marks so it isn't confused as a variable. When a display involves both text and a variable, use a comma in between. For spaces between words, use [ALPHA] [#] above the zero key.

Note: A message longer than 16 characters will not display completely on the home screen. If you want to include a longer message, break it into pieces and use several **Disp** commands. You might also want to research the **Output(** command, which allows you to display text or values at a specific location on the screen, and **ClrHome(**, which clears the home screen before displaying information.

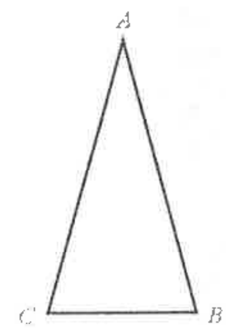
1. Explain each of the following problems in terms of the model of hot and cold cubes. Each explanation should include a statement of how the temperature changes overall.
   a. $4 - (-3)$
   b. $4 \cdot (-5)$
2. a. Find the missing entries for the In-Out table.
   b. Describe the function represented by the table both in words and by using an algebraic expression.

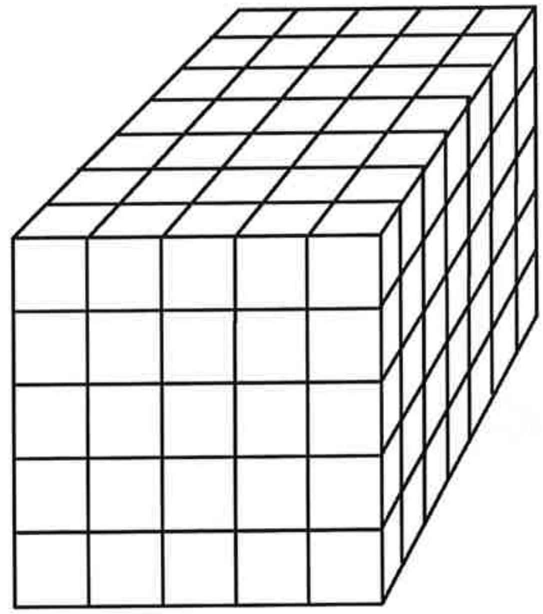
In	Out
4	7
8	15
–2	–5
10	?
–5	?
?	31

3. An **isosceles triangle** is a triangle in which at least two of the angles are equal.

   The figure at right represents a general isosceles triangle in which $\angle B$ and $\angle C$ are equal.

   a. Suppose $\angle B$ is 60°. Find the size of $\angle A$.
   b. Suppose $\angle B$ is 72°. Find the size of $\angle A$.
   c. Make an In-Out table and develop an expression that will tell you the size of $\angle A$ in terms of the size of $\angle B$. That is, $\angle B$ should be the *In* and $\angle A$ should be the *Out*.





This problem is like the border problem, but it involves cubes instead of squares.

Imagine that you have a rectangular solid with dimensions 5 inches by 5 inches by 7 inches. This large rectangular solid is made up of smaller cubes. Each small cube is 1 inch on every edge.

Someone comes along and paints the large rectangular solid on all of its faces, including the bottom. None of the paint leaks to the inside.

> How many of the small cubes have paint on them?
>
> How many have only one face painted?
>
> How many have two faces painted?

Your write-up of this problem should have these components:

- *Problem Statement:* Restate the problem in your own words.
- *Process:* Show all the work you did in solving this problem. If you used tables or diagrams, include them.
- *Solution:* Give your solution and justify it so that someone else will be convinced that your solution is correct.
- *Extra credit:* Suppose the dimensions of the rectangular solid are 5 inches by 5 inches by n inches. Find a formula that gives the number of small cubes with only one face painted.

I. The Antique Probability Cubes

Kate and Alejandro are studying probability in school. Their teacher brought in a collection of old dice that includes two rather peculiar dice.

One of these dice has a 0 where the 4 usually is. Its faces are 1, 2, 3, 0, 5, and 6. The other die has a 0 where the 1 usually is. Its faces are 0, 2, 3, 4, 5, and 6. Like normal dice, each face has an equal chance of being rolled.

The teacher has challenged Kate and Alejandro to investigate the probability of rolling various sums with this pair of dice.

Imagine that you have accepted this challenge.

1. Find the probability of rolling a sum of 7 with these dice. Show all your work.
2. Which sum is the most likely with this pair of dice? Explain why.

II. Another Coin Game

Consider this coin game.

You flip a coin. If the coin comes up heads, you lose, and your game is over. If the coin comes up tails, you then roll a regular die. Whatever number you roll is the number of dollars you win.

1. Make an area model to illustrate the possible outcomes of this game and their probabilities.
2. Compute the expected value for this game. Explain your work.
3. Suppose a county fair charges you \$2 each time you play the game.

   In the long run, who will come out ahead—you or the county fair? By how much per game? Explain your answer.
4. Suppose the county fair does not want its profit to average more than 20¢ per game. Describe a specific way to change the game so the fair could still make a profit charging \$2 per game.

III. Field Trip Expressions

Examine this list of variables. The variables represent the amounts of various items related to a class field trip to a museum.

V	the number of buses the school district has
F	the number of buses used for the school field trip
B	the number of boys on each bus
G	the number of girls on each bus
M	the number of miles to the museum
P	the number of miles per gallon each bus gets
C	the cost of a gallon of gas
T	the number of teachers on each bus
S	the number of gallons of gas a bus's gas tank will hold
D	the cost per person of admission to the museum

1. What, if anything, does each expression represent? Use a summary phrase, if possible.

   a. $B + G$ **b.** TV **c.** CSV

2. Write an expression for each of these summary phrases.

   a. The amount of money the museum will collect from the field trip

   b. The number of buses that did not go on the field trip

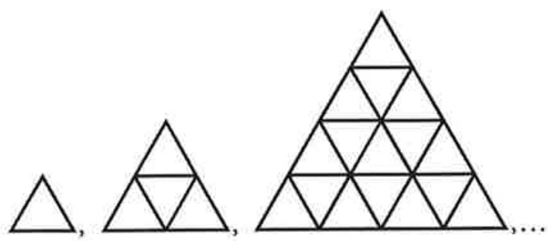
   c. The cost of gas for the field trip

3. Make up a "field trip expression" of your own using the list of variables above. Explain what your expression means.

IV. Mini-POW

Solve this problem. Then write up your results, describing your process and solution as you would for a Problem of the Week.

Consider this sequence of diagrams.



The first diagram is made of three line segments of equal length. You can think of the second diagram as made of nine segments of that same length, and so on.

1. Find the number of segments of the given length that would be needed for the 10th diagram in this sequence.
2. Explain how you would find the number of segments of the given length that would be needed for the 100th diagram in this sequence.

I. What's It All About?

For this problem, choose *two* of these three concepts.

a. similarity of polygons

b. standard deviation

c. In-Out table

Follow these steps for *each* of your *two* choices.

- Explain what the concept means.
- Describe a situation in which the concept could be used.
- Write a question that might arise in the situation you described that the concept would help to answer.
- Show the calculations you would need to make to answer the question. Make up sample numbers to illustrate how to do this.

II. The Phone Decision

The Beep-Beep cell phone company has two special calling plans for college students. Here are the prices for each plan.

Plan A

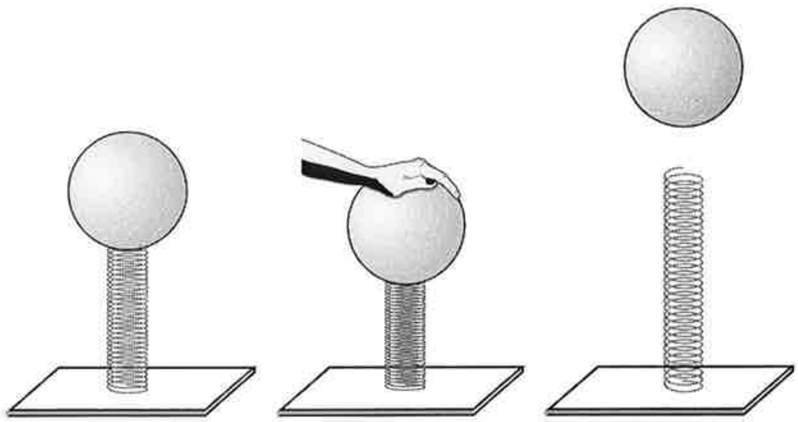
$8.00	base fee per month
$0.10	charge per minute

Plan B

$20.00	base fee per month
$0.03	charge per minute

1. a. Devon makes about 250 minutes of calls per month. Which calling plan is better for her? Why?
   b. Dylan makes about 100 minutes of calls per month. Which calling plan is better for him? Why?
2. a. Make an In-Out table that the Beep-Beep company could use to give advice to college students about Plan A. Your table should compare the number of minutes with the corresponding cost.
   b. Write a rule to describe the relationship in your table.
   c. Make a graph from this information.
3. a. Make an In-Out table that the Beep-Beep company could use to give advice to college students about Plan B.
   b. Write a rule to describe the relationship in your table.
   c. On the same set of axes that you used to make a graph for Plan A, now graph the information for Plan B.
4. Which calling plan do you think would be better for you? Why?
5. Imagine that you are responsible for helping new students to determine which calling plan might best suit their needs. What would you tell students about choosing a calling plan? Why?

III. Boingggg!!!!!



Students in a science class were studying how springs behave. They mounted a large spring on a table and rested a ball on top of the spring. Then they pushed the ball down. As they did so, they tightened, or *compressed,* the spring. When they let go, the ball was shot into the air.

The students did experiments in which they varied how far down they pushed the ball (the *amount of compression*). They measured how high the ball went, compared to its resting position, as carefully as they could. They collected the following data.

Amount of compression (in cm)	Height of ball from resting point (in cm)
3	10
5	30
8	70
10	110

1. Make a graph of this information.
2. Use any methods you think are appropriate to find a rule describing how the height depends on the amount of compression. Describe your method and state your results clearly.
3. Predict how high the ball would rise above its resting position if the spring were compressed 15 centimeters. Explain carefully how you made your prediction.
4. List other variables that might affect the height a ball rises from a spring. Describe experiments you could do to study the effects of these other variables.

IV. Mini-POW

Solve this problem. Then write up your results in a POW style. Describe your process, your solution, and any extensions or generalizations.

Matt ate a total of 100 raisins over a five-day period. After the first day, he ate six more raisins each day than on the previous day. How many raisins did he eat that first day?

Mathematics Unit Plan

Unit Title: Fireworks

Designed by: D. Fendel, D. Resek, L. Alper, S. Fraser, 2011, *Interactive Mathematics Year 2, Second Edition* (Emeryville, CA: Key Curriculum Press).

Grade: 9/10

Time Frame (Number of Lessons): 13 days

Summary of Unit

This unit uses a variety of contexts—projectile motion, areas and volumes, the Pythagorean theorem, and economics—to develop students' understanding of quadratics functions and their representations, as well as methods for solving quadratic equations.

The central problem involves a rocket used to launch a fireworks display. The height of the rocket is described by a quadratic function, and the questions involve vertices and *x*-intercepts, which are fundamental features of the graphs of quadratic functions.

Over the course of the unit, students strengthen their abilities to work with algebraic symbols and to relate algebraic representations to problem situations. Specifically, they see that rewriting quadratic expressions in special ways, either in factored form or in vertex form, provides insight into the graphs of the corresponding functions. Establishing this connection between algebra and geometry is a primary goal of the unit.

UNIT OVERVIEW

This unit addresses the following Common Core Standards for Math:

MATHEMATICAL PRACTICE STANDARDS

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

CONTENT STANDARDS

- CC.A-SSE.2: Use the structure of an expression to identify ways to rewrite it.
- CC.A-SSE.3a: Factor a quadratic expression to reveal the zeros of the function it defines.
- CC.A-SSE.3b: Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
- CC.A-APR.1: Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
- CC.A-APR.2: Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the reminder on division by x-a is p(a), so p(a) = 0 if an only if (x-a) is a factor of p(x).
- CC.A-APR.3: Indentify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.
- CC.A-REI.4: Solve quadratic equations in one variable.
- CC.A-REI.4a: Use the method of completing the square to transform any quadratic equation in x into an equation of the form (x-p)2 = q that has the same solutions. Drive the quadratic formula from this form.
- CC.A-REI.4b: Solve quadratic equations by inspection, taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as a +/- bi for real numbers a and b.
- CC.F-IF.7a: Graph linear and quadratic functions and show intercepts, maxima, and minima.
- CC.F-IF.7c: Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.
- CC.F-IF.8: Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
- CC.F-IF.8a: Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
- CC.F-IF.9: Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

- CC.F-BF.3: Identify the effect on the graph of replacing f(x0 by f(x) + k, k f(x), f(kx), and f(x+k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Big Idea// Unit Essential Question:

How to calculate how long a fireworks rocket will take to reach the top of its trajectory, how high it will be when in reaches the top, and how long it will take to fall back to the ground.

Unit Enduring Understanding(s):

- ✓ Quadratic Functions and their representations
- ✓ Graphs of quadratic functions
- ✓ Solving quadratic equations
- ✓ Understand the connection between algebra and geometry.

Knowledge and Skills:

- ✓ Express real world situations in terms of functions and equations.
- ✓ Use graphs to understand and solve quadratic equations.
- ✓ Understand the role of the vertex and the x-intercept in the graphs of quadratic equations.
- ✓ Know the significance of the sign of the x2 term in determining the orientation of the graph of a quadratic function.
- ✓ Use an area model to understand multiplication of binomials, factoring of quadratic expressions, and completing the square of quadratic expressions.
- ✓ Transform quadratic expressions into vertex form.
- ✓ Simplify expressions involving parentheses.
- ✓ Interpret quadratic equations in terms of graphs and vice versa.
- ✓ Estimate x-intercepts using a graph.
- ✓ Find roots of an equation using the vertex form of the corresponding function.
- ✓ Use the zero product rule of multiplication to solve equations by factoring.

Assessments:

Teachers will have a variety of opportunities to formatively assess student understanding, in addition to end of unit summative assessments. Each of these assessment tools are included within the lesson activities or at the end of this unit.

- ✓ Problem of the Week
- ✓ Written/Oral Assignments:
  - Using Vertex Form
  - Squares and Expansions
  - How Much Can they Drink?
  - Another Rocket
  - A Firework's Summary
  - A Quadratic Summary

- ✓ End of Unit In Class
- ✓ End of Unit Take Home

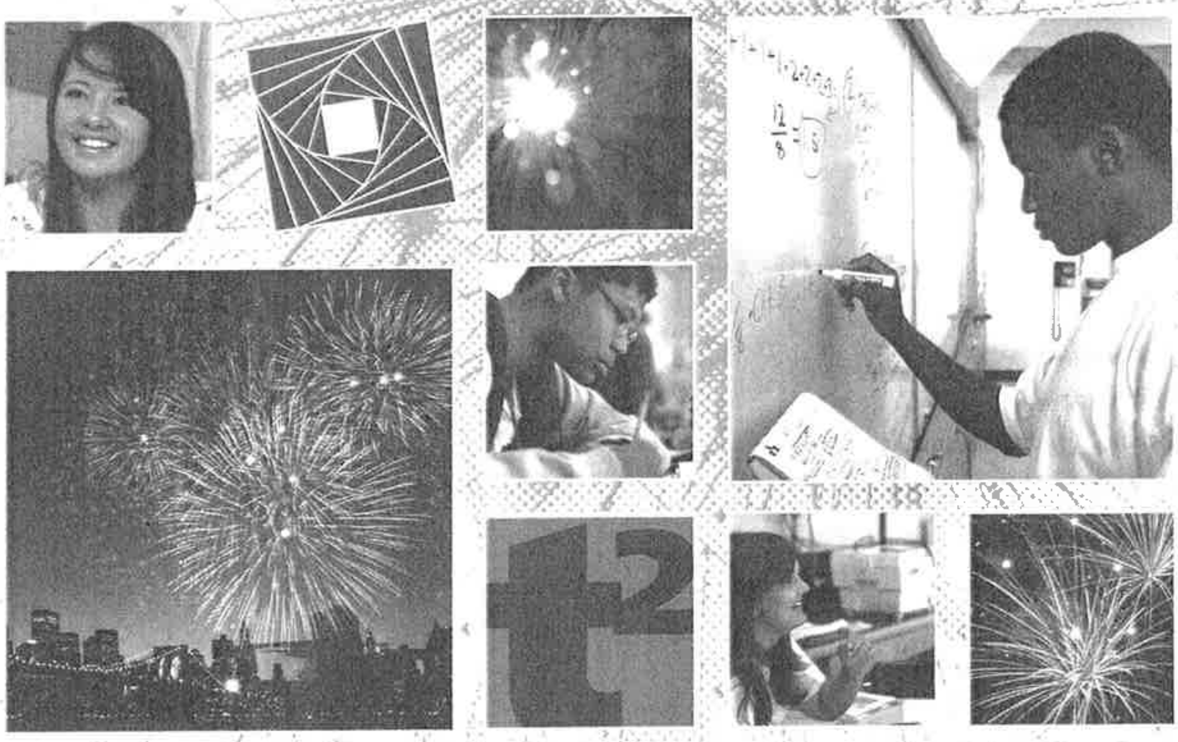
- ✓ Student Portfolio

Accommodations/Differentiation

IMP offers both extension and reinforcement activities to address the varying needs of students in the classroom. All of these activities are noted and including in the lesson activity pages.

Fireworks

Quadratic Functions, Graphs, and Equations



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Fireworks Unit Overview

Intent

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The central problem involves a rocket used to launch a fireworks display. The height of the rocket is described by a quadratic function, and the questions involve vertices and x-intercepts, which are fundamental features of the graphs of quadratic functions.

Over the course of the unit, students strengthen their abilities to work with algebraic symbols and to relate algebraic representations to problem situations. Specifically, they see that rewriting quadratic expressions in special ways, either in factored form or in vertex form, provides insight into the graphs of the corresponding functions. Establishing this connection between algebra and geometry is a primary goal of the unit.

Mathematics

Fireworks focuses on the use of quadratic functions to represent a variety of real-world situations and on the development of algebraic skills for working with those functions. Experiences with graphs play an important role in understanding the behavior of quadratic functions.

The main concepts and skills students will encounter and practice during the unit are summarized here.

Mathematical Modeling
- Expressing real-world situations in terms of functions and equations
- Applying mathematical tools to models of real-world problems
- Interpreting mathematical results in terms of real-world situations

Graphs of Quadratic Functions
- Understanding the roles of the vertex and x-intercept in the graphs of quadratic functions
- Recognizing the significance of the sign of the x^2 term in determining the orientation of the graph of a quadratic function
- Using graphs to understand and solve problems involving quadratic functions

Working with Algebraic Expressions
- Using an area model to understand multiplication of binomials, factoring of quadratic expressions, and completing the square of quadratic expressions

- Transforming quadratic expressions into vertex form
- Simplifying expressions involving parentheses
- Identifying certain quadratic expressions as perfect squares

Solving Quadratic Equations

- Interpreting quadratic equations in terms of graphs and vice versa
- Estimating x-intercepts using a graph
- Finding roots of an equation using the vertex form of the corresponding function
- Using the zero product rule of multiplication to solve equations by factoring

Progression

The unit begins with a graphical treatment of quadratic functions. The area model for multiplication is then used as the basis for the development of a set of manipulative skills. Students use these skills and graphs to solve problems drawn from a variety of real-world contexts. In each case, the interplay between symbolic and graphical representations of quadratic functions is emphasized. In addition, there are two POWs in the unit and the option of including a third from the list of supplemental activities.

Pacing Guides

50-Minute Pacing Guide (20 days)

Day	Activity	In-Class Time Estimate
	A Quadratic Rocket	
1	*Victory Celebration*	35
	POW 10: Growth of Rat Populations	10
	Homework: *A Corral Variation*	5
2	Discussion: *A Corral Variation*	15
	Parabolas and Equations I	20
	Parabolas and Equations II	15
	Homework: *Rats in June*	0
3	Discussion: *Rats in June*	10
	Parabolas and Equations III	15
	Vertex Form for Parabolas	25
	Homework: *Using Vertex Form*	0
4	Discussion: *Using Vertex Form*	15
	Crossing the Axis	25
	Homework: *Is It a Homer?*	10
	The Form of It All	
5	Discussion: *Is It a Homer?*	15
	A Lot of Changing Sides	30
	Homework: *Distributing the Area I*	5

6	Discussion: *Distributing the Area I*	15
	Views of the Distributive Property	35
	Homework: *Distributing the Area II*	0
7	Discussion: *Distributing the Area II*	15
	Square It!	25
	Homework: *Squares and Expansions*	10
8	Discussion: *Squares and Expansions*	15
	Presentations: *POW 10: Growth of Rat Populations*	20
	POW 11: Twin Primes	10
	Homework: *Vertex Form to Standard Form*	5
9	Discussion: *Vertex Form to Standard Form*	40
	Homework: *How Much Can They Drink?*	10
10	Discussion: *How Much Can They Drink?*	15
	Putting Quadratics To Use	
	Revisiting Leslie's Flowers	30
	Homework: *Emergency At Sea*	5
11	Discussion: *Emergency At Sea*	15
	Here Comes Vertex Form	30
	Homework: *Finding Vertices and Intercepts*	5
12	Discussion: *Finding Vertices and Intercepts*	15
	Another Rocket	30
	Homework: *Profiting from Widgets*	5
13	Discussion: *Profiting from Widgets*	15

	Pens and Corrals in Vertex Form	35
	Homework: *Vertex Form Continued*	0
14	Discussion: *Vertex Form Continued*	15
	Back to Bayside High	
	Fireworks in the Sky	30
	Homework: *Coming Down* and *A Fireworks Summary*	5
15	Discussion: *Coming Down* and *A Fireworks Summary*	25
	Intercepts and Factoring	
	Factoring	25
	Homework: *Let's Factor!*	0
16	Discussion: *Let's Factor!*	15
	Solve That Quadratic!	35
	Homework: *Quadratic Choices*	0
17	Discussion: *Quadratic Choices*	25
	Presentations: *POW 11: Twin Primes*	20
	Homework: *A Quadratic Summary*	5
18	Discussion: *A Quadratic Summary*	15
	"Fireworks" Portfolio	35
19	*In-Class Assessment*	45
	Homework: *Take-Home Assessment*	5
20	Exam Discussion	30
	Unit Reflection	20

90-minute Pacing Guide (13 days)

Day	Activity	In-Class Time Estimate
	A Quadratic Rocket	
1	*Victory Celebration*	35
	A Corral Variation	45
	POW 10: Growth of Rat Populations	10
	Homework: *Rats in June*	0
2	Discussion: *Rats in June*	10
	Parabolas and Equations I	20
	Parabolas and Equations II	15
	Parabolas and Equations III	15
	Vertex Form for Parabolas	25
	Homework: *Using Vertex Form*	5
3	Discussion: *Using Vertex Form*	20
	Crossing the Axis	25
	Is It a Homer?	45
	The Form of It All	
4	*A Lot of Changing Sides*	30
	Distributing the Area I	35
	Views of the Distributive Property	25
5	*Views of the Distributive Property* (continued)	15
	Distributing the Area II	35
	Square It!	30
	Homework: *Squares and Expansions*	10
6	Discussion: *Squares and Expansions*	10
	Presentations: *POW 10: Growth of Rat Populations*	15
	POW 11: Twin Primes	10
	Vertex Form to Standard Form	50
	Homework: *How Much Can They Drink?*	5

	Putting Quadratics to Use	
7	Discussion: *How Much Can They Drink?*	15
	Revisiting Leslie's Flowers	35
	Emergency At Sea	35
	Homework: *Here Comes Vertex Form*	5
8	Discussion: *Here Comes Vertex Form*	15
	Finding Vertices and Intercepts	40
	Another Rocket	30
	Homework: *Profiting from Widgets*	5
9	Discussion: *Profiting from Widgets*	10
	Pens and Corrals in Vertex Form	25
	Vertex Form Continued	25
	Back to Bayside High	
	Fireworks in the Sky	25
	Homework: *Coming Down* and *A Fireworks Summary*	5
10	Discussion: *Coming Down*	25
	Intercepts and Factoring	
	Factoring	25
	Let's Factor!	35
	Homework: *Solve That Quadratic!*	5
11	Discussion: *Solve That Quadratic!*	10
	Quadratic Choices	50
	Presentations: *POW 11: Twin Primes*	20
	Homework: *A Quadratic Summary* and *"Fireworks" Portfolio*	10
12	Discussion: *A Quadratic Summary* and *"Fireworks" Portfolio*	30
	In-Class Assessment	50
	Homework: *Take-Home Assessment*	10
13	Exam Discussion	30
	Unit Reflection	30
	A Quadratic Rocket	

Materials and Supplies

All IMP classrooms should have a set of standard supplies and equipment, and students are expected to have materials available for working at home on assignments and at school for classroom work. Lists of these standard supplies are included in the section "Materials and Supplies for the IMP Classroom" in *A Guide to IMP.* There is also a comprehensive list of materials for all units in Year 2.

Listed here are the supplies needed for this unit. General and activity-specific blackline masters are available for presentations on the overhead projector or for student worksheets. The masters are found in the *Fireworks* Unit Resources under Blackline Masters.

Fireworks

- No extra materials needed

More About Supplies

- Graph paper is a standard supply for IMP classrooms. Blackline masters of 1-Centimeter Graph Paper, ¼-Inch Graph Paper, and 1-Inch Graph Paper are provided so that you can make copies and transparencies for your classroom. (You'll find links to these masters in "Materials and Supplies for Year 2" in the Year 2 guide and in the Unit Resources for each unit.)

Assessing Progress

Fireworks concludes with two formal unit assessments. In addition, there are many opportunities for more informal, ongoing assessment throughout the unit. For more information about assessment and grading, including general information about the end-of-unit assessments and how to use them, consult the *Year 2: A Guide to IMP* resource.

End-of-Unit Assessments

This unit concludes with in-class and take-home assessments. The in-class assessment is intentionally short so that time pressures will not affect student performance. Students may use graphing calculators and their notes from previous work when they take the assessments.

Ongoing Assessment

Assessment is a component in providing the best possible ongoing instructional program for students. Ongoing assessment includes the daily work of determining how well students understand key ideas and what level of achievement they have attained in acquiring key skills.

Students' written and oral work provides many opportunities for teachers to gather this information. Here are some recommendations of written assignments and oral presentations to monitor especially carefully that will offer insight into student progress.

- *Using Vertex Form* will illustrate students' ability to pull together and use the various components of the vertex form of a quadratic.
- *Squares and Expansions* will demonstrate students' developing understanding of the technique of completing the square.
- *How Much Can They Drink?* will provide information on students' developing understanding of how to find the maximum value of a quadratic function to find the solution to a problem in context.
- *Another Rocket* will show how well students are prepared to address the unit problem.
- *A Fireworks Summary* is a reflective piece in which students summarize their work on the unit problem.
- *A Quadratic Summary* is a reflective piece in which students summarize their understanding of the big ideas of the unit.

Supplemental Activities

Fireworks contains a variety of activities at the end of the student pages that you can use to supplement the regular unit material. These activities fall roughly into two categories.

- **Reinforcements** increase students' understanding of and comfort with concepts, techniques, and methods that are discussed in class and are central to the unit.
- **Extensions** allow students to explore ideas beyond those presented in the unit, including generalizations and abstractions of ideas.

The supplemental activities are presented in the teacher's guide and the student book in the approximate sequence in which you might use them. Below are specific recommendations about how each activity might work within the unit. You may wish to use some of these activities, especially the later ones, after the unit is completed. In addition to these activities, you may want to use supplemental activities from Patterns that were not assigned or completed.

***What About One?* (reinforcement)** This problem-solving activity may be assigned anytime during in the unit. As in *POW 10: Growth of Rat Populations,* students have to be organized.

***Quadratic Symmetry* (extension)** Assign anytime after *Vertex Form For Parabolas*. Quadratic graphs are symmetrical about a vertical line through their vertex. In this activity, students use the general vertex form of a parabola to explore the relationship between the coordinates of corresponding points on either side of the line of symmetry.

***Subtracting Some Sums* (reinforcement)** You might use this activity with students who are having difficulty simplifying with negatives.

***Subtracting Some Differences* (reinforcement)** You might use this activity with students who are having difficulty simplifying with negatives.

***Choosing Your Intercepts* (reinforcement)** Assign anytime after *Crossing the Axis*. This activity gives students practice with finding an equation in vertex form for a parabola given the vertex and x-intercepts.

***A Lot of Symmetry* (reinforcement)** Assign anytime after *Distributing the Area II*. The special binomial product $(x + n)(x - n) = x^2 - n^2$ is introduced here in the context from *A Lot of Changing Sides*.

***Divisor Counting* (extension)** This problem-solving activity is best assigned after students have worked on *POW 11: Twin Primes*. The activity asks students to look for numbers that have a given number of divisors.

***The Locker Problem* (extension)** This classic problem makes a good follow-up to *Divisor Counting.*

***Equilateral Efficiency* (extension)** Assign this activity, which introduces Hero's formula for finding the area of a triangle in terms of its side lengths, after *Revisiting Leslie's Flowers*. With a little algebra, Hero's formula can be derived using the same approach used for finding the altitude of a triangle in *Revisiting Leslie's Flowers.*

***Check It Out!* (extension)** Use this activity with *Finding Vertices and Intercepts* or anytime during the unit. The activity introduces the notion that solving radical equations like $\sqrt{2x-3}=-5$ by squaring both sides may introduce extraneous roots.

***The Quadratic Formula* (extension)** Assign after *Coming Down*. The activity asks students to apply and then derive the quadratic formula. You may want to go through the derivation with them.

***Let's Factor Some More!* (reinforcement)** Assign after *Let's Factor!* This activity encourages students to use an area model to factor quadratics in which the coefficient of the x^2 term isn't 1.

***Vertex Form Challenge* (extension)** Assign after *Solve That Quadratic!* The activity gives students practice changing quadratic functions in standard form with leading coefficients other than 1 or −1 into vertex form. The problems involve fractions and decimals.

***A Big Enough Corral* (extension)** Assign after *Solve That Quadratic!* This activity explores quadratic inequalities. To do Question 2, students must be able to factor quadratics.

***Factors of Research* (extension)** Assign after *Solve That Quadratic!* The activity suggests further areas of exploration in the topic of factoring. Question 2 asks for a generalization of the difference of squares introduced in the supplemental activity *A Lot of Symmetry*.

***Make Your Own Intercepts* (extension)** Assign after *Quadratic Choices*. The activity builds on the idea that students can now easily find a quadratic that has two given x-intercepts. For example, for intercepts $x = 4$ and $x = 2$, the quadratic $y = (x - 4)(x - 2)$ will do. However, it isn't the *only* quadratic with those intercepts. All quadratics $y = a(x - 4)(x - 2)$ for any real number a will also work.

***Quadratic Challenges* (reinforcement)** Assign after *Quadratic Choices*. The activity offers three more quadratic equations for students to solve. (A graphing calculator would make finding the requested decimal answers too easy.)

***Standard Form, Factored Form, Vertex Form* (reinforcement)** Assign after *Quadratic Choices*. The activity pulls together the relationships among standard form, factored form, x-intercepts, vertex, and vertex form and makes a good group assignment.

A Quadratic Rocket

Intent

In these activities, students are introduced to the unit problem and begin to explore the properties of the parabolic graphs of quadratic functions.

Mathematics

Victory Celebration introduces a classic problem in projectile motion in which a projectile travels along a path determined by the function $y = 160 + 92x - 16x^2$. This is a **quadratic function,** and its graph is a **parabola**. From an algebraic point of view, the key features of a parabola are

- its symmetry
- its turning point, called the *vertex*
- its intercepts

Other features, such as the focus and directrix, are important from a geometric point of view but are not studied in this unit.

In physics, quadratic functions are written with the constant first—in this case the height off the ground, 160 feet. In mathematics, quadratic functions are typically written with the x^2 term first. The physics format is used when referring to the unit problem, but the mathematics format is used elsewhere.

Students first explore the unit problem informally. Then they analyze the key features of parabolas and how they connect to the values of the parameters a, h, and k in the **vertex form** of the equation of a quadratic function, $y = a(x - h)^2 + k$.

Progression

The early activities introduce quadratic functions. The remaining activities ask students to make connections between the symbolic and graphical representations of these functions. In addition, students will begin work on the unit's first POW.

Victory Celebration

POW 10: Growth of Rat Populations

A Corral Variation

Parabolas and Equations I

Parabolas and Equations II

Rats in June

Parabolas and Equations III

Vertex Form for Parabolas

Using Vertex Form

Crossing the Axis

Is It a Homer?

Victory Celebration

Intent

This activity engages students in the unit problem. Students sketch the rocket situation, paraphrase the questions needed to solve the unit problem, and use a formula to find height values for specific time values.

Mathematics

The unit problem is a classic projectile-motion problem. By engaging in the unit problem in an informal way, students will explore some input and output numbers connected by the formula

$$h(t) = 160 + 92t - 16t^2$$

They will use the new vocabulary term **parabola** to describe the shape of the rocket's path and **quadratic function** to describe equations of the form $f(x) = ax^2 + bx + c$.

Progression

After reviewing the activity as a class, students work in groups on the questions. They then present their work, in the form of posters and oral reports, to the class.

Approximate Time

35 minutes

Classroom Organization

Whole class, then groups, followed by whole-class presentations

Doing the Activity

Have volunteers read the activity aloud. You might ask whether anyone has launched rockets in science classes or as a hobby.

Assign each group to prepare a poster displaying the information asked for in Questions 1 through 4.

Discussing and Debriefing the Activity

Have a few groups present the information on their posters to the class. As part of their work, they will probably have come up with steps like these:

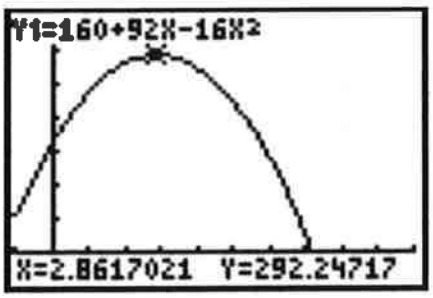
- Find the value of t that makes $h(t)$ a maximum.
- Substitute this value for t in the function $h(t)$ to find the rocket's maximum height.
- To find out how long the rocket will be in the air, solve the equation $h(t) = 0$.

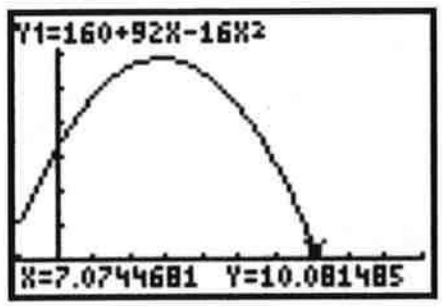
Have groups share their methods and results for Question 4. Students probably found (by graphing or guess-and-check) that the rocket will reach its maximum height in a little under 3 seconds and that the maximum height will be about 290 feet.

If students haven't yet graphed the height function, have them do so. Discuss how the graph confirms or corrects their ideas about the maximum height.

At which point on the graph is the function at its maximum? This means finding both coordinates of that point. At the maximum point in the rocket's path, the exact value of t is 2.875. (Students might not get exact answers.)

What does the landing time for the rocket mean in terms of the graph? Students likely found that the rocket would reach the ground after a little more than 7 seconds. This estimate involves interpreting the rocket's landing in terms of the function $h(t)$.





Students' explanations of these estimates will probably reflect that they tried to make $h(t)$ close to zero. Make this issue explicit by asking for an equation whose solution will give the time needed for the rocket to return to the ground. **What equation could you set up whose solution would tell you when the rocket hits the ground?** Students should see that trying to get $h(t)$ equal to zero is the same as solving the equation $160 + 92t - 16t^2 = 0$. (The exact value for the time for the rocket to land is irrational, so students cannot get the exact answer by graphing or guess-and-check.)

Key Questions

At which point on the graph is the function is at its maximum? This means finding both coordinates of that point.

What does the landing time for the rocket mean in terms of the graph?

What equation could you set up whose solution would tell you when the rocket hits the ground?

POW 10: Growth of Rat Populations

Intent

This POW provides an opportunity for students to organize data, devise solution methods, and describe and justify their work in writing.

Mathematics

In the problem, a female rat will produce a litter of six (three males and three females) every 40 days. Her female offspring will do the same, starting 120 days after their birth, and their female offspring will do the same. The time period in question, one year, allows for several generations of females to begin reproducing. The key mathematical skill students will encounter as they try to chart the growth of the rat population is the systematic organization and representation of data.

Progression

After a whole-class introduction, students work on the POW individually. In the upcoming activity *Rats in June,* they will partially solve this POW, finding the number of rats as of June 1.

Approximate Time

10 minutes for introduction

1–3 hours (at home)

20 minutes for presentations

Classroom Organization

Whole-class introduction with small-group discussion, then individuals, followed by whole-class presentations

Doing the Activity

You might have volunteers read the POW aloud and then give students about 10 minutes for discussion in groups. Schedule presentations for about seven days from today if you are on a 50-minute schedule.

Discussing and Debriefing the Activity

Though there is a unique, correct answer to this POW (1808), there are many ways to organize the data. After the presentations, give other students time to explain their organizational schemes. Focus the discussion on the process of working on the problem, without getting bogged down in the arithmetic.

Key Question

What were some schemes you used to keep the data organized?

Supplemental Activity

What About One? (reinforcement), a problem-solving activity that may be assigned anytime during the unit, requires students to be organized.

A Corral Variation

Intent

This activity presents another real-world situation involving a quadratic function, this one related to the corral problems in the unit *Do Bees Build It Best?*

Mathematics

In this classic area-maximization problem, students derive an expression for the area of the family of rectangles created by a fixed length of fence along three sides, with the fourth side provided by another long fence. Students will use this quadratic expression in one variable to calculate areas for several dimensions and then try to find the dimensions that enclose the maximum area.

Progression

After a brief introduction, students work on this activity individually and share results in a class discussion. The discussion should include how this problem relates to *Victory Celebration*.

Approximate Time

5 minutes for introduction

25 minutes for activity (in class or at home)

15 minutes for discussion

Classroom Organization

Individuals, followed by a whole-class discussion

Doing the Activity

Have students read through the activity. Answer any questions they have.

Discussing and Debriefing the Activity

For the discussion of Question 2, you might want to create a class table with headings "Length," "Width," and "Area" and collect answers.

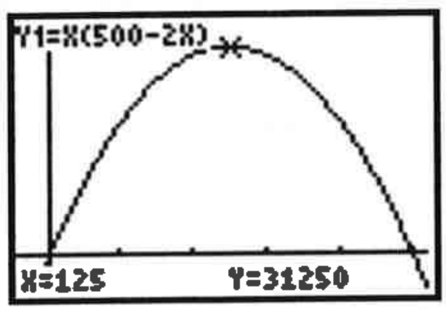
For Question 3, there may be several versions of the area expression, including these:

$x(500 - 2x) \qquad (500 - 2x)x \qquad 500x - 2x^2 \qquad 2[x(250 - x)]$

For now, don't try to show these to be algebraically equivalent; the distributive property will be presented in *The Form of It All*. Instead, simply show that each gives the same output for any input.

Now explore with the class what the activities *Victory Celebration* and *A Corral Variation* have in common, in terms of both the graphs and the algebra. Here are some issues to address about the graphs of these problems.

- The graphs are not linear; instead, they go up and then down.
- The graphs each have a unique maximum. In both problems, a primary goal is to find this unique maximum, both its x- and y-coordinates. (In *Victory Celebration,* another major goal is to find the positive x-intercept.) Mention that some real-world problems might involve a minimum rather than a maximum, and draw a graph to illustrate a quadratic function with a minimum point. Identify this point on the graph, whether maximum or minimum, as the **vertex**. You might use the phrase "extreme value" to refer to this point generically, without specifying whether it is a maximum or minimum.
- The graphs are symmetrical. (For *Victory Celebration,* this may not be as clear, as students have probably looked only at the portion of the graph from $x = 0$ to the positive x-intercept.) You might ask students to illustrate how this symmetry works. They might point out that in each graph, there are generally two x-values for each y-value and that the two x-intercepts for the corral graph are such a "matched pair."



Here are some issues to address about the algebra of these problems.

- Both problems involve an expression with an x^2 term. As needed, multiply out the expressions students present to show that each has an x^2 term, reviewing that x^2 means simply $x \cdot x$ and that $x \cdot 2x$ is $2x^2$. (All that is needed is a general awareness that the product has an x^2 term. The topic of multiplying binomials will be treated more fully in upcoming activities.) Note that one way to get an x^2 term is by multiplying two x terms or, more generally, by multiplying two linear expressions.
- Identify an expression like these as a **quadratic expression** and the function it defines as a **quadratic function**. Without getting too formal, clarify that a quadratic has at most three terms: an x^2 term, perhaps an x term, and perhaps a constant term.
- Introduce the term **standard form** for a quadratic function written as $y = ax^2 + bx + c$.

Key Questions

What dimensions give the maximum area? What is the maximum area?

Parabolas and Equations I

Intent

This activity and those that follow introduce the term **parabola** and familiarize students with parabolic-shaped graphs and their properties.

Mathematics

In the sequence of activities *Parabolas and Equations I, II,* and *III,* students use calculators to explore what happens to the graph of $y = x^2$ when the parameters a, k, and h in $y = ax^2$, $y = x^2 + k$, and $y = (x - h)^2$ are changed. The graphing exploration culminates in *Vertex Form for Parabolas*, in which the ideas from all three activities are pulled together to present the general vertex form, $y = a(x - h)^2 + k$.

In this first activity, students explore the relationship between changes in the parameter a in $y = ax^2$ and corresponding changes in the graph.

Progression

Students work on the activity individually, with help from group mates as needed.

Approximate Time

20 minutes

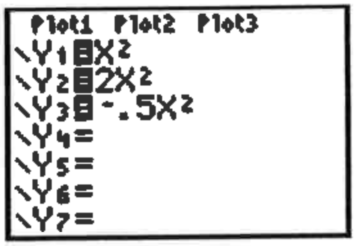
Classroom Organization

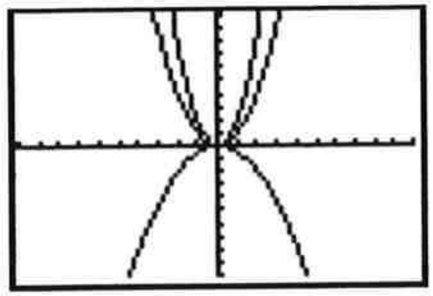
Whole-class introduction, followed by individual work within small groups

Doing the Activity

You might begin by demonstrating the use of a graphing calculator to graph the rocket's height function, $h(x) = 160 + 92x - 16x^2$. Point out that this is one member of a family called **quadratic functions.** The general form of this family is $y = ax^2 + bx + c$, where a, b, and c represent numbers. Ask, **What are the values of *a*, *b*, and *c* in the rocket function?**

Have students reset their calculators to the standard viewing window, $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. Have them store in Y_1 the simplest member of this family, $Y_1 = x^2$. You can suggest they keep $Y_1 = x^2$ as they work and put other graphs in Y_2, Y_3, and so on, as done in the following illustrations. This makes it easier to compare each graph with the simplest parabola and address the question, **With each change in *a*, what has changed in the graph?**





Circulate as students work, answering questions about calculator operation and reminding students to record what they find, including making sketches of the graphs. Make sure students are working through the material on their own, as using a graphing calculator is a basic mathematics skill that can be mastered only by practice.

Discussing and Debriefing the Activity

During the discussion, introduce the idea of *concavity*. Help students relate the terms *concave up* and *concave down* to the graphs and the equations.

If time allows, you might ask students to create original designs of their own using no more than five equations, including both linear and quadratic functions.

Key Questions

What are the values of *a*, *b*, and *c* in the rocket function?

With each change in *a*, what has changed in the graph?

Parabolas and Equations II

Intent

This activity continues the exploration begun in *Parabolas and Equations I*, in which students familiarized themselves with parabolic graphs.

Mathematics

In this second activity of the sequence *Parabolas and Equations I, II,* and *III,* students explore the relationship between changes in the parameter k in $y = x^2 + k$ and corresponding changes in the graph.

Progression

After successfully completing *Parabolas and Equations I,* students should begin work on this activity individually, with help from group mates as needed.

Approximate Time

15 minutes

Classroom Organization

Individuals within small groups

Doing the Activity

Remind students to reset their calculators to the standard viewing window, $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$.

Discussing and Debriefing the Activity

After most students have completed this activity, call the class together and have volunteers share their answers to Question 2. There is more than one way to make each design.

Then say, **Now we know how a and k change the graphs of $y = ax^2$ and $y = x^2 + k$. Put these two equations together, and we have $y = ax^2 + k$.**

Ask students to predict, without actually graphing, how the graph of each of the following equations would compare with the simplest graph, $y = x^2$.

$$y = 2x^2 + 3$$

$$y = -2x^2 - 3$$

$$y = 0.5x^2 - 4$$

$$y = -0.01x^2 + 5$$

Have students check their predictions by graphing the equations.

Rats in June

Intent

This activity will clarify some of the issues in *POW 10: Growth of Rat Populations,* as well as assure that students get started on the problem.

Mathematics

The key mathematical skill students will encounter as they try to chart the growth of the rat population is the systematic organization and representation of data.

Progression

In this activity, students will partially solve the POW, finding the number of rats as of June 1.

Approximate Time

40 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

Doing the Activity

Tell students that this activity will get them started on the POW and that as they work they should jot down any questions they have about the problem.

Discussing and Debriefing the Activity

Give students a few minutes to share results in their groups. Then ask, **How many rats, male and female, are there as of June 1?**

The goal of this discussion is to clear up any misinterpretations of the problem. In particular, students should see that the original female will have four litters by June 1 and that the females in the first litter will themselves each have a litter by that date. (As of June 1, there are 22 female and 22 male rats.)

Parabolas and Equations III

Intent

This activity continues the work from *Parabolas and Equations I* and *II,* in which students familiarized themselves with parabolic graphs.

Mathematics

In this third activity in the sequence *Parabolas and Equations I, II,* and *III,* students explore the relationship between changes in the parameter h in $y = (x - h)^2$ and corresponding changes in the graph.

Progression

After successfully completing *Parabolas and Equations I* and *II,* students should begin work on this activity individually, with help from group mates as needed. In the next activity, *Vertex Form for Parabolas*, the ideas from these three activities are pulled together when students work with vertex form, $y = a(x - h)^2 + k$.

Approximate Time

15 minutes

Classroom Organization

Individuals within small groups

Doing the Activity

Once again, remind students to reset their viewing windows to standard.

Discussing and Debriefing the Activity

After most students have completed this final activity in the sequence, call the class together and have volunteers share their answers to Question 2.

Then say, **Now we know how a, k, and h change the graphs of $y = ax^2$, $y = x^2 + k$, and $y = (x - h)^2$. Put these three equations together, and we have $y = a(x - h)^2 + k$.**

Ask students to predict, without actually graphing, how the graph of each of the equations here would compare with the simplest graph, $y = x^2$.

$$y = 2(x - 4)^2 + 3$$

$$y = -2(x - 4)^2 - 3$$

$$y = 0.5(x - 2)^2 - 4$$

$$y = -0.01(x + 2)^2 + 5$$

Have students check their predictions by graphing the equations.

If time allows, you might ask students to create another original design using no more than five equations, including both linear and quadratic functions.

Vertex Form for Parabolas

Intent

In this activity, the ideas from *Parabolas and Equations I, II,* and *III* are pulled together as students work with the general vertex form for quadratics, $y = a(x - h)^2 + k$.

Mathematics

The **vertex form** for quadratics, $y = a(x - h)^2 + k$, allows one to treat a parabola as a transformation of the "parent" graph $y = x^2$. The vertex form gives the parabola's vertex at the point (h, k). If $a > 0$, the parabola will be concave up; if $a < 0$, the parabola will be concave down. The absolute value of a will determine how vertically stretched or compressed the graph looks in the viewing window.

The activities in *A Quadratic Rocket* focus only on the relationship between the parameters in vertex form and the corresponding graphs. Later in the unit, students will learn how to convert quadratics in standard form, $y = ax^2 + bx + c$, into vertex form by completing the square.

Progression

After an opening class discussion, students work individually on this activity to re-create graphical designs.

Approximate Time

25 minutes

Classroom Organization

Whole class, then individuals within small groups

Doing the Activity

To introduce the activity, you might lead the class in a brief discussion.

We know how a, k, and h change the graphs of $y = ax^2$, $y = x^2 + k$, and $y = (x - h)^2$. Put these three equations together, and we have $y = a(x - h)^2 + k$.

Discussing and Debriefing the Activity

Have students share their equations in their groups. Then have each group choose one set of equations for each design and display them for presenting to the class. As solutions are presented, ask the class to comment on how closely the graphs of each set of equations match the given designs.

Although many answers are possible for each design, in Questions 1 and 2 the values of the parameters should reflect the symmetries in the graphs. For example, in Question 1, the vertex values for the parabolas on the upper left and upper right should have the same y-coordinate and the x-coordinates should be negatives of each other. The equations $y_1 = (x + 3)^2 + 2$, $y_2 = x^2$, and $y_3 = (x - 3)^2 + 2$ will work, with vertices at (−3, 2), (0, 0), and (3, 2).

Supplemental Activity

Quadratic Symmetry (extension) further explores quadratic graphs, which are symmetrical about a vertical line through their vertex. Students use the general vertex form of a parabola to explore the relationship between the coordinates of corresponding points on either side of the line of symmetry.

Using Vertex Form

Intent

Students apply the ideas developed in the last few activities about the graphical effects of the parameters in the vertex form of a quadratic equation.

Mathematics

The vertex form of the equation of a quadratic function, $y = a(x - h)^2 + k$, allows one to determine the vertex of the graph, the direction of the graph's concavity, and the relative amount of vertical stretch or compression of the parabola. In this activity, students derive equations to produce a particular graphical design and read the vertex coordinates from several equations.

Progression

Students work individually on the activity and discuss their findings in class.

Approximate Time

25 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

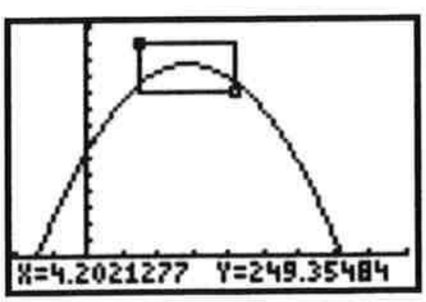
Individuals, followed by whole-class discussion

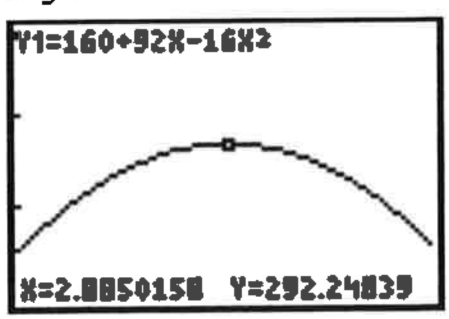
Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

Have several students or groups present their solutions to Question 3. The problem was to be done by estimating the coordinates of the vertex from the graph. The Zoom feature can be helpful in accurately estimating these values.





How might you write the rocket equation in vertex form? Based on the estimate for the vertex shown in the previous graph, students might propose an equation like

$$y = -(x - 2.88)^2 + 292$$

This will produce a parabola with the correct vertex and pointing in the right direction, but that still needs to be stretched vertically (by increasing $|a|$) to match the graph of $y = 160 + 92x - 16x^2$.

Crossing the Axis

Intent

In this activity, students begin to develop an understanding of the processes through which they might find the x-intercepts of the graph of a quadratic function.

Mathematics

In this activity, students will use knowledge about the location of the vertex and whether the value of a in $y = a(x - h)^2 + k$ is positive or negative (indicating the graph's concavity) to determine the number of x-intercepts a parabola has. In general, there are three cases: one, two, or no x-intercepts.

Students will also find the equation for a parabola given the vertex and one x-intercept. The values of a parabola's vertex determine the parameters h and k in the equation $y = a(x - h)^2 + k$. If, in addition, an x-intercept (or "zero") of the parabola is given, the value of the parameter a can be found by substituting 0 for y and the x-intercept value for x and solving.

Progression

Students work on the activity in groups and then share their work with the whole class.

Approximate Time

25 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

Students may need help with the logic for Questions 1 to 4. You may also want to work through the example preceding Question 5 with the whole class.

Discussing and Debriefing the Activity

Elicit answers for Questions 1 to 4. The goal is for students to recognize that the location of the vertex and the concavity of the graph determine the number of x-intercepts the graph of a quadratic function has.

Have volunteers share their work for Questions 5 and 6. Students might use the approach modeled in the student book to find the value of the parameter a.

Supplemental Activities

Subtracting Some Sums (reinforcement) gives students practice simplifying with negatives.

Subtracting Some Differences (reinforcement) gives students practice simplifying with negatives.

Choosing Your Intercepts (reinforcement) offers students practice with finding an equation in vertex form for a parabola given the vertex and *x*-intercepts.

Is It a Homer?

Intent

Students apply ideas from *Using Vertex Form* and *Crossing the Axis* in a real-world context.

Mathematics

In this activity, students treat the path of a baseball as a parabola, which allows them to assume that the flight of the ball is symmetrical: its highest point occurs halfway along its path from the ground up to that maximum height and back to the ground again. Given the location of this maximum point and one x-intercept, and treating the ground as the x-axis, students will use this information to find an equation for the parabolic path. Then they will use the equation to find a particular output (the height of the ball) for a given input (the location of the center field fence). Finally, they will determine whether the ball will clear the 15-foot fence.

Progression

Students work individually on this activity and discuss their findings in class. The discussion will also refer to their work on the previous exploratory activities.

Approximate Time

10 minutes for introduction

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

You might suggest that students read the activity and sketch the parabolic path of the ball from home plate.

Consider home plate to be (0, 0) with the x-axis running along the ground directly beneath the path of the ball and the y-axis indicating the height of the ball above the ground. Some students may want to place (0, 0) about two feet above home plate, where the bat meets the ball. Either positioning is fine.

Discussing and Debriefing the Activity

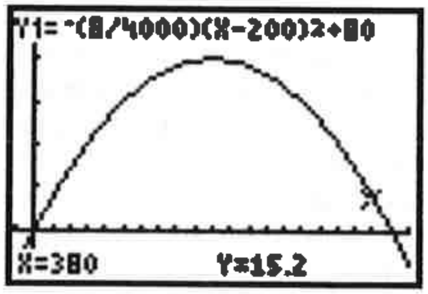
Have individuals or groups present their solutions. As needed, ask questions such as these listed here.

What are the coordinates of the vertex of the ball's parabolic path?

How did you find an equation for the ball's path?

Once you have an equation, how can you decide whether the ball will clear the fence for a home run?

The function is approximately $y = -0.002(x - 200)^2 + 80$. When $x = 380$, the value of y is 15.2. So, will the ball clear a 15-foot fence? Yes. In fact, because the batter will actually hit the ball from above home plate, the ball would clear the fence by an even greater margin. Note that these calculations do not take into consideration the effect of any wind.



You might complete this discussion by asking students to make up a similar problem for another context, such as a soccer kick at goal or a football kick for a field goal.

Key Questions

What are the coordinates of the vertex of the ball's parabolic path?

How did you find an equation for the ball's path?

Once you have an equation, how can you decide whether the ball will clear the fence for a home run?

The Form of It All

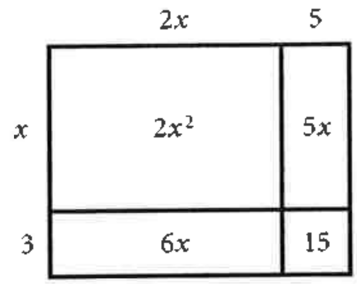
Intent

These activities focus on the symbolic representation of quadratic functions as well as some of the algebraic skills needed to multiply and factor quadratic expressions.

Mathematics

The distributive property is at the heart of the "doing and undoing" skills developed in *The Form of It All*. Students will use an area model to connect multiplication of polynomials to multiplication of numbers and to develop procedures for multiplying expressions of the forms $(x + a)(x + b)$ and $(x + c)^2$. For example, to multiply $(2x + 5)(x + 3)$,

$$\begin{array}{r} 2x + 5 \\ \underline{\times \quad x + 3} \\ 15 \\ 6x \\ 5x \\ \underline{+ \quad 2x^2} \\ 2x^2 + 11x + 15 \end{array}$$



Students will use these techniques to convert quadratic functions from vertex form to standard form.

Progression

The activities begin with a review of the distributive property in a numeric context and then extend it to work with polynomials. The area model is introduced as a tool for multiplying polynomials and completing the square. Students move back and forth between graphical and symbolic representations of quadratic functions. In addition, they present their solutions to the unit's first POW and begin work on the second.

A Lot of Changing Sides

Distributing the Area I

Views of the Distributive Property

Distributing the Area II

Square It!

Squares and Expansions

POW 11: Twin Primes

Vertex Form to Standard Form

How Much Can They Drink?

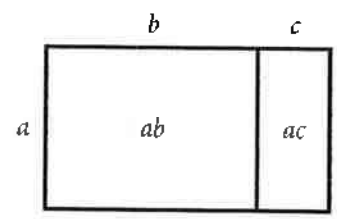
A Lot of Changing Sides

Intent

This activity introduces the area model for multiplying binomials.

Mathematics

The Form of It All emphasizes building meaning into the symbolic manipulations associated with multiplying and factoring polynomials. The key idea here is the distributive property of multiplication over addition: $a(b + c) = ab + ac$. Given the natural connection between multiplication and the area of a rectangle, this property is easily represented in this manner:



Total area: $a(b + c) = ab + ac$

This activity provides students with this geometric model for the process of multiplying binomials (and other polynomials). In this activity, students will use symbols to express the areas of rectangles created by altering squares with sides of length X. As the unit develops, they will also use this area model for completing the square and for factoring.

Progression

Students work as a whole class on Question 1 and then tackle the remaining five questions in their groups.

Approximate Time

30 minutes

Classroom Organization

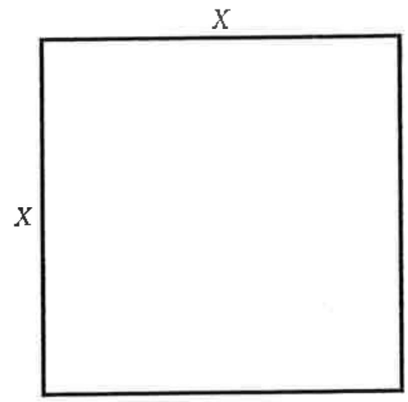
Whole-class introduction, followed by small groups

Doing the Activity

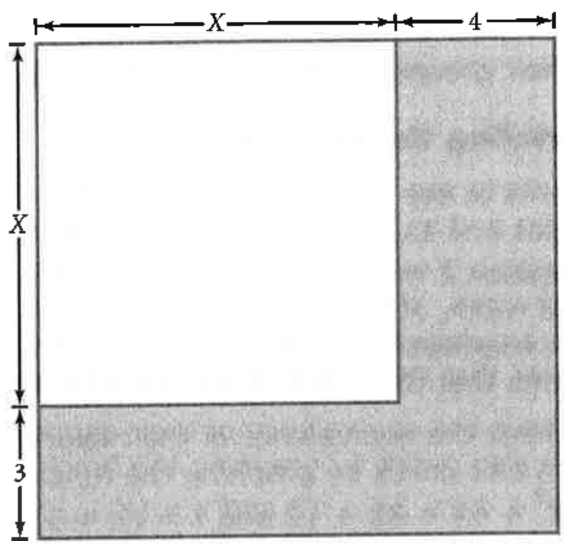
If students mention that this activity seems to be a digression from work with vertex form, you can suggest that knowing how to multiply two binomials is necessary to transform quadratics written in vertex form, like $y = (x - 4)^2 - 7$, into an equivalent equation in standard form, like $y = x^2 - 8x + 9$.

Have students read the introduction and Question 1. Then ask them to explain what, exactly, they are being asked to do. Be sure they identify the two steps of drawing a sketch and finding an expression for the new area.

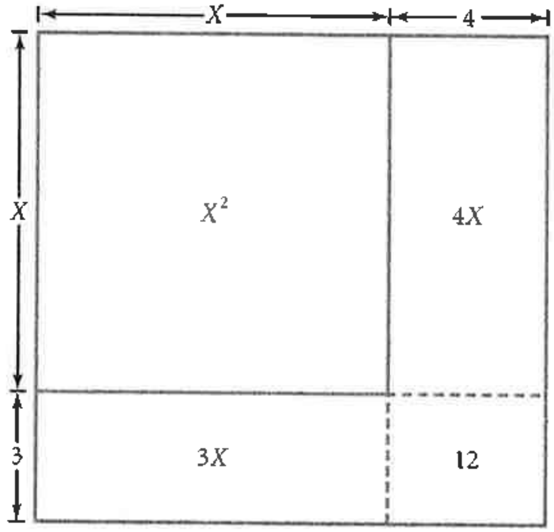
To get them started, ask them to sketch and label the original lot. They should get something simple like this.



Then have them discuss in their groups how to sketch and label the lot described in Question 1. They should come up with something like the next diagram, in which the shaded area represents the "new" portion of the lot.



Ask groups to come up with two expressions for the area of the new lot, as described in the activity. Students should see that the area is the product $(X + 3)(X + 4)$. For the second expression, you may have to suggest that they divide the diagram into rectangles. As the next diagram shows, the area is also the sum $X^2 + 4X + 3X + 12$. Students do not need to combine terms, but it's okay if they do.



It is important that the expression without parentheses comes from the diagram as a sum of areas, even if some students know how to multiply out the product $(X + 3)(X + 4)$.

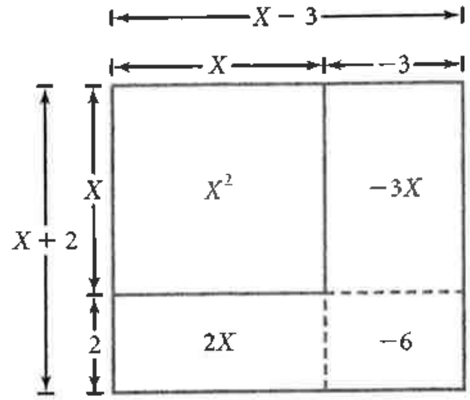
Have students work in their groups on the rest of the activity.

Discussing and Debriefing the Activity

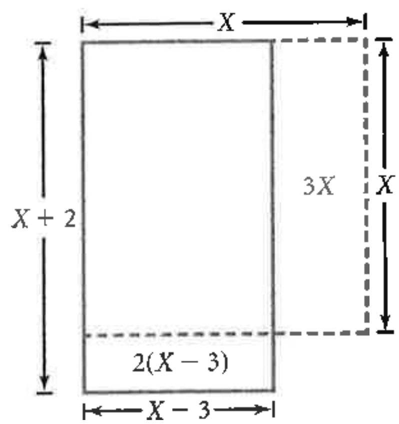
The key goal is for students to see the connection between the area expressed as a sum of separate rectangles and as the product of the new length and width. For example, the area in Question 2 might be expressed as the sum $X^2 + 5X$ or as the product of the length and width, $X(X + 5)$. You might ask students, **What is the term for two algebraic expressions that represent the same thing?** If necessary, remind students that these are called *equivalent expressions.*

You can have students check the equivalence of their expressions by substituting numeric values. They can also check by graphing the functions defined by the equations, such as $Y = X^2 + 4X + 3X + 12$ and $Y = (X + 3)(X + 4)$ for Question 1. They should see that the graphs are identical. Be sure they understand that neither of these "checks" proves the equivalence.

The area model falls apart somewhat when it comes to dealing with negative numbers, as in Questions 5 and 6, but students may have some creative ways of dealing with "negative area." Most students are comfortable with a diagram like the next one for Question 5, even though it shows both negative lengths and negative areas.



Another way students might draw the situation in Question 5 is shown here.



In this case, the two ways to describe the area symbolically are

$$(X + 2)(X - 3) \qquad \text{and} \qquad X^2 - 3X + 2(X - 3)$$

In the second expression, students begin with the original square's area of X^2, subtract the rectangle with area $3X$, and add the rectangle with area $2(X - 3)$. Question 6 can be modeled geometrically using these same ideas, though it is a bit trickier.

If time permits, you may want to do some more examples like those in *A Lot of Changing Sides,* focusing on the issue of signs when one or both of the constant terms is negative.

Distributing the Area I

Intent

Students apply the area model to the multiplication of numeric and algebraic expressions.

Mathematics

This activity lays the foundation for understanding the area model for multiplying binomials. Students will multiply the binomials $(a + b)$ and $(c + d)$, using specific values for the variables, to find areas. In some cases, they will be given values for area and must work backward to find the factors whose product gives that area. Working backward, or "undoing," prepares students to understand factoring with polynomials.

Progression

Students work individually or in groups on the activity and share results in a brief class discussion.

Approximate Time

5 minutes for introduction

15 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Review the use of the area model, presented in the introduction to this activity in the student book, for multiplying algebraic expressions.

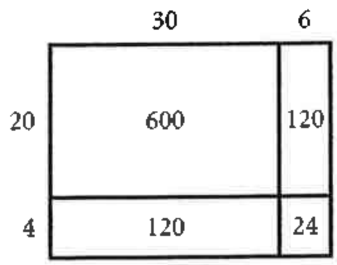
Discussing and Debriefing the Activity

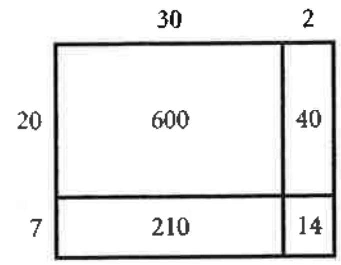
Have students or groups present their solutions.

Question 1 is a straightforward task of labeling an area model and computing areas.

Starting with Question 2, students need to reason to find missing lengths and areas. In Question 2, if area IV is 6 square units and length d is 2, then length b must be 3. The question being addressed is, **If I know the area of a rectangle, what must the dimensions be?** This is analogous to the question, **If I know the product of two numbers, what must the numbers be?** Students are "undoing" multiplication, or factoring.

Question 6 is challenging and involves some guess-and-check. If the total area is 864 square units, the overall dimensions of the rectangle must be factors of 864. Students know that $a = 30$ and $c = 20$, so they need to find lengths b and d such that the three remaining areas add to 264 square units. There are only two whole-number factor pairs of 864 with one number greater than 30 and the other greater than 20: 32 and 27, and 36 and 24. So, there are two possible answers: $b = 2$ and $d = 7$, or $b = 6$ and $d = 4$.





Views of the Distributive Property

Intent

This activity gives students some ways to think about the distributive property in arithmetic, both in terms of familiar algorithms and using an area model. Students then apply these ideas to the multiplication of algebraic expressions.

Mathematics

The distributive property for multiplication over addition is written symbolically as $a(x + y) = ax + ay$. This property is more generally applied to the multiplication of polynomials as follows:

$$(a + b)(c + d) = (a + b)(c) + (a + b)(d) = ac + bc + ad + bd$$

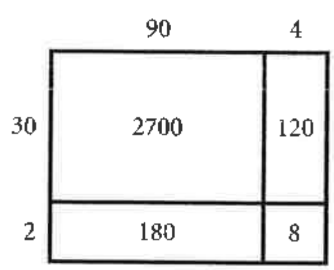
In this activity, students are asked to consider that they have been using this property since they learned how to multiply two multidigit numbers. Once the familiar algorithm is expanded to show (using the area model) the partial products and how they are combined to form the product, students are asked to generalize this approach to algebraic expressions. For example,

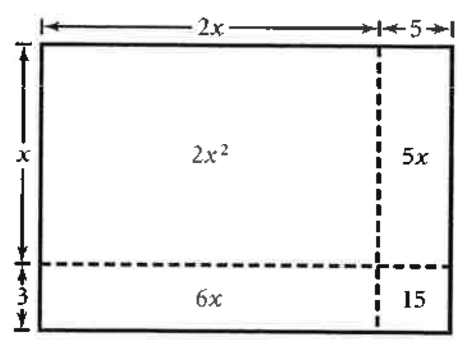
To multiply 32 • 94,

$$\begin{array}{r} 30 + 2 \\ \underline{\times\ 90 + 4} \\ 8 \\ 120 \\ 180 \\ \underline{+\ \ 2700} \\ 3008 \end{array}$$

To multiply (2x + 5)(x + 3),

$$\begin{array}{r} 2x + 5 \\ \underline{\times\ \ x + 3} \\ 15 \\ 6x \\ 5x \\ \underline{+\ \ 2x^2} \\ 2x^2 + 11x + 15 \end{array}$$





Progression

Students read the activity and then work on the tasks in groups, with a follow-up discussion focused on arithmetic–algebra connections and combining terms.

Approximate Time

35 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

Have students read the expository material in the student book and then work on the questions in their groups.

Discussing and Debriefing the Activity

Although Questions 1 and 2 may seem elementary, they serve as a valuable foundation for upcoming work with variables. In discussing the long form of multiplication, you may need to review some basics of place value so students see where the zeros are coming from. You might also bring out that the standard multiplication algorithm (the "short form") involves combining some terms mentally. This is indicated in the student book, but may be worth repeating.

For Question 3, focus on the facts that the multiplication involves six partial products and that the corresponding area model has six separate rectangles.

Students are likely to have different methods for Question 4a (preferably including at least one long-form and one short-form method) and various ideas about how (or whether) to line up the terms. For instance, they might produce either of these forms.

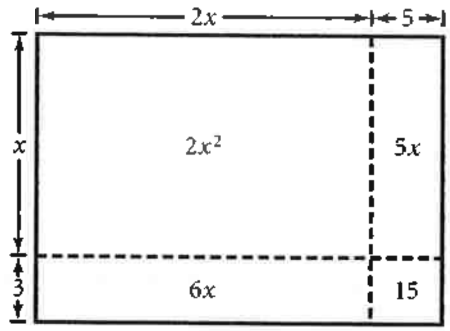
$$\begin{array}{r} 2x + 5 \\ \underline{\times \quad x + 3} \\ 15 \\ 6x \\ 5x \\ \underline{+ \quad 2x^2} \end{array} \qquad \text{or} \qquad \begin{array}{l} \quad\quad\;\; 2x + 5 \\ \underline{\times \qquad x + 3} \\ \qquad\qquad 15 \\ \quad\;\; 6x \\ \quad\;\; 5x \\ \underline{+ \; 2x^2 \qquad} \end{array}$$

Others may follow a pattern that looks more like the standard multiplication algorithm.

$$\begin{array}{r} 2x + 5 \\ \underline{\times \quad x + 3} \\ 6x + 15 \\ \underline{+ 2x^2 + 5x} \end{array} \qquad \text{or} \qquad \begin{array}{r} 2x + 5 \\ \underline{\times \qquad\qquad x + 3} \\ 6x + 15 \\ \underline{+ \; 2x^2 + 5x \qquad\quad} \end{array}$$

Emphasize that because the purpose of a written format is to make the work as clear and as easy to do as possible, people may prefer different methods.

The diagram for Question 4b should look something like this.



If needed, point out the analogy between this diagram and the diagram in Question 2, which involves numeric multiplication, including the connection between the rectangles in the diagram and the partial products in the long form of the multiplication. Also reinforce the idea that a product of two binomials will have four terms, although it's likely that two of them can be combined.

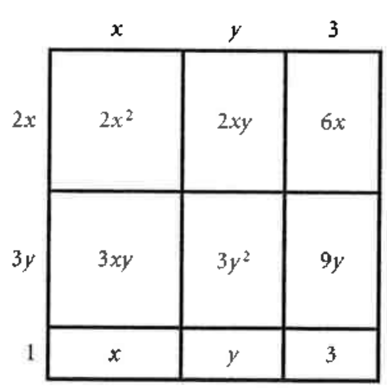
Combining Terms

Whatever method students use for getting partial products or partial sums, they need to know how to combine the terms into a single final expression. Review the idea of combining like terms, perhaps using the analogy of place value to bring out that multiples of x are like 10s, multiples of x^2 are like 100s, and so on.

Ask, **In general, how do you multiply two sums? What are the partial products? Where do the partial products come from?** Students should be able to articulate that the partial products are all the products obtained by multiplying a term of one sum by a term of the other sum.

Have students explain the process using an area model. Emphasize the value of being able to go back and forth between the symbol manipulation and the area model for multiplication. The goal is for students to understand this multiplication process, rather than simply memorizing a rule such as FOIL (*first, outside, inside, last*).

You may want to review multiplication examples in which one or both factors involve subtraction, such as $(x + 3)(x - 2)$ or $(x - 4)(x - 6)$. You may also want to have students work out an example in which each factor has more than two terms, such as $(x + y + 3)(2x + 3y + 1)$.



Distributing the Area II

Intent

This activity continues the themes of using an area model to represent multiplication and factoring of numbers, and connecting arithmetic multiplication to polynomial multiplication.

Mathematics

Students return to the area model introduced in *Distributing the Area I* and use it to represent multiplication of two binomials and to find the binomials that produce a quadratic product. They also continue to use the vertical form introduced in *Views of the Distributive Property* to multiply two polynomials.

Progression

Students work on the activity individually. The subsequent class discussion will clear up any lingering questions and emphasize the arithmetic-algebra connection.

Approximate Time

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, then groups, followed by whole-class discussion

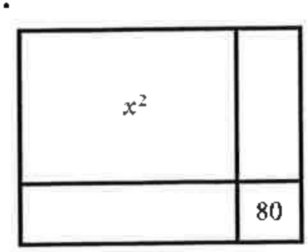
Doing the Activity

No introduction is necessary for this activity.

Discussing and Debriefing the Activity

In Question 4, students are asked for the first time to "undo" a quadratic expression that is the answer to a multiplication problem. Rather than showing them a procedure for factoring, this is a chance for students to use the model they have been developing to reason their way through the process.

If the total area is $x^2 + 18x + 80$ and the area has four parts, students might begin by labeling a rectangle this way.



From this diagram, they can reason that the dimensions of the upper-left square are x and x and the dimensions of the lower-right square are two numbers whose product is 80. In addition, the two factors of 80 must be numbers that will create remaining areas that add to 18*x*. The area model can then be completed as follows.

	x	8
x	x^2	$8x$
10	$10x$	80

When multiplying polynomials, students should simplify their final answers by combining like terms. Traditionally, terms are written in order of descending powers of the variable.

Supplemental Activity

A Lot of Symmetry (reinforcement) introduces the special binomial product $(x + n)(x - n) = x^2 - n^2$ in the context from *A Lot of Changing Sides*.

Square It!

Intent

Students apply the multiplication skills they have been developing to the task of rewriting quadratic functions from vertex form into standard form.

Mathematics

Students focus on rewriting quadratic functions in vertex form, $y = a(x - h)^2 + k$, in standard form, $y = ax^2 + bx + c$. First they focus on using an area model to square binomials. After rewriting functions in standard form, they combine this work with previous content to use the vertex and intercepts of the graph of a parabola to write its vertex-form equation and then rewrite the equation in standard form.

Progression

Students work on the activity in groups and discuss their findings as a whole class.

Approximate Time

25 minutes

Classroom Organization

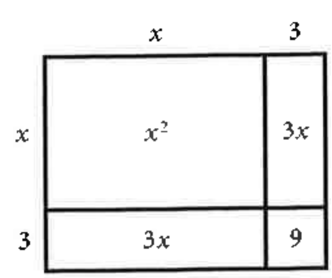
Groups, followed by whole-class discussion

Doing the Activity

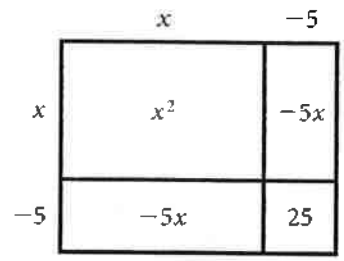
Tell students that they will now use what they have learned about multiplying binomials to change quadratics written in vertex form into standard form.

Discussing and Debriefing the Activity

As you discuss Question 1, you might have students draw area models to illustrate what's going on and use the diagrams to explain any apparent patterns. For instance, they might note that they always get two same-area rectangles. In the diagram here for Question 1a, these are the lower-left and upper-right rectangles. This leads to the observation that the coefficient of x in the square of a binomial is twice the constant term of the expression being squared with the same sign as that constant term. Similarly, the area of the lower-right corner is the square of the constant term of the expression being squared, and this value is the constant term in the result.



You might want to reassure students that they can use the area model even if the constant term is negative. For instance, the next diagram shows that $(x - 5)^2$ is equal to $x^2 - 10x + 25$.



You might have students compare this area approach with the process of expanding the expressions in Question 1 using the vertical-multiplication form from *Distributing the Area II* and bring out again that these products are illustrative of the distributive property.

Question 2a is straightforward, but students may have questions about 2b and 2c. These entail three steps, designed to follow the order of operations.

- Square the binomial.
- Apply the distributive property, multiplying each term of the resulting trinomial by the given coefficient (in Question 2b by 3 and in Question 2c by 0.5).
- Add like terms and write the complete answer.

In Question 3, to find an equation from a graph, students need to estimate the coordinates of the vertex and at least one x-intercept and then use the techniques from *Crossing the Axis* and *Is it a Homer?* to find the equation in vertex form. They should check their answers by graphing their functions and comparing them with the pictured graphs.

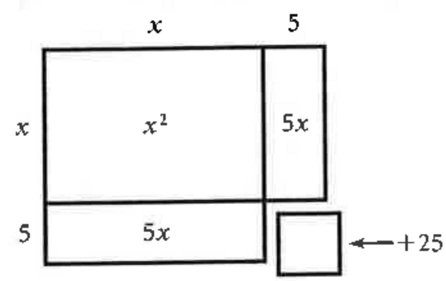
Squares and Expansions

Intent

Students use the area model for multiplication to begin to develop, in a meaningful way, a familiar symbolic procedure.

Mathematics

In this activity, students try the first step of the technique traditionally called **completing the square,** which later will be applied to changing standard form into vertex form and to deriving the quadratic formula. Using an area model, students recognize that they are literally "completing a square" when they create a perfect square. For example, given the expression $x^2 + 10x$, adding 25 completes the square, resulting in the expression $x^2 + 10x + 25 = (x + 5)^2$.



Progression

Students will work on the three parts of this activity—completing the square, expanding expressions, and sketching a graph—individually and share their results in a class discussion.

Approximate Time

10 minutes for introduction

25 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

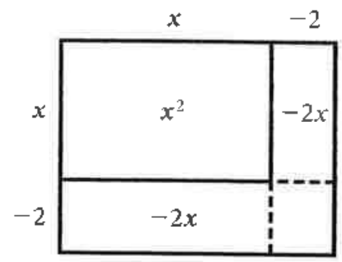
Individuals, followed by whole-class discussion

Doing the Activity

Read the activity and work through Questions 1a and 1b as a class.

Discussing and Debriefing the Activity

Focus the discussion of Question 1 on the use of area models to explain the process. As each diagram is presented, be sure students write the corresponding expression as the square of a linear expression. For instance, they might get a preliminary diagram like this one for Question 1d and thus need to add 4 to get a perfect square. They can then write the expression as $(x - 2)^2$.



You may want to help students articulate that in each part of Question 1, the constant term of the binomial being squared is half the coefficient of x in the original expression. For instance, −2, from $x - 2$, is half the coefficient −4 in $x^2 - 4x$.

As students present their work for Question 2, you might focus their attention on being careful about signs.

Question 3 will remind students of the usefulness of vertex form in sketching graphs. They should immediately realize from the equation that the vertex is at (5, 10) and that this point is a maximum, so the graph opens downward. To sketch the graph, the only further information they need is a rough idea of where the x-intercepts are. They should reason that they want $(x - 5)^2 = 10$, so $x - 5 = \pm\sqrt{10}$. That is, $x - 5$ is about 3 or −3, so x is approximately 8 or 2. This is a good opportunity to get students to articulate the symmetry principle for parabolas once again.

POW 11: Twin Primes

Intent

In exploring and then justifying an interesting property of the number system, students will also be applying some of the skills they are developing in this unit, including the manipulation of algebraic expressions.

Mathematics

Twin primes are prime numbers that are 2 units apart, such as 5 and 7, 41 and 43, 71 and 73, 137 and 139, and 281 and 283. This POW centers on an interesting property of twin primes. If you add 1 to their product, the result is a perfect square and a multiple of 36. In this activity, students look for twin primes, explore this property, and then try to prove that it is always true.

Progression

After a brief introduction, students will work on this problem individually outside of class and then submit detailed write-ups of their solutions. Presentations will follow.

Approximate Time

10 minutes for introduction

1–3 hours (at home)

20 minutes for presentations

Classroom Organization

Whole class, then individuals, followed by whole-class presentations

Doing the Activity

To introduce this POW, review the definition of *twin prime*. To be sure students understand the properties being presented, ask them to find all the twin primes between 50 and 100.

If students have done any calculator programming, you may want to suggest they write a program that will give all twin primes between two input numbers.

Discussing and Debriefing the Activity

The POW asks students to use a variable and prove that multiplying twin primes and adding 1 results in a perfect square. If students use x and $x + 2$ to represent a pair of twin primes, the task is then to show that $x(x + 2) + 1$ is a perfect square. As students may have discovered, this expression is a perfect square for any integer x, because it is equal to $(x + 1)^2$.

If some students got this algebraic representation but did not succeed in showing that the expression is also a multiple of 36, you might ask, **If a perfect square is a multiple of 36, what is true about the square root of that number?** This might lead them to recognize that they need to show that $x + 1$ (which is the number between the two twin primes) is a multiple of 6.

Showing that the result is a multiple of 36 builds on the fact that if a prime greater than 3 is divided by 6, the remainder must be either 1 or 5. In symbolic terms, this means that a prime is of the form $6k + 1$ or $6k + 5$.

Supplemental Activities

Divisor Counting (extension) asks students to look for numbers that have a given number of divisors. This problem-solving activity is best assigned after students have worked on *POW 11: Twin Primes*.

The Locker Problem (extension) is a classic problem that makes a good follow-up to *Divisor Counting*.

Vertex Form to Standard Form

Intent

This activity gives students more practice with algebraic manipulation and parabolas in another real-world context.

Mathematics

One definition of a **parabola** is the set of points that are equidistant from a fixed line and a fixed point not on that line. The fixed line is called the *directrix,* and the fixed point is called the *focus* of the parabola. One important property of parabolas is that they concentrate incoming energy—light or sound, for example—at the focus. This reflective property is the real-world context for Question 9, in which students are to find, graph, and then use the equation of a parabolic mirror with its vertex at the origin and passing through the point (40, 10).

Progression

Students will work on this activity individually and share ideas in a class discussion.

Approximate Time

5 minutes for introduction

25 minutes for activity (at home or in class)

40 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

To introduce this activity, you might mention some of the types of objects that have parabolic cross sections, such as automobile headlights, TV dishes, makeup mirrors, and sound amplifiers. You might also suggest that students may want attempt to build a solar cooker out of cardboard parabolic slices and aluminum foil.

Discussing and Debriefing the Activity

You might have groups prepare presentations on each of Questions 1 to 8.

Then discuss how students approached Question 9. The only way to get the length for the cardboard support is to find the equation for a parabola with vertex at (0, 0) and using the point (40, 10) to find the value of *a*.

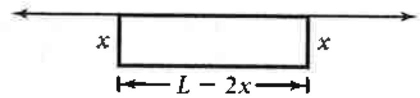
How Much Can They Drink?

Intent

This final activity of *The Form of It All* revisits a context first encountered in the unit *Do Bees Build It Best?* in which students look for the vertex of a quadratic function to find the dimensions of a prism with the maximum possible volume.

Mathematics

In *A Corral Variation*, students learned that the area of a rectangle formed by a fixed length L of fence along three sides is given by $A = x(L - 2x)$. This area function's graph is a parabola, and its vertex can be used to find the maximum possible area.



In *Do Bees Build It Best?* students learned that the volume of a prism is the product of the area of the prism's base and the prism's height. They now encounter a watering trough with a rectangular cross section formed just as the rectangular fence above and are asked to find the dimensions of the trough that will give the maximum volume.

Progression

Students work on this activity individually and discuss their results in class.

Approximate Time

10 minutes for introduction

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Whole class, then individuals, followed by whole-class discussion

Doing the Activity

You might want to review the idea that the volume of a rectangular solid is the product of the solid's length, width, and height by posing some volume problems.

How many cubes 1 inch on an edge will a box measuring 5 inches by 7 inches by 10 inches hold?

A wave table in the physics lab is an open-top box with a rectangular base. It holds 9996 cubic centimeters and is 42 centimeters long and 28 centimeters wide. What is its height?

A farmer has 36 feet of fencing and wants to enclose the maximum rectangular area for his llamas. Find the dimensions of three possible areas he could enclose. What do you think the maximum area is? Why?

Discussing and Debriefing the Activity

You might begin the discussion with Question 3 and return to Questions 1 and 2 if needed to illustrate the process of finding the volume.

Some students may need help developing and making sense of the expression $40 - 2x$ for the width. You might have them make a table of specific examples, including the one in the activity with $x = 5$. Once they have this expression, it should be straightforward to get a formula for the volume equivalent to $V = 80(40 - 2x)x$.

Because the volume expression as given here involves three factors, including the constant 80, students might not recognize that they are working with a quadratic function. Have them multiply out the expression and rearrange the terms (getting $-160x^2 + 3200x$) to emphasize this. Seeing the expression as a quadratic will allow students to connect "finding the maximum" with the idea of finding the vertex of the corresponding graph.

Students might find the vertex (and thus find the maximum volume) by a guess-and-check approach or by tracing on their calculator graphs. They might also use the symmetry of the parabola and conclude that the x-coordinate of the vertex is halfway between x-coordinates of the x-intercepts. Once they have the x-coordinate of the vertex, they can get the volume by substitution.

Bring out that students can find the x-intercepts easily by tracing. But they can also find them symbolically, because the function is expressed as a product of linear terms. The product is zero whenever one of the factors is zero, namely, when $x = 0$ or $x = 20$. If needed, write the explicit step $40 - 2x = 0$ and go from that to $x = 20$.

To foreshadow later work, point out that the factored form of the volume equation makes finding the x-intercepts easy and also allows one to find the vertex using the symmetry.

Putting Quadratics to Use

Intent

These activities focus on problems that can be solved using a quadratic function in vertex form and the algebra skills needed to transform quadratic functions into vertex form.

Mathematics

In *The Form of It All,* students modeled the multiplication of polynomials using the area of a rectangle. The activities in *Putting Quadratics to Use* build on that work and on the general emphasis on the "doing and undoing" aspect of algebraic thinking present throughout the curriculum. Students learn how to undo the process of transforming quadratic functions in vertex form into standard form, using the method typically known as "completing the square."

Students also continue to derive quadratic functions for a variety of real-world contexts, some of which they have encountered in earlier units. With their developing skill of transforming functions into vertex form, students find and interpret key points on the related graphs, including vertices and intercepts.

Progression

In *Putting Quadratics to Use,* students work with six contexts that lead to quadratic functions. By the end of these activities, they will have had many opportunities to transform quadratic functions back and forth between standard form and vertex form, and to use vertex form to maximize or minimize functions and solve quadratic equations.

Revisiting Leslie's Flowers

Emergency at Sea

Here Comes Vertex Form

Finding Vertices and Intercepts

Another Rocket

Profiting from Widgets

Pens and Corrals in Vertex Form

Vertex Form Continued

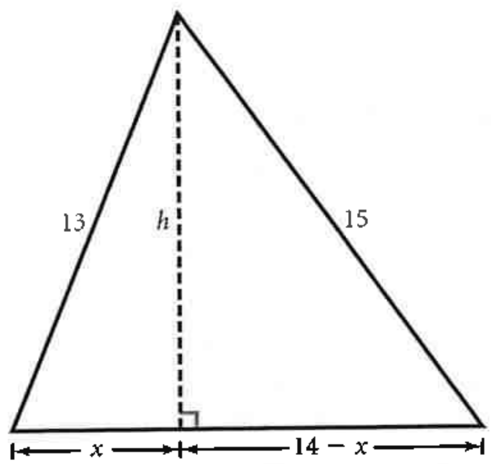
Revisiting Leslie's Flowers

Intent

This activity uses a context first encountered in the unit *Do Bees Build It Best?* as an opportunity for students to apply their new skills for working with quadratic equations.

Mathematics

In *Leslie's Fertile Flowers*, students applied the Pythagorean theorem for right triangles twice to find the dimensions and area of a triangular flower bed. In this activity, students will write and solve the quadratic equation they derive symbolically by solving each of the equations below for h^2 and setting the two expressions equal.



$$x^2 + h^2 = 13^2 \qquad (14 - x)^2 + h^2 = 15^2$$

Progression

Students work on this activity in groups.

Approximate Time

30 minutes

Classroom Organization

Groups, followed by whole-class discussion

Materials

Students' notes from the unit *Do Bees Build it Best?*

Doing the Activity

This activity uses the situation presented in *Leslie's Fertile Flowers,* from *Do Bees Build It Best?* In that activity, students found the value of x by a guess-and-check approach. They did this by applying the Pythagorean theorem to the two right triangles created by the altitude of the triangle. The goal now is for students to work through the algebra needed to solve the equation directly. You might choose to work through Question 1 as a class, both to review the situation and to help students set up the algebraic representation.

One key step is using $14 - x$ to represent the second portion of the 14-foot side. Letting h represent the altitude, the left-hand triangle then leads to the equation $13^2 - x^2 = h^2$ while the right-hand triangle leads to the equation $15^2 - (14 - x)^2 = h^2$. Combining the two gives the one-variable equation

$$13^2 - x^2 = 15^2 - (14 - x)^2$$

Before setting students off on the challenge of solving this equation, you might point out that expanding the expression on the right side is precisely the skill they have been developing.

You might also note that the two expressions each represent the vertex form for a parabola, so solving this equation is like finding the place where two parabolas intersect.

Discussing and Debriefing the Activity

The first stage of the solution will likely be to remove the parentheses, probably getting

$$169 - x^2 = 225 - 196 + 28x - x^2$$

One issue that may arise is how to deal with the x^2 terms. Although the mechanics are the same as with linear terms, students may be less comfortable with a quadratic expression. It may be worth taking a moment to discuss why "adding x^2 to both sides" is a legitimate action that yields an equivalent equation.

In one sequence of steps or another (and you may want to get more than one sequence), students should be able to simplify the equation to something like $140 = 28x$ and see that $x = 5$.

Have students confirm this answer in at least a couple of ways. One confirmation is to check that this value satisfies the original version of the equation, $13^2 - x^2 = 15^2 - (14 - x)^2$. Another way is to verify that splitting the side of length 14 into parts of length 5 and 9 gives the same result for the length of h. As $13^2 - 5^2$ and $15^2 - 9^2$ are both equal to 144, both give $h = 12$.

Supplemental Activity

Equilateral Efficiency (extension) introduces Hero's formula for finding the area of a triangle given the lengths of its sides. With a little algebra, Hero's formula can be derived with the same approach used for finding the altitude of the triangle in *Revisiting Leslie's Flowers*.

Emergency at Sea

Intent

This activity gives students another opportunity to set up and solve a quadratic equation for a real-world context.

Mathematics

As in *Revisiting Leslie's Flowers*, students will use the Pythagorean theorem on the two right triangles created by dividing a scalene triangle along an internal altitude. The solution to the resulting quadratic equation will give the length of the third side of the triangle, and substitution will give the triangle's height.

Progression

Students work on this activity individually and share their results with the class.

Approximate Time

5 minutes for introduction

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

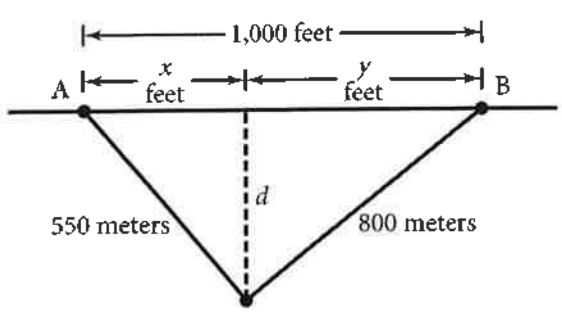
Individuals, followed by whole-class discussion

Doing the Activity

Read the activity as a class, making sure students understand the diagram.

Discussing and Debriefing the Activity

If students go through a process similar to that used for *Revisiting Leslie's Flowers,* they might create a diagram like the one here, in which $y = 1000 - x$. This will likely lead to the equation $550^2 - x^2 = 800^2 - (1000 - x)^2$.



Students should see that their recent work expanding expressions with parentheses is directly applicable here. Have them apply what they have learned to get a parentheses-free expression equivalent to $(1000 - x)^2$. Once they have done so, the equation will probably look like this.

$$550^2 - x^2 = 800^2 - (1{,}000{,}000 - 2000x + x^2)$$

You may need to point out that the entire expression for $(1000 - x)^2$, namely, $1{,}000{,}000 - 2000x + x^2$, is being subtracted, which is indicated by enclosing the expression in parentheses.

When 550^2 and 800^2 are replaced by their numeric values, the equation becomes

$$302{,}500 - x^2 = 640{,}000 - (1{,}000{,}000 - 2000x + x^2)$$

The equation simplifies as follows:

$$302{,}500 - x^2 = 640{,}000 - 1{,}000{,}000 + 2000x - x^2$$

$$302{,}500 = 640{,}000 - 1{,}000{,}000 + 2000x$$

$$302{,}500 = -360{,}000 + 2000x$$

$$302{,}500 + 360{,}000 = 2000x$$

$$662{,}500 = 2000x$$

$$x = 331.25 \text{ feet (distance to tower A)}$$

Thus the distance to tower B is 668.75 feet. For Question 4, students can find the distance d from the boat to the shore from the equation $d^2 = 550^2 - 331.25^2$, which gives $d \approx 439.06$, and then verify that this value also satisfies the equation $d^2 + 668.75^2 = 800^2$.

Here Comes Vertex Form

Intent

This activity emphasizes reversing, or undoing, the process of changing a quadratic function from vertex form to standard form.

Mathematics

Students use the process of completing the square to transform quadratic expressions in standard form, $ax^2 + bx + c$, into vertex form. To simplify things, the expressions all have a leading coefficient of 1 or −1. The groundwork for this process was laid in *Squares and Expansions*, where students learned that expressions of the form $x^2 + kx$ can be made into a "square," geometrically and symbolically, by adding the constant term $\left(\frac{k}{2}\right)^2$.

Progression

Students work on the activity individually or in groups. In Questions 1 to 4, they change quadratic functions from standard form to vertex form and interpret the results. In Questions 5 and 6, they are given functions in vertex form and are asked to find their x-intercepts.

Approximate Time

30 minutes

Classroom Organization

Groups or individuals, followed by whole-class discussion

Doing the Activity

You may need to demonstrate the technique of completing the square. The idea underlying the process seems obvious: adding 0 to any quantity leaves the quantity's value unchanged. But 0 can take many forms, such as 5 − 5 and −3.65 + 3.65. There are many approaches to the mechanics of this process, and it may be helpful for students to see a variety of methods.

Discussing and Debriefing the Activity

Have volunteers share their methods for transforming each equation. Here are two ways students might approach the expression $x^2 + 4x + 1$ from Question 1, for example.

- Begin by putting the constant term in front to get $1 + (x^2 + 4x)$. Then look at $x^2 + 4x$. Completing the square for this expression requires adding 4, but if we add 4, we must also subtract 4 (for a net change of 0), so we write $x^2 + 4x$ as $x^2 + 4x + 4 - 4$. The original expression thus becomes $1 + (x^2 + 4x + 4) - 4$, which can then be written in vertex form as $(x + 2)^2 - 3$.

- Look at the initial two terms of the expression, $x^2 + 4x$, and think about how to complete the square. The expression needs a constant term of 4 to complete the square and obtain $(x + 2)^2$. The original expression, $x^2 + 4x + 1$, has a constant term of 1 and is thus "3 short," so we rewrite it as $(x + 2)^2 - 3$. Similarly, in Question 2, the constant term of 15 is "6 too much," and the vertex form is $(x - 3)^2 + 6$.

Whatever method students use, you might have them substitute one or two values for x to verify that the vertex form they get is equivalent to the original expression. This is also another opportunity to look at how to get the vertex from the vertex form.

In Questions 3 and 4, students need to exercise care to get the signs correct. For Question 3, they might begin by writing the equation as $y = 3 - (x^2 - 8x)$, as in the first method described earlier, or as $-(x^2 - 8x - 3)$, based on the second method, and proceed with the expression in parentheses as follows:

- The expression in parentheses, $x^2 - 8x$, needs 16 to be a perfect square, so we can write it as $x^2 - 8x + 16 - 16$ or, equivalently, $(x - 4)^2 - 16$. So the overall expression becomes $3 - [(x - 4)^2 - 16]$, which is equivalent to $3 - (x - 4)^2 + 16$. This simplifies, in vertex form, to $-(x - 4)^2 + 19$.
- Looking at the expression in parentheses, $x^2 - 8x - 3$, we see that the initial portion, $x^2 - 8x$, needs 16 to be a perfect square, $(x - 4)^2$. The expression has a constant term of -3, which is "19 short" of 16, so the expression in parentheses can be thought of as $(x - 4)^2 - 19$. The original expression is therefore $-[(x - 4)^2 - 19]$, which is equivalent to $-(x - 4)^2 + 19$ in vertex form.

The next activity, *Finding Vertices and Intercepts*, gives further practice with completing the square. Students will have more opportunities to work with the process later, so it is not necessary for them to master the skill of turning quadratics into vertex form at this time.

Finding Vertices and Intercepts

Intent

Students transform functions into vertex form and use the result to find vertices and x-intercepts.

Mathematics

One advantage of having a quadratic function in vertex form is being able to find graphical features like the coordinates of the vertex and the direction of concavity by inspection. Another advantage, emphasized here, is being able to use this form to solve a quadratic equation symbolically. For example, if the vertex form of a quadratic function is $y = (x + 2)^2 - 4$, it is relatively easy to find the x-value(s) when $y = 0$, if such solutions exist: $(x + 2)^2 - 4 = 0$ is equivalent to $(x + 2)^2 = 4$, so $x + 2 = \pm 2$, and $x = -4$ or 0.

Progression

Students work individually on the activity and share their findings in a class discussion.

Approximate Time

5 minutes for introduction

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This is a good time to review two key ideas.

- "Finding an x-intercept" means making y equal to zero.
- The two square roots of a given number are found by setting up and solving two separate equations.

Discussing and Debriefing the Activity

Questions 1 and 3 both have whole-number x-intercepts.

In Question 1, the vertex form is $y = (x + 2)^2 - 1$, so students want $(x + 2)^2 - 1 = 0$ or, equivalently, $(x + 2)^2 = 1$. Splitting this into two cases, $x + 2 = 1$ and $x + 2 = -1$, gives $x = -1$ and $x = -3$.

You might take this opportunity to bring out that the vertex has an x-coordinate of -2, which is the midpoint between the intercepts at $x = -1$ and $x = -3$, thus reviewing the idea that parabolas are symmetric about a vertical line drawn through the vertex. The two x-intercepts are 1 unit on either side from -2.

For Question 2, the vertex form is $y = (x - 3)^2 + 3$, and students should see that there are no intercepts. You might ask them to explain what's going on with the graph. For instance, they might say the graph "starts" at $y = 3$ and goes up from there because the expression $(x - 3)^2$ cannot be negative.

Question 4 presents the first case in which students are solving to get irrational intercepts. The vertex form is $y = -(x + 1)^2 + 5$, so students want $(x + 1)^2 = 5$.

Students might get an intuitive sense of how this works if they begin with a decimal approximation for $\sqrt{5}$. They can conclude from $(x + 1)^2 = 5$ that $x + 1$ is approximately either 2.24 or −2.24, giving the approximations $x + 1 \approx 2.24$ and $x + 1 \approx -2.24$, so $x \approx 1.24$ or -3.24.

Then have them look at how they can do something similar using the radical $\sqrt{5}$ to see that $x = \sqrt{5} - 1$ or $-\sqrt{5} - 1$. They can verify that these expressions are consistent with the earlier approximations. You might bring out that even in this case, the x-coordinate of the vertex is halfway between the two x-intercepts.

Supplemental Activity

Check It Out! (extension) introduces the notion that solving radical equations such as $\sqrt{2x - 3} = -5$ by squaring both sides may introduce extraneous roots. You can use this activity with *Finding Vertices and Intercepts* or anytime during the unit.

Another Rocket

Intent

Students use their developing methods for solving quadratic equations in the context of the unit problem, with a somewhat simpler function.

Mathematics

Students use the technique of completing the square once again to find the vertex and x-intercepts of a quadratic function. However, this time, the coefficient of the x^2 term is not 1.

Progression

Students work in groups on the activity. Initial questions about how to proceed with functions in which the leading coefficient is not 1 might lead to a class discussion.

Approximate Time

30 minutes

Classroom Organization

Groups

Doing the Activity

Allow students to begin without any preliminary discussion. Some will quickly notice that the coefficient of x^2 is not 1, so this problem is different from those they have been doing.

To answer their questions, you may want to do an example with the class. For instance, consider the task of putting $y = 2x^2 + 12x + 6$ into vertex form. There are various approaches for taking into account the coefficient 2 (from $2x^2$), each with advantages. Here are two such options.

- Factor out 2, getting $2(x^2 + 6x + 3)$, and proceed with the expression in parentheses as in one of the earlier methods to get $(x + 3)^2 - 6$. As a final step, simplify $2[(x + 3)^2 - 6]$ into vertex form.
- Isolate the constant term, writing the expression as $6 + 2(x^2 + 6x)$. Then write $x^2 + 6x$ as $(x + 3)^2 - 9$, and proceed to simplify $6 + 2[(x + 3)^2 - 9]$ into vertex form.

The first approach may be more effective for demonstrating that any quadratic expression can be put into vertex form: factor out the coefficient, put the rest into vertex form, and then undo the original factoring. The second approach sets up the expression initially to look more like vertex form and then only requires completing the square and dealing with the additional constant term.

Discussing and Debriefing the Activity

Once groups get a solution, they may ask you whether it is correct. Rather than answering directly, you might instead ask them to graph the original function on their calculators and then use the Zoom feature to estimate the x-intercepts. If their answers are correct, they should agree with the calculator values.

Later in the unit, in *Fireworks Height Revisited,* students will put the equation $h(t) = 160 + 92t - 16t^2$ from the unit problem in *Victory Celebration* into vertex form.

Profiting from Widgets

Intent

Students are introduced to another application of quadratic equations and are asked to find the maximum value of a quadratic function.

Mathematics

This activity presents an economics context for the mathematical work of the unit. Given an expression for the number of "widgets" sold as a linear function of price, students are to derive a revenue function, in which revenue is found by multiplying the number of widgets sold by the price per widget. This revenue function is quadratic, and students can use the skills they have been developing to find its vertex.

Progression

Students work on the activity individually and compare methods and results in a class discussion.

Approximate Time

5 minutes for introduction

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals or groups, followed by whole-class discussion

Doing the Activity

Have students read the activity in their groups and discuss what kind of equation they might use to represent the sales revenue.

Discussing and Debriefing the Activity

Begin by having students explain the equation for sales revenue, which should look something like

$$R = d(1000 - 5d)$$

You might give the class additional examples similar to Question 1 to analyze whether students are clear about the meaning and basis of this formula.

Students might have found the maximum revenue by various means, including guess-and-check and estimating from a graph. Another approach is to find the two x-intercepts ($d = 0$ and $d = 200$) and use the fact that the d-coordinate of the vertex is the midpoint of the d-intercepts. Because the expression is in factored form, students may find it easy to get the intercepts.

If students didn't do so on their own, ask them explicitly to find the maximum by putting the revenue function in vertex form. Here is a possible sequence of steps.

$$
\begin{aligned}
R &= d(1000 - 5d) \\
&= 1000d - 5d^2 \\
&= -5(d^2 - 200d) \\
&= -5(d^2 - 200d + 10{,}000 - 10{,}000) \\
&= -5[(d - 100)^2 - 10{,}000] \\
&= -5(d - 100)^2 + 50{,}000
\end{aligned}
$$

This final form gives the vertex as (100, 50,000).

Ask students to interpret this information in terms of the context. Acme will make the maximum possible profit by charging $100 per widget. At that price, they will sell 500 widgets and make $50,000 in revenue.

For Question 4, students should recognize that the marketing director's formula has flaws, such as that if the price is set above $200 per widget, the formula says the company will sell a negative number of widgets.

Pens and Corrals in Vertex Form

Intent

Students revisit two contexts from earlier in the curriculum, in which solutions were found informally, and solve them again using the more formal methods of this unit.

Mathematics

The two problem contexts in this activity come from *A Corral Variation* and from *Don't Fence Me In* in the unit *Do Bees Build It Best?*, in which students found that the rectangle with the greatest area for a given perimeter is a square. Students are asked to prove the correctness of their previous solutions using algebra. In each case, they will write a quadratic function, transform it into vertex form, and read the vertex coordinates to find the maximum value of the function.

Progression

Students work on the activity in groups and share their results in a class discussion.

Approximate Time

35 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

For Question 1, ask for volunteers to present the steps of putting the expression into vertex form and how to use it to find the maximum area. In the steps shown here, the term added to create a perfect square is left as 125^2 until the last step. This technique might help students focus on the big picture rather than on the arithmetic.

$$\begin{aligned} y &= x(500 - 2x) \\ &= -2x^2 + 500x \\ &= -2(x^2 - 250x) \\ &= -2(x^2 - 250x + 125^2 - 125^2) \\ &= -2[(x - 125)^2 - 125^2] \\ &= -2(x - 125)2 + 2 \,.\, 1252 \\ &= -2(x - 125)^2 + 31{,}250 \end{aligned}$$

Connect this algebra with the real-world context by asking about the units for x and y (feet and square feet).

Students might prefer to justify the answer here based on the symmetry of the graph, which tells them that the vertex is halfway between the intercepts at $x = 0$ and $x = 250$. If so, use this as the stimulus for a discussion about the advantages and disadvantages of different methods. You might point out that the symmetry approach works well here because the function is in factored form, but that this is not the case for the rocket situation.

The algebra for Question 2 is even simpler, because the coefficient of x^2 is −1 instead of −2. The work might look like this.

$$\begin{aligned} y &= x(100 - x) \\ &= -x^2 + 100x \\ &= -(x^2 - 100x) \\ &= -(x^2 - 100x + 50^2 - 50^2) \\ &= -[(x - 50)^2 - 50^2] \\ &= -(x - 50)^2 + 50^2 \\ &= -(x - 50)^2 + 2500 \end{aligned}$$

If students find the "second side" of the rectangle as $\frac{200 - 2}{2}$ (subtracting the two sides of length x from the total perimeter of 200 and then dividing by 2 to get the other sides), point out that this expression is equivalent to $100 - x$.

By now students should be comfortable explaining, based on this final expression, that the maximum area occurs when x is 50 meters, so the rectangle is a square with an area of 2500 square meters.

Vertex Form Continued

Intent

This activity offers students additional experience with the types of problems they have been solving throughout *Putting Quadratics to Use*.

Mathematics

In this activity, students will transform functions in standard form into vertex form and vice versa. They will also find the symbolic representation of a quadratic function from its vertex and an additional point. Students should be growing more fluent with manipulating quadratic expressions. At this point, they are not expected to have mastered the process of transforming quadratics from standard form into vertex form when the coefficient of x^2 is other than 1 or −1. They should be able to work through such transformations as a group, though, as they did in activities like *Another Rocket*.

Progression

Students work on this activity individually and share results in a class discussion.

Approximate Time

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

You might follow up the work on Question 1 by asking about the x-intercepts of these quadratic functions. Ask whether students see a way to get the intercepts for Question 1b without putting the functions into vertex form. They should see that the function in Question 1c does not have any x-intercepts.

Question 2 provides practice with squaring binomials and manipulating expressions involving parentheses. You may want to encourage students to use intermediate steps, such first writing the expression in Question 2a as

$$-2(x^2 - 4x + 4) + 4$$

and then, after removing the parentheses, as

$$-2x^2 + 8x - 8 + 4$$

You may want to point out that this is an application of the distributive property. Also be on the alert for sign errors.

Questions 3 and 4 give more practice with finding an equation given the vertex and a point. You might ask students to rewrite each function in standard form.

For Question 4, the arc shape of the cross section of the cable forming the main span of the Golden Gate Bridge is actually a catenary curve, but a parabola shape is very close. The equation for the arc, using (0, 0) as the vertex and passing through the point (2100, 500), is $y = (500/2100)^2 \bullet x^2$. The height at the halfway point, where $x = 1050$, is $y = 125$.

Back to Bayside High

Intent

In *Back to Bayside High,* students return to and solve the unit problem.

Mathematics

These activities allow students to pull together all the ideas that have been at play throughout the unit:

- graphical and symbolic representations of quadratic functions, including connections between the vertex form of a function and key features of the graph, such as its vertex and direction of concavity
- properties of parabolic graphs, including symmetry and the vertex
- symbolic manipulations, many based on applications of the distributive property

Specifically, students will be transforming the function in the unit problem into vertex form and then using this form to find and interpret the graph's vertex and x-intercepts.

Progression

Students will solve the unit problem in the first two activities. In the final activity, they will summarize their solutions and methods in writing.

Fireworks in the Sky

Coming Down

A Fireworks Summary

Fireworks in the Sky

Intent

Students are now ready to return to the unit problem, presented in *Victory Celebration.* As they will see, this activity considers only part of the problem—the questions relating to the rocket's highest point. The question of when the rocket hits the ground is examined in the next activity, *Coming Down.*

Mathematics

Armed with the skills needed to transform a function in standard form into vertex form, students are now ready to tackle the unit problem. They will find the exact maximum height of the soccer team's rocket, and the exact time at which this height is reached, by reading the vertex coordinates from the vertex form of the function. The algebra of putting the expression $160 + 92t - 16t^2$ into vertex form is similar to students' work in *Another Rocket,* but the numbers make it somewhat more difficult.

Progression

Students work on the activity in groups. The class discussion that follows will focus on the procedures they used, as well as why they used them and why they work.

Approximate Time

30 minutes

Classroom Organization

Groups, followed by whole-class discussion

Doing the Activity

Before groups embark on this activity, you might remind them that they estimated the maximum height of the rocket at the beginning of the unit and emphasize that now they are to find it exactly, using vertex form.

Discussing and Debriefing the Activity

If some groups worked through the algebra, let them give presentations, but move slowly to be sure they don't lose the rest of the class. If students were not able to get all the details on their own, you may want to lead them through it so they can see the process in full. Ask volunteers to provide the ideas for individual steps as the whole class contributes by checking the mechanics.

Here are two possible sequences of steps for the algebraic transformation of the expression into vertex form, based on the second approach in the discussion of *Another Rocket.* Other sequences of steps are also possible. The steps in the two methods are the same except that the first uses fractions, while the second uses decimals. The steps are numbered for easy reference in the subsequent discussion.

If students use a decimal approach, emphasize that they are not to round off the decimals. (The fact that the denominator of the original fraction is a power of 2 implies it has a finite decimal expansion, so it's feasible to keep the values precise.)

Description	Fraction representation
Step 1: Begin with the original expression.	$160 + 92t - 16t^2$
Step 2: Isolate the constant term.	$160 - (16t^2 - 92t)$
Step 3: Factor out the coefficient of t^2 and simplify the fraction $\frac{92}{16}$.	$160-16\left(t^2-\frac{23}{4}t\right)$
Step 4: Add and subtract to complete the square.	$160-16\left[t^2-\frac{23}{4}t+\left(\frac{23}{8}\right)^2-\left(\frac{23}{8}\right)^2\right]$
Step 5: Write the perfect square as the square of a binomial.	$160-16\left[\left(t-\frac{23}{8}\right)^2-\left(\frac{23}{8}\right)^2\right]$
Step 6: Multiply through by 16.	$160-16\left(t-\frac{23}{8}\right)^2+16\bullet\left(\frac{23}{8}\right)^2$
Step 7: Combine the constant terms.	$-16\left(t-\frac{23}{8}\right)^2+\frac{1169}{4}$

The same process, using decimals, looks like this.

Description	Decimal Representation
Step 1: Begin with the original expression.	$160 + 92t - 16t^2$
Step 2: Isolate the constant term.	$160 - (16t^2 - 92t)$
Step 3: Factor out the coefficient of t^2 and write $92 \div 16$ as 5.75.	$160 - 16(t^2 - 5.75t)$
Step 4: Add and subtract to complete the square.	$160 - 16(t^2 - 5.75t + 2.875^2 - 2.875^2)$
Step 5: Write the perfect square as the square of a binomial.	$160 - 16[(t - 2.875)^2 - 2.875^2]$
Step 6: Multiply through by 16.	$160 - 16(t - 2.875)^2 + 16 \bullet 2.875^2$
Step 7: Combine the constant terms.	$-16(t - 2.875)^2 + 292.25$

If you need to lead the class through much of this algebra, here are some questions you might ask.

- Steps 2 and 3: **What initial steps did you use in other problems, especially when the coefficient of the squared term wasn't 1?**
- Step 4: **What needs to be added to complete the square?** Remind students, if needed, to subtract in order to compensate for adding. Whether students work with fractions or decimals, they might benefit from leaving the expression as the square of some number, either $\left(\frac{23}{8}\right)^2$ or 2.875^2, rather than finding the numeric value of the expression.

Even if they have difficulty working out all the steps, students should be able to follow the general process, to describe what's going on, and to use the final result.

Once students have the vertex form, you might want to give groups time to think about how to use it to answer the questions about the rocket. You can then have students compare the exact answers they get with the approximate solutions found at the beginning of the unit.

Here are the answers to this part of the unit problem.

- The rocket reaches its maximum height exactly $\frac{23}{8}$ seconds, or $2\frac{7}{8}$ seconds, or 2.875 seconds after launch.
- At its highest point, the rocket is exactly $\frac{1169}{4}$ feet, or $292\frac{1}{4}$ feet, or 292.25 feet above the ground.

Coming Down

Intent

This activity completes work on the unit problem begun in *Fireworks in the Sky*, as students find the time it takes for the rocket to return to the ground.

Mathematics

Having transformed the quadratic function in the unit problem from standard form to vertex form, students are now ready to find the exact amount of time it will take the rocket to return to the ground. To do this, they will set the vertex form of the function equal to zero and solve for t.

Progression

Students work on the activity individually and then participate in a whole-class discussion of the process of solving the equation.

Approximate Time

5 minutes for introduction

20 minutes for activity (at home or in class)

25 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Read through the activity as a class. Make sure students have a general idea of the process for solving the problem.

Discussing and Debriefing the Activity

After students give an articulate statement of the connection between the algebra and the context, you may want to focus the discussion on the broad outline of the steps needed to get from vertex form to the x-intercept. These are the key steps for both the decimal and fractional forms of the function.

Description	Decimal representation	Fraction representation
Set up the equation.	$-16(t - 2.875)^2 + 292.25 = 0$	$-16\left(t - \frac{23}{8}\right)^2 + \frac{1169}{4} = 0$
Begin to isolate the variable.	$16(t - 2.875)^2 = 292.25$	$16\left(t - \frac{23}{8}\right)^2 = \frac{1169}{4}$

Divide both sides by 16.	$(t - 2.875)^2 = 18.265625$	$\left(t-\frac{23}{8}\right)^2=\frac{1169}{4}$
Take the (positive) square root of both sides.	$t-2.875=\sqrt{18.265625}$	$t-\frac{23}{8}=\sqrt{\frac{1169}{64}}$
Add to both sides to get *t*.	$t=\sqrt{18.265625}+2.875$	$t=\frac{\sqrt{1169}}{8}+\frac{23}{8}$

Either route leads to a solution of approximately 7.15 seconds, which students should identify as the time it takes for the rocket to hit the ground.

You can use this as an occasion to look back at the ideas from *Simply Square Roots* in *Do Bees Build It Best?,* asking students how they might simplify $\sqrt{\frac{1169}{64}}$. The purpose is not to get a "better" answer (the expression $\frac{\sqrt{1169}}{8}$ is really not any better), but to review some general principles about square roots.

Vertex Form for the General Quadratic

If it seems worthwhile and appropriate, you might have students look at putting the general quadratic function, $y = ax^2 + bx + c$, into vertex form, using their work on the height function from *Fireworks in the Sky* as a model. Even if you have to lead them through the process, they will see that the steps can be done for any quadratic function.

The steps might go like this:

$$y = ax^2 + bx + c$$

$$= c + a\left[x^2 + \left(\frac{b}{a}\right)x\right]$$

$$= c + a\left[x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right]$$

$$= c + a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right]$$

$$= c - a\left(\frac{b}{2a}\right)^2 + a\left(x + \frac{b}{2a}\right)^2$$

$$= \frac{4ac - b^2}{4a} + a\left(x + \frac{b}{2a}\right)^2$$

(The combining of terms in the last step is not essential to the big picture.) Students can then identify the vertex itself from the vertex form. (Setting the final expression equal to zero and solving for x is all that is left to derive the quadratic formula.)

Supplemental Activity

The Quadratic Formula (extension) asks students to apply and then derive the quadratic formula. You may want to go through the derivation with them.

A Fireworks Summary

Intent

Students summarize their work on the central unit problem, presented in *Victory Celebration.*

Mathematics

Students articulate the methods they used to address the unit problem.

Progression

This summative activity, to be done by students working individually, will produce a summary of the unit problem's solutions and solution methods.

Approximate Time

5 minutes for introduction

25 minutes for activity (at home)

Classroom Organization

Individuals

Doing the Activity

Tell students that this activity will be part of their unit portfolios. As with other summative activities, this is a chance for students to review what they have been doing for several weeks, to organize it in their minds and on paper, and to clarify any questions they might still have.

Intercepts and Factoring

Intent

These activities introduce another symbolic method for solving quadratic equations and ask students to summarize their work over the entire unit.

Mathematics

Factoring polynomials plays a large role in classical mathematics, persists as a major topic in traditional mathematics curricula, and appears on some standardized tests. Factoring is included in this unit so students can become familiar with the concept. It has been proven mathematically that almost all polynomials are prime and hence do not factor. For solving equations in applied situations outside of school mathematics, factoring is seldom used, and computers quickly estimate the roots of polynomials as accurately as needed. However, some polynomials can be factored, and when they can, it is easy to find the x-intercepts.

In *Intercepts and Factoring,* students use the area model for multiplication to factor quadratic expressions with leading coefficient of 1 or −1. They combine factoring with the **zero product rule** ($ab = 0$ if and only if $a = 0$ or $b = 0$) to solve quadratic equations and connect the solutions to the x-intercepts of the quadratic function. If a quadratic equation can be factored as a product of two linear terms, then it is easy to solve because of the zero product rule.

Progression

The activities support the development of factoring as a method for solving quadratics and then turn to completion of the unit portfolio.

Factoring

Let's Factor!

Solve That Quadratic!

Quadratic Choices

A Quadratic Summary

Fireworks Portfolio

Factoring

Intent

This activity introduces the technique of solving quadratic equations by factoring.

Mathematics

Factoring, when it is possible, is one way to find the solutions of quadratic equations. Factoring reduces a quadratic expression to the product of two linear expressions. This makes solving the equation easier because of the **zero product rule:** $ab = 0$ if and only if $a = 0$ or $b = 0$. If $ax^2 + bx + c = 0$ can be rewritten, in factored form, as

$$(px + q)(rx + s) = 0$$

then $px + q = 0$ or $rx + s = 0$. These linear equations are easy to solve. For simplicity, the work in this unit on factoring focuses on quadratic expressions in which the coefficient of x^2 is 1.

Progression

After a whole-class discussion of factoring, students work on this activity in groups.

Approximate Time

25 minutes

Classroom Organization

Whole class, then groups

Doing the Activity

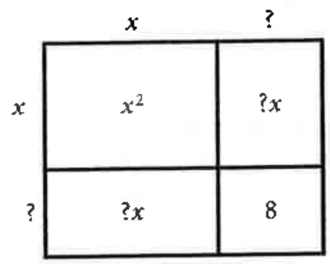
Students may be familiar with the term *factoring* from work with whole numbers, such as writing 20 as 4 • 5. Many will recall drawing "factor trees" in middle school. They know, from working on the POW *Twin Primes,* that if a whole number has as factors only itself and 1, then it is prime. Numbers that are not prime can be written as a product of two or more factors greater than 1.

Explain that factoring in algebra has a meaning similar to factoring with whole numbers, but in algebra factoring applies to polynomials with whole-number coefficients. A polynomial is factored when it is written as a product of polynomials of equal or lesser degree. For example, students learned in *Distributing the Area II* that $x^2 + 3x - 10$ can be factored as $(x + 5)(x - 2)$ and in *Square It!* that $x^2 + 14x + 49$ is equivalent to $(x + 7)^2$.

Unlike multiplying, there is often no direct approach to factoring. Factoring polynomials usually involves making an educated guess about the factors and then checking the answer by multiplying the factors to see if they give the original polynomial.

Drawing an area diagram and entering what is known can help one to make an educated guess.

If students have trouble getting started, you might work through Question 1 together. To factor $x^2 + 6x + 8$, start with an area diagram like that shown in the student book and fill in the upper-left and lower-right rectangles. The upper-right and lower-left rectangles must then have a total area of $6x$.



With some guess-and-check, students should see that if $x + 2$ and $x + 4$ are chosen as the dimensions, they will get the desired outcome. They can thus factor $x^2 + 6x + 8$ as $(x + 2)(x + 4)$.

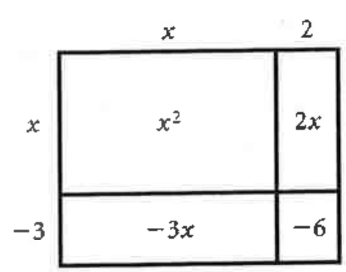
Once we know the factors, it is easy to find the x-intercepts—the values of x for which the product is zero—by applying the zero product rule: a set of factors multiply to give zero if and only if one or more of the factors equals zero.

For $(x + 2)(x + 4)$, the first factor is 0 when $x = -2$. Then $(-2 + 2)(-2 + 4) = 0(2) = 0$. The second factor is 0 when $x = -4$. Then $(-4 + 2)(-4 + 4) = -2(0) = 0$. Therefore, the values of x that make the original quadratic $x^2 + 6x + 8$ equal 0 are -2 and -4. Of course, these values can also be found by completing the square.

Discussing and Debriefing the Activity

Allow time for several presentations that focus on the thinking that went into each solution.

You may want to give students another example in which the constant term is negative so that the x term must be thought of as a sum of a positive and a negative portion. For example, for the expression $x^2 - x - 6$, students will have to think of $-x$ as the sum $-3x + 2x$, leading to a diagram like this.



Also offer an example such as $x^2 + 3x + 1$, for which it is fairly clear that the expression cannot be factored, at least not using integer coefficients.

At the conclusion of this discussion, ask students whether there are any patterns or regularities that suggest a method for factoring quadratics.

Let's Factor!

Intent

This activity is a continuation of *Factoring.*

Mathematics

Students continue to develop their understanding of the factoring process as they use factoring to locate the x-intercepts of quadratic functions—the values of x for which a quadratic function has a y-value of zero.

Progression

Students work on the activity individually and share results with the class.

Approximate Time

20 minutes for activity (at home or in class)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

No introduction is needed for this activity.

Discussing and Debriefing the Activity

For the examples that can be factored (Questions 1a, 1b, 1d, and 1e), be sure students see how to find the x-intercepts (Questions 2a, 2b, 2d, and 2e) from the factored expressions. For Question 1e, you might bring out that the two x terms in the related area model must total 0. Emphasize that if a quadratic expression can be factored, the graph of its related quadratic equation must have x-intercepts.

For Questions 1c and 1f, ask students to use vertex form or graphs to explain why the expressions can't be factored. Bring out these observations.

- The vertex form for Question 1c is $(x + 3)^2 + 1$ so the graph has its vertex at $(-3, 1)$ and is concave up. It therefore has no x-intercepts and thus can't be factored.
- The vertex form for Question 1f is $(x - 5)^2 - 19$ so the graph does have intercepts, but they are irrational, so factoring is not useful. The intercepts can be found by setting $(x - 5)^2 - 19$ equal to 0 and solving to get $x = 5 + \sqrt{19}$ and $x = 5 - \sqrt{19}$.

This unit is not intended to provide students with a full exposure to factoring techniques. The goal is for them to become aware of factoring as one possible method for solving some quadratic equations. They should realize that this method is very convenient if it works, but that it works only when the intercepts are rational (not square roots). If the intercepts are irrational, one can always use the vertex form approach.

Supplemental Activity

Let's Factor Some More! (reinforcement) encourages students to use an area model to factor quadratics in which the coefficient of the x^2 term isn't 1.

Solve That Quadratic!

Intent

This activity follows up on the second question of *Let's Factor!*, in which students connect solutions of quadratic equations to x-intercepts of the related graphs.

Mathematics

The values that make a quadratic equation equal to zero are the same as the values of the x-intercepts of the related graph. In this activity, students synthesize algebraic and geometric approaches to solving quadratic equations.

Progression

Students will work on this activity in groups to find solutions to quadratic equations and then to solve a quadratic equation drawn from a familiar context.

Approximate Time

5 minutes for introduction

20 minutes for activity (at home or in class)

10 minutes for discussion

Classroom Organization

Individuals or groups, followed by whole-class discussion

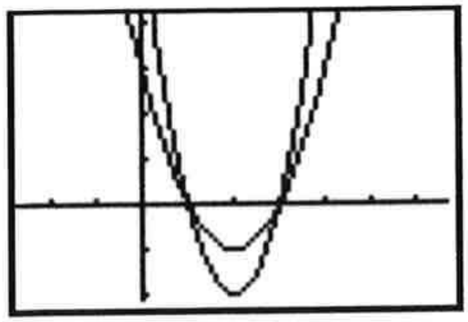
Doing the Activity

You might introduce this activity by reviewing the connection between solving quadratic equations and finding x-intercepts. Bring out that just as the number of x-intercepts for the graph of a quadratic function can be 0, 1, or 2, the number of solutions to a quadratic equation can be 0, 1, or 2.

Discussing and Debriefing the Activity

Students might approach the equations in Question 1 in various ways, but be sure the use of factoring is addressed. Questions 1a and 1b provide simple illustrations of that approach.

You can use Question 1c to illustrate the technique of simplifying an equation by dividing both sides by a constant. You might also bring out that although the equations $y = 2x^2 - 8x + 6$ and $y = x^2 - 4x + 3$ have different graphs, they have the same x-intercepts.



For Question 1d, be sure students explain why the equation has no solutions. They might do this using the graph or using the vertex form.

For Question 2, if students use the same labeling as in *Pens and Corrals in Vertex Form,* they should get the equation $x(500 - 2x) = 20{,}000$ for Question 2a. They might solve this in various ways, such as using a graph, vertex form, or factoring. Students should see several perspectives, such as that the line $y = 20{,}000$ crosses the graph of $y = x(500 - 2x)$ in two places.

Ask, **What do the two solutions represent in terms of the pen?** In this case, the two solutions represent different shapes for the pen. The solution $x = 50$ gives a 50-by-400 pen and the solution $x = 200$ gives a 200-by-100 pen.

You might lead a general discussion here about how students might decide what method to use for solving a quadratic equation. Among the considerations are these: Do they need an exact answer? How difficult is the algebra for the different methods? For instance, if an approximate answer will suffice, graphing may be the simplest approach. If students are doing a problem using an algebraic method, they might consider the factoring method if the expression is already in factored form or if the numbers seem easy to work with. On the other hand, as the vertex form method will always work, they might simply rely on this as their standard approach.

Supplemental Activities

Vertex Form Challenge (extension) gives students practice changing quadratic functions in standard form with leading coefficients other than 1 or −1 into vertex form. The problems involve fractions and decimals.

A Big Enough Corral (extension) explores quadratic inequalities. To do Question 2, students must be able to factor quadratics.

Factors of Research (extension) suggest further areas of exploration in the topic of factoring. Question 2 asks for a generalization of the difference of squares introduced in the supplemental activity *A Lot of Symmetry*.

Quadratic Choices

Intent

Students step back from the details of the two methods they have been learning for finding the x-intercepts of a quadratic function to reflect on when they might choose one method over the other.

Mathematics

Students now have three methods from which they can choose to find the x-intercepts of a quadratic function:

- Graphing, which will show how many intercepts and give their approximate values
- Vertex form, which will always give exact values for the x-intercepts, if they exist
- Factoring, which in certain limited cases can provide a quick way to find exact values for x-intercepts

The focus of this activity is on when to use each method.

Progression

Students will work on this activity individually. In Question 4, which will be the focus of the follow-up class discussion, students make general observations about the values of a, b, and c and how they might affect the choice of methods.

Approximate Time

25 minutes for activity (at home or in class)

25 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

This activity requires little or no introduction.

Discussing and Debriefing the Activity

Students will presumably want to use vertex form for Question 1 and factored form for Question 2, as the functions are in the given form. But there may be an interesting discussion for Question 3 and for the broader Question 4.

There are no right answers, and students may have legitimate disagreements about what's easiest. Be sure they see that the vertex form method will always work, even when factoring may be simpler.

Generalizing About x-Intercepts

For the general vertex form $y = a(x - h)^2 + k$, the number of x-intercepts depends only on a and k. This leads to these three general principles.

- If $k = 0$, there is exactly one x-intercept (no matter what a is).
- If k and a have the same sign, there are no x-intercepts.
- If k and a have opposite signs, there are two x-intercepts.

The teacher commentary for *Coming Down* notes that when the general quadratic function, written in standard form, is $y = ax^2 + bx + c$, then in vertex form it is

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Students can apply the three general points above to this situation, where k is $\frac{4ac - b^2}{4a}$. You might bring out that the denominator $4a$ has the same sign as a, so the existence of x-intercepts depends only on the sign of $4ac - b^2$. If $4ac - b^2$ is 0, there is one x-intercept; if $4ac - b^2$ is positive, there are no x-intercepts; and if $4ac - b^2$ is negative, there are two x-intercepts.

Traditional presentations state these conclusions like this:

- If $b^2 - 4ac = 0$, there is one x-intercept.
- If $b^2 - 4ac > 0$, there are two x-intercepts.
- If $b^2 - 4ac < 0$, there are no x-intercepts.

Supplemental Activities

Make Your Own Intercepts (extension) builds on the idea that students can now easily find a quadratic that has two given x-intercepts. For example, for intercepts $x = 4$ and $x = 2$, the quadratic $y = (x - 4)(x - 2)$ will do. However, it isn't the *only* quadratic with those intercepts. All quadratics $y = a(x - 4)(x - 2)$ for any real number a will also work.

Quadratic Challenges (reinforcement) offers three more quadratic equations for students to solve. (A graphing calculator would make finding the requested decimal answers too easy.)

Standard Form, Factored Form, Vertex Form (reinforcement) pulls together the relationships among standard form, factored form, x-intercepts, vertex, and vertex form. This activity makes a good group assignment.

A Quadratic Summary

Intent

Students begin their portfolio preparation by summarizing the big ideas of the unit.

Mathematics

Some of the ideas students may discuss in their summaries are listed here.

- The graphs of quadratic functions, which are parabolas with symmetry and a turning point
- The connections between symbolic and graphical representations of quadratic functions
- Several methods for manipulating symbols to find key features of a function's graph, including the vertex and x-intercepts
- Methods for solving quadratic equations and their relationship to graphs of the corresponding functions

Progression

After a brief introduction, each student will work on his or her own to write a summary of the unit's big ideas.

Approximate Time

5 minutes for introduction

25 minutes (at home)

15 minutes for discussion

Classroom Organization

Individuals, followed by whole-class discussion

Doing the Activity

Tell students that this summary of the unit's big ideas will be part of their unit portfolios. To help them start thinking about the summary, ask volunteers to share ideas about some of the key concepts from the unit. Also mention the reference page in the student book, *A Summary of Quadratics and Other Polynomials,* which students may find helpful.

Discussing and Debriefing the Activity

As one aspect of the summary, ask students to discuss what sorts of situations they have seen in which quadratic functions occurred. Here are the main contexts from this unit; students may mention other possibilities as well.

- Gravitational fall, as in the unit problem, *Victory Celebration*
- Area (or volume with one dimension fixed), as in *A Corral Variation*

- Problems involving right triangles and the Pythagorean theorem, as in *Revisiting Leslie's Flowers*
- Problems involving maximizing profit, using the assumption that sales quantity is a linear function of price, as in *Profiting from Widgets*

Fireworks Portfolio

Intent

Students compile their unit portfolios and write their cover letters.

Mathematics

Portfolios for this unit include students' reflective work on the two activities *A Fireworks Summary* (in which they looked back over their work on the unit problem) and *A Quadratic Summary* (in which they reviewed the unit's big mathematical ideas). It also includes activities that helped them understand the value of vertex form in solving real-world problems and activities that helped them become comfortable with the mechanics of working with quadratic expressions.

Progression

Students start work on their portfolios in class by reading the instructions in the student book. They then work independently to review their work in the unit, select samples, reflect on the evidence of their learning, and write cover letters.

Approximate Time

10 minutes for introduction

35 minutes for activity (at home or in class)

Classroom Organization

Individuals

Doing the Activity

Have students read the instructions in the student book carefully.

By this point in the curriculum, students are familiar with the process of assembling a unit portfolio. Stress that this is their chance to identify any difficulties or questions they might still have about the mathematics of the unit. Remind students that their cover letters are important components of their portfolios, as they communicate with you what they have learned in the unit.

Discussing and Debriefing the Activity

You may want to have students share their portfolios in their groups, comparing what they wrote about in their cover letters and the activities they selected.

Assessments

In-Class Assessment

1. Write an equation for a quadratic function whose graph has its vertex at the point (−5, 3).

2. Write the expression $x^2 + 8x + 11$ in vertex form. Show how to use your work to get the vertex of the graph of the function $f(x) = x^2 + 8x + 11$.

3. What needs to be added to $x^2 + 18x$ to make it a perfect square? Explain your answer using an area diagram.

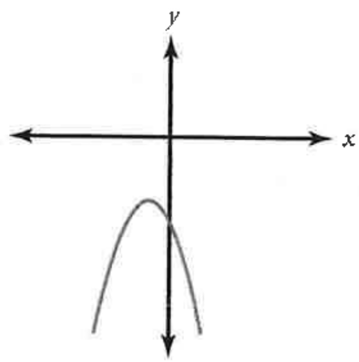
4. Write the quadratic expression $(x - 3)^2 + 8$ in standard form.

Take-Home Assessment

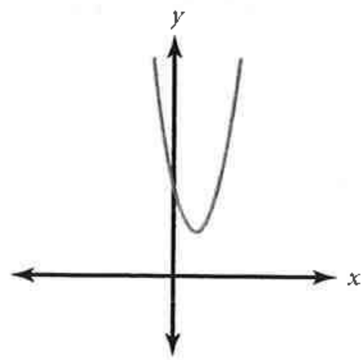
1. The zero product rule says that if a product of two numbers is zero, then at least one of those numbers must equal zero. Explain how the zero product rule is useful in solving equations.

2. Each graph here is the graph of a quadratic function. No scales are given, so all you know about a particular point is which quadrant it is in. The scales on the four graphs might not be the same.

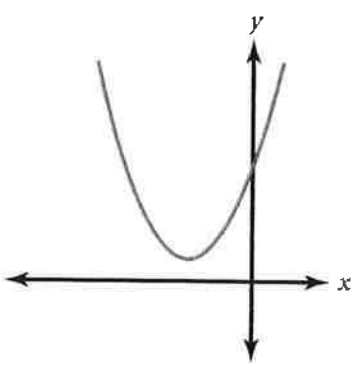
a.



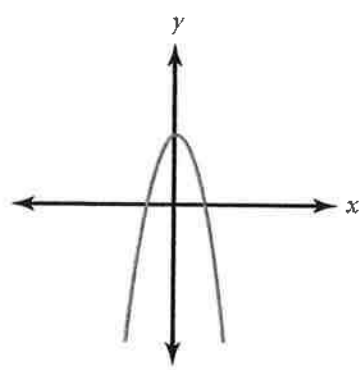
b.



c.



d.



Match each graph with one of the equations listed here. For each graph, explain why that equation is the only one from the list that is appropriate.

i. $y = 3x^2 + 5$

ii. $y = 2(x - 1)^2 + 2$

iii. $y = -(x + 1)^2 - 3$

iv. $y = -x^2 + 6$

v. $y = -3(x - 4)^2 + 2$

vi. $y = \frac{1}{2}(x + 3)^2 + 1$

In Acapulco, Mexico, there is famous place called La Quebrada, or "The Break in the Rocks," where divers jump from high on the cliffs into the Pacific Ocean. Once a diver jumps, there is a relationship between how far from the cliff the diver is and how high above the water the diver is.

The relationship is given by the function $y = -(x - 0.5)^2 + 27.25$. In this function, x is the diver's distance from the cliff and y is the diver's height above the water. Both distances are in meters.

3. How high is the diver above the water when the diver is 3 meters from the cliff?

4. How far from the cliff is the diver when the diver is at the highest point above the water? How high is the diver at that time?

5. How far from the cliff is the diver when the diver is 7 meters above the water?

Fireworks Guide for the TI-83/84 Family of Calculators

This Calculator Guide gives suggestions for calculator use with selected activities of the Year 2 unit *Fireworks*. The associated Calculator Notes contain detailed calculator keystroke instructions that you can distribute to your students. NOTE: If your students have the TI-Nspire handheld, they can attach the TI-84 Plus keypad (from Texas Instruments) and use the calculator notes for the TI-83/84.

The *Fireworks* unit provides many opportunities for students to explore functional relationships using the graphing calculator, and to recall and practice many of the skills needed to work with this tool. Students will graph and create tables to investigate quadratic functions. Students can learn other features, like finding an x-intercept or a root, as they work through the curriculum.

Although much of the mathematics in this unit is traditional in nature, the opportunities for students to develop their understanding are enhanced by the use of the graphing calculator. Strong connections are made between symbolic and graphical representations of quadratic situations. Furthermore, students contemplate and assess the importance and distinction of approximate versus exact results.

Parabolas and Equations I: In the sequence of lessons *Parabolas and Equations I, II,* and *III,* students use graphing calculators to explore what happens to the graph of a parabola when the parameters a, k, and h in $y = ax^2$, $y = ax^2 + k$, and $y = (x - h)^2$ are changed. In this first activity, students explore the graphs of quadratic equations, and investigate how changes in the coefficients of a quadratic function affect its graph—that is, how the graph of $y = ax^2$ changes when the parameter a is changed. See the discussion in the *Teacher's Guide* activity notes for this activity. Additionally, this activity is an opportunity to refresh skills with graphing on the calculator. If your students need support with this, distribute the Calculator Note "Function Graphing".

Parabolas and Equations II: In this activity, students use the graphing calculator to explore how the graph of $y = ax^2 + k$ changes when the parameter k is changed.

Parabolas and Equations III: In this third activity, students use the graphing calculator to explore how the graph of $y = (x - h)^2$ changes when the parameter h is changed. Students have now done sufficient exploration to predict the appearance of any graph of an equation in the form $y = a(x - h)^2 + k$. They can quickly verify their predictions using the graphing calculator.

Vertex Form for Parabolas: In this activity, students use a graphing calculator to find the equation that produces a given graph. In this way, students gain facility with predicting both a graph based on an equation and an equation based on a graph.

Using Vertex Form: Students use the graphing calculator to further explore the relationships between graph and equation, and identifying the vertex. In Question 3, students estimate the coordinates of a vertex using a calculator graph. In doing this, it will be useful to adjust the window. This can be done quickly using the Zoom feature, and students will also use the Trace feature—both these features are discussed in the Calculator Note "Function Graphing."

Is It a Homer? In this activity, students will benefit from verifying their solution by graphing.

Square It!: In this activity, students change quadratic equations from vertex form to standard form. They can check their answers by graphing the two equations and verifying that they are equivalent—they should have the same graph. In Question 3, students can verify their answers by graphing.

Squares and Expansions: Students can verify their answers to Question s 2 and 3 by graphing.

POW: Twin Primes: If your students have done any calculator programming, you may want to suggest they write a program that gives all twin primes between two input numbers.

Squares and Expansions: Students can verify their answers by graphing.

How Much Can They Drink?: Students might use the graphing calculator to find the vertex and x-intercepts by tracing the graph.

Finding Vertices and Intercepts: Be sure that students practice finding vertices and intercepts using symbolic methods. The graphing calculator is useful for checking answers, but you may want to instead reinforce algebraic ways of verifying solutions at this point.

Another Rocket: To verify answers, you might have students graph the function on their calculators and then use the Zoom feature to estimate the x-intercepts. If their answers are correct, their answers should agree with the calculator values.

Coming Down: In this activity, students solve an equation to find when a firework hits the ground. You might wish to introduce students to estimating intercepts using a graphing calculator, using the Calculator Note "Determining x-Intercepts." You can have students check their work in future activities using these processes.

Solve That Quadratic!: This activity is a good opportunity to use graphs to reinforce the connection between solving quadratic equations and finding x-intercepts. Point out that just as the number of x-intercepts for the graph of a quadratic function can be 0, 1, or 2, so also the number of solutions for a quadratic equation can be 0, 1, or 2. Graph some of the equations and elicit that the solution to the quadratic equation are the values of the x-intercept.

Quadratic Choices: A third choice for finding the x-intercepts of a quadratic function is by graphing, which will show how many x-intercepts there are and will give approximate values for them. You may need to reiterate that you expect students to be able to find x-intercepts using symbolic methods, although graphing can be useful for verification.

Supplemental Problems:

Quadratic Challenges: There are three more quadratic equations to solve. The use of a graphing calculator makes finding the decimal answers too easy.

Calculator Notes

Function Graphing

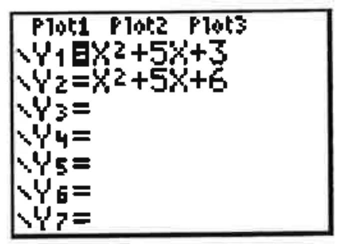
These instructions describe the five basic graphing keys. You will find these keys immediately below the calculator's screen.

Using the [Y=] Editor

Press [Y=] to display this screen and enter functions. Use the [X,T,Θ,n] key for the independent variable *x* when you enter a function.

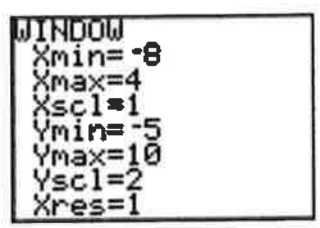
To remove a function from this list, move the cursor to that line (to the right of the = sign) and press [CLEAR].

You can also make a function inactive without removing it. To do this, move the cursor over the = sign for that line and press [ENTER]. The = sign will no longer be highlighted. In the example shown here, the function **Y1=X^2+5X+3** is active and the function **Y2=X^2+5X+6** is inactive. The calculator will graph only active functions. To make the function active again, move the cursor to the = sign and press [ENTER] again.



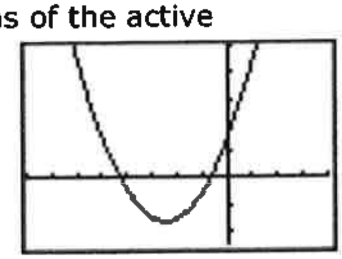
Setting the Viewing Window

Press [WINDOW]. This is where you tell the calculator what part of the graph to show and how to scale the axes. The **Xmin** and **Xmax** values determine the left and right bounds of the graph. The **Ymin** and **Ymax** values determine the bottom and top set at 1.bounds.The **Xscl** and **Yscl** values determine the frequency of the tick marks on each axis. **Xres** should be



Displaying a Graph

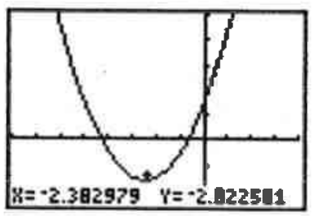
Press [GRAPH]. This tells the calculator to draw the graphs of the active functions in the [Y=] edit screen. The portion of the graph you see will match the settings in the [WINDOW] display.

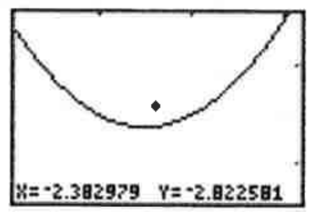


Exploring Graphs with ZOOM

Press ZOOM. The Zoom features are other ways to tell the calculator what part of the graph to show, and they automatically adjust the window settings.

6:ZStandard resets the window to a standard setting—*x* and *y* ranging from –10 to 10, with tick marks every 1 unit. To use **2:Zoom In** or **3:Zoom Out**, highlight your choice and press ENTER. You will return to the graph view, and a blinking cursor will appear at the center of the graph. Move the cursor to the spot you want to zoom in to (or out from) and press ENTER again. In the example below, **2:Zoom In** has been selected, and the cursor has been moved to near the parabola's vertex. Pressing ENTER zooms in on the vertex. Note that the coordinates of the cursor, shown at the bottom of the screen, remain the same.

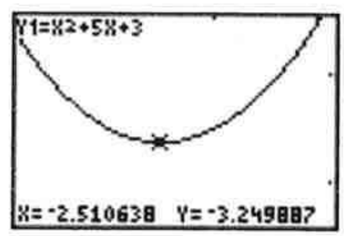




ZSquare adjusts your window to make the *x* and *y* units on the screen equally-sized, to keep your graphs from being stretched or distorted.

Exploring Graphs with TRACE

Once you have a graph displayed, press TRACE and press the left and right arrow keys to move the cursor along the graph. The *x*- and *y*-coordinates of the cursor are shown at the bottom of the screen. If you have graphed more than one function, use the up and down arrow keys to move the cursor between the different functions. The equation of the function the cursor is on is shown at the top of the screen.

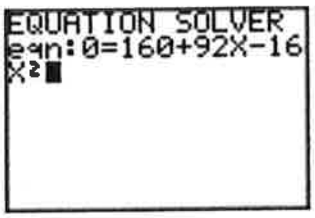


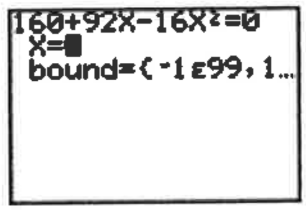
You can use **Zoom In** in conjunction with TRACE to find the coordinates of a point on the graph, such as a vertex, with greater accuracy.

Determining x-Intercepts

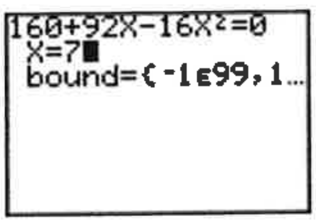
You can use the calculator to solve equations when one side of the equation is zero. The examples here show several methods for finding (at least approximately) an x-intercept of the function $h(t) = 160 + 92t - 16t^2$ from the activity *Coming Down*.

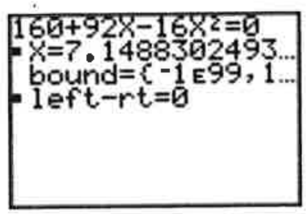
To use the SOLVER feature, press MATH and then scroll down and select **0:Solver...**. The first screen should say EQUATION SOLVER across the top. If not, press the up arrow key once. Below the words EQUATION SOLVER, you'll find **eqn:0=**. Enter your function here. After typing in **160 + 92X – 16X²**, press ENTER.



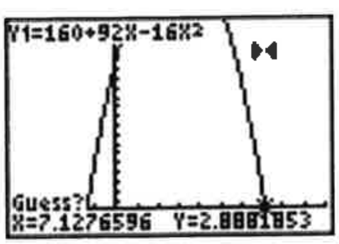


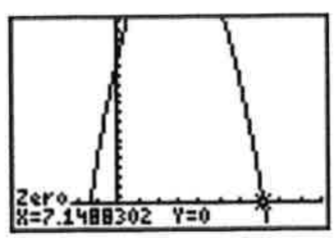
Your cursor should be at the **X=** as shown on the screen. Enter an estimate of where an x-intercept might be and then press ALPHA [SOLVE]. The next screens illustrate an example of what might result. Investigate what happens when you adjust the estimate you entered.





A second method of finding an x-intercept is to instruct the calculator to determine the x-intercept while working from a graph of the function. To do this, press 2ND [CALC] and select **2:zero**. Next, the calculator will request guidance for its calculation algorithm by asking first for a left and right bound and then for a guess between the two. Move the cursor along the graph using the arrow keys, and press ENTER in response to each prompt. After these three prompts are entered, the calculator will display the x-value of the intercept.





[–5, 10, 1, –20, 200, 10]

Assessments

In-Class Assessment

1. Write an equation for a quadratic function whose graph has its vertex at the point (–5, 3).

2. Write the expression $x^2 + 8x + 11$ in vertex form. Show how to use your work to get the vertex of the graph of the function $f(x) = x^2 + 8x + 11$.

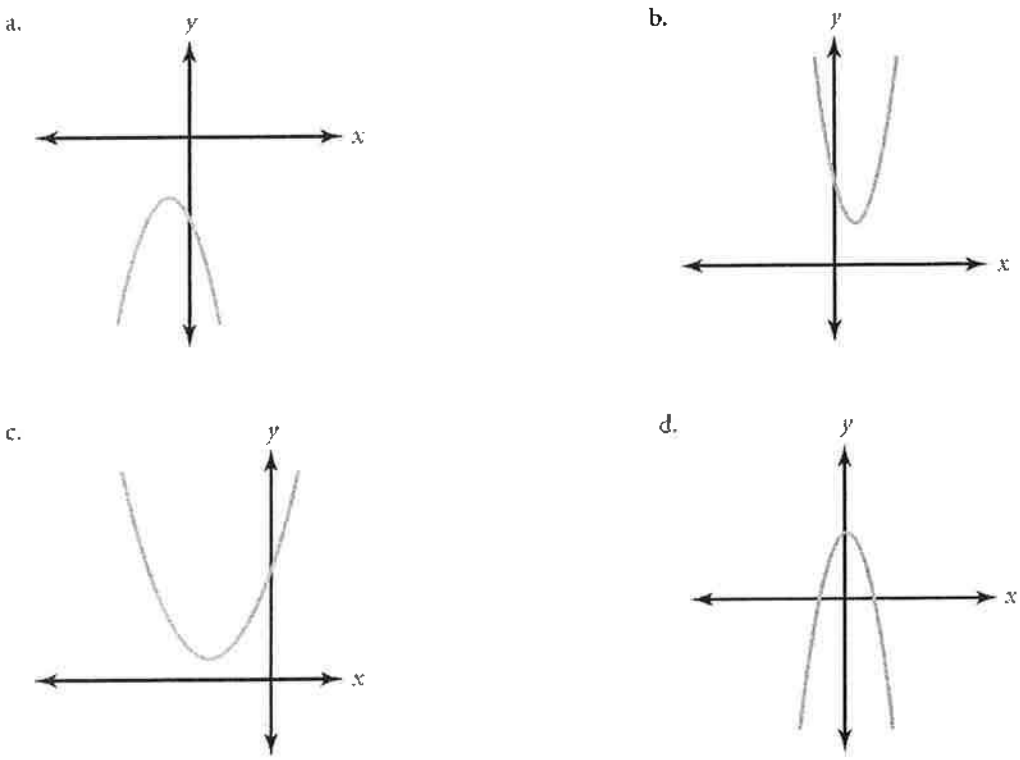
3. What needs to be added to $x^2 + 18x$ to make it a perfect square? Explain your answer using an area diagram.

4. Write the quadratic expression $(x - 3)^2 + 8$ in standard form.

Take-Home Assessment

1. The zero product rule says that if a product of two numbers is zero, then at least one of those numbers must equal zero. Explain how the zero product rule is useful in solving equations.

2. Each graph here is the graph of a quadratic function. No scales are given, so all you know about a particular point is which quadrant it is in. The scales on the four graphs might not be the same.



Match each graph with one of the equations listed here. For each graph, explain why that equation is the only one from the list that is appropriate.

i.	$y = 3x^2 + 5$	iv.	$y = -x^2 + 6$
ii.	$y = 2(x - 1)^2 + 2$	v.	$y = -3(x - 4)^2 + 2$
iii.	$y = -(x + 1)^2 - 3$	vi.	$y = \frac{1}{2}(x + 3)^2 + 1$

In Acapulco, Mexico, there is famous place called La Quebrada, or "The Break in the Rocks," where divers jump from high on the cliffs into the Pacific Ocean. Once a diver jumps, there is a relationship between how far from the cliff the diver is and how high above the water the diver is.

The relationship is given by the function $y = -(x - 0.5)^2 + 27.25$. In this function, x is the diver's distance from the cliff and y is the diver's height above the water. Both distances are in meters.

3. How high is the diver above the water when the diver is 3 meters from the cliff?

4. How far from the cliff is the diver when the diver is at the highest point above the water? How high is the diver at that time?

5. How far from the cliff is the diver when the diver is 7 meters above the water?