

Mathematics IV Frameworks Student Edition

Unit 4 Introduction to Trigonometry

1st Edition

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Mathematics IV – Unit 4

Introduction to Trigonometry

Student Edition

INTRODUCTION:

This unit is entitled *Introduction to Trigonometry*. In fact, students were first introduced to right-triangle trigonometry and its applications in Mathematics II. In this unit, students will expand their understanding of trigonometry as they are asked to use the circle to define the trigonometric functions. They will come to understand angle measurement in both degrees and radians and will be provided with experiences in applying the six trigonometric functions as functions of angles in standard position and as functions of arc lengths on the unit circle. Contextual situations will provide the need to find values of trigonometric functions by using given points on the terminal sides of standard position angles. At the conclusion of the unit 4 study, students should be able to find values of trigonometric functions by using the unit circle.

ENDURING UNDERSTANDINGS:

- Understand the relationship between right triangle trigonometry and unit circle trigonometry
- Use angle measure in degrees and radians interchangeably
- Recognize and use the reciprocal relationships of the six trigonometric functions
- Use the unit circle to define trigonometric functions

KEY STANDARDS ADDRESSED:

MM4A2. Students will use the circle to define the trigonometric functions.

- a. Define and understand angles measured in degrees and radians, including but not limited to 0° , 30° , 45° , 60° , 90° , their multiples, and equivalences.
- b. Understand and apply the six trigonometric functions as functions of general angles in standard position.
- c. Find values of trigonometric functions using points on the terminal sides of angles in the standard position.
- d. Understand and apply the six trigonometric functions as functions of arc length on the unit circle.
- e. Find values of trigonometric functions using the unit circle.

RELATED STANDARDS ADDRESSED:**MM4P1. Students will solve problems (using appropriate technology).**

- a. Build new mathematical knowledge through problem solving.
- b. Solve problems that arise in mathematics and in other contexts.
- c. Apply and adapt a variety of appropriate strategies to solve problems.
- d. Monitor and reflect on the process of mathematical problem solving.

MM4P2. Students will reason and evaluate mathematical arguments.

- a. Recognize reasoning and proof as fundamental aspects of mathematics.
- b. Make and investigate mathematical conjectures.
- c. Develop and evaluate mathematical arguments and proofs.
- d. Select and use various types of reasoning and methods of proof.

MM4P3. Students will communicate mathematically.

- a. Organize and consolidate their mathematical thinking through communication.
- b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
- c. Analyze and evaluate the mathematical thinking and strategies of others.
- d. Use the language of mathematics to express mathematical ideas precisely.

MM4P4. Students will make connections among mathematical ideas and to other disciplines.

- a. Recognize and use connections among mathematical ideas.
- b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
- c. Recognize and apply mathematics in contexts outside of mathematics.

MM4P5. Students will represent mathematics in multiple ways.

- a. Create and use representations to organize, record, and communicate mathematical ideas.
- b. Select, apply, and translate among mathematical representations to solve problems.
- c. Use representations to model and interpret physical, social, and mathematical phenomena.

Unit Overview:

The individual tasks contain the skills and a variety of activities to provide hands on experience with the unit circle. All concepts are initially developed using degree measures. Radians are introduced in the “What is a Radian” activity and then previous concepts are revisited. The culminating task highlights multiple applications of the concepts in this unit through rides at a typical county fair. There are many extensions and further applications that interested students or teachers could utilize.

Vocabulary and Formulas:

An angle is in ***standard position*** when the vertex is at the origin and the initial side lies on the positive side of the x-axis.

The ray that forms the ***initial side*** of the angle is rotated around the origin with the resulting ray being called the ***terminal side*** of the angle.

An angle is ***positive*** when the location of the terminal side results from a counterclockwise rotation. An angle is ***negative*** when the location of the terminal side results from a clockwise rotation.

Angles are called ***coterminal*** if they are in standard position and share the same terminal side regardless of the direction of rotation.

A ***reference angle*** is the angle formed between the terminal side of an angle in standard position and the closest side of the x-axis. All reference angles measure between 0° and 90°.

The ***unit circle*** is a circle with a radius of 1.

The ***radian*** is the length of the arc divided by the radius of the arc for a plane angle subtended by a circular arc. As the ratio of two lengths, the ***radian*** is a "pure number" that needs no unit symbol. The ***radian*** is a unit of angular measure defined such that an angle of one radian subtended from the center of a unit circle produces an arc with arc length 1.

$$\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\csc \theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cot \theta = \frac{\text{adjacent}}{\text{opposite}}$$

$$\text{Angular Velocity} = \frac{\theta}{t}$$

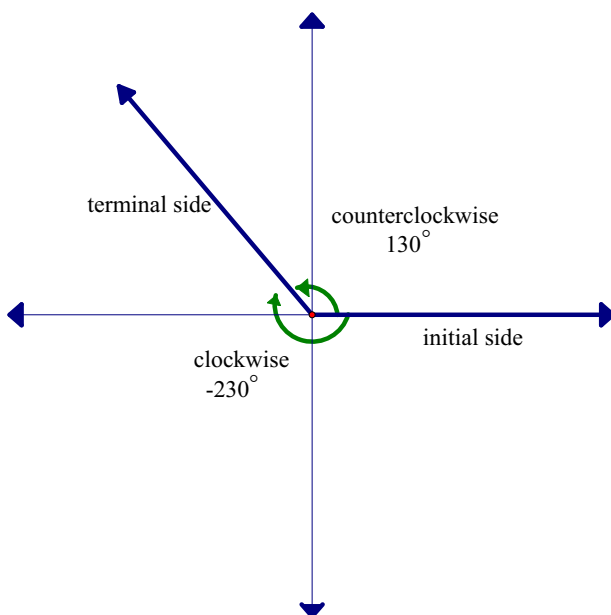
$$\text{Linear Velocity} = r \cdot \frac{\theta}{t}$$

Getting Started with Trigonometry and the Unit Circle Learning Task:

An angle is in **standard position** when the vertex is at the origin and the initial side lies on the positive side of the x-axis.

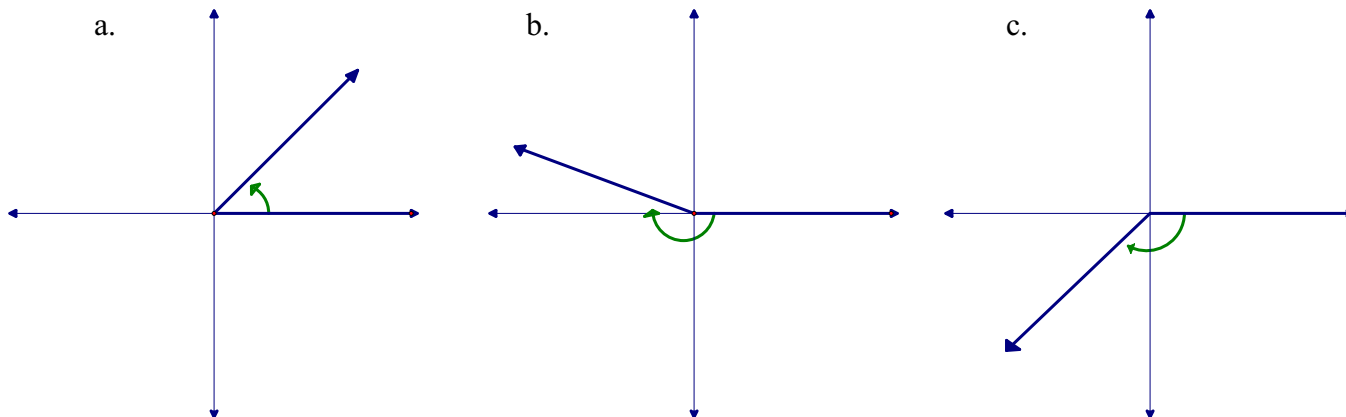
The ray that forms the **initial side** of the angle is rotated around the origin with the resulting ray being called the **terminal side** of the angle.

An angle is **positive** when the location of the terminal side results from a counterclockwise rotation. An angle is **negative** when the location of the terminal side results from a clockwise rotation.

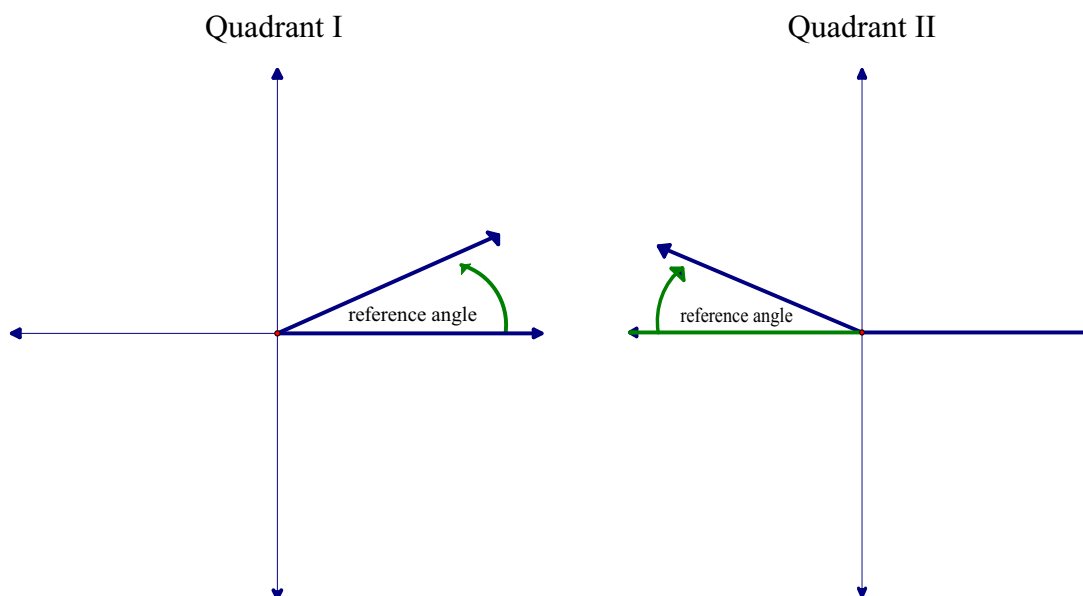


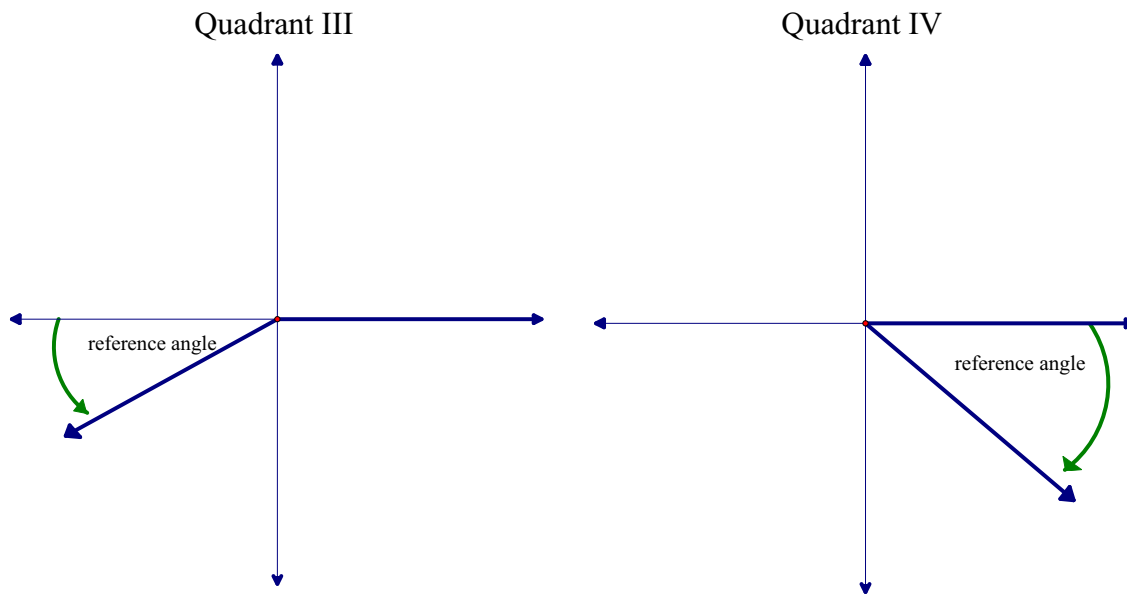
The two angles above are called **coterminal** because they are in standard position and share the same terminal side. Angles are also coterminal when they share terminal sides as the result of complete rotations. For example, 20 degree and 380 degree angles in standard position are coterminal.

1. Measure each of the angles below. Determine three coterminal angles for each of the angles.



Reference angles are the angle formed between the terminal side of an angle in standard position and the closest side of the x-axis. All reference angles measure between 0° and 90° .





2. Determine the reference angle for each of the following positive angles.

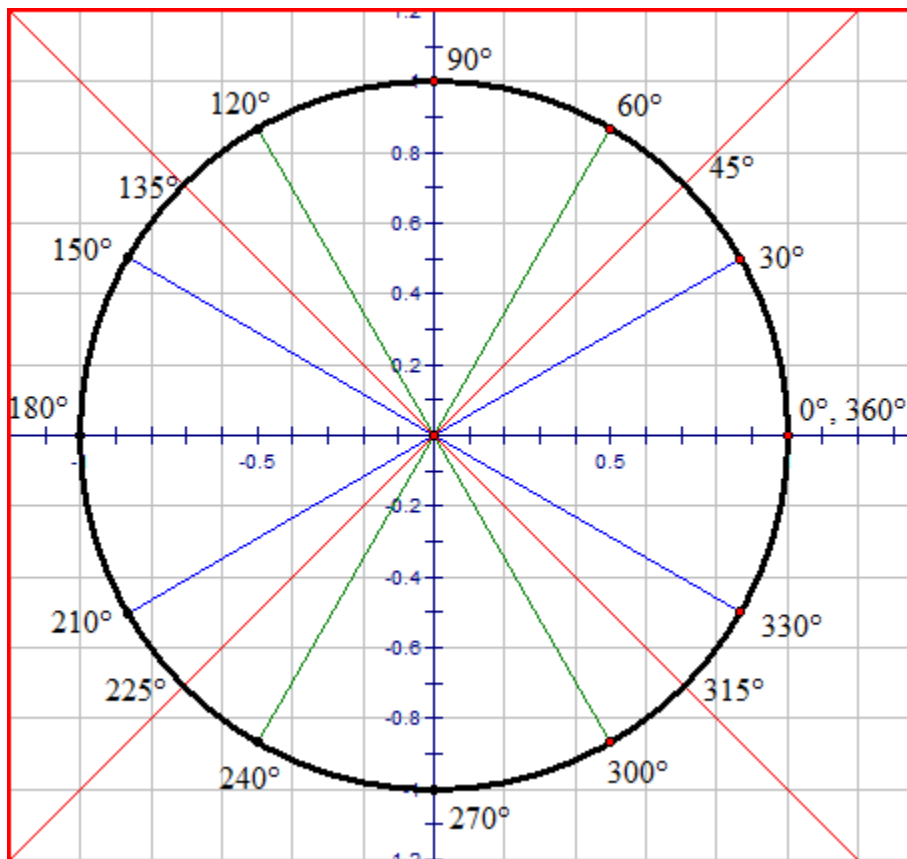
- | | |
|----------------|----------------|
| a. 300° | d. 210° |
| b. 135° | e. 585° |
| c. 30° | f. 870° |

3. Determine the reference angle for each of the following negative angles.

- | | |
|-----------------|------------------|
| a. -45° | d. -405° |
| b. -120° | e. -330° |
| c. -240° | f. -1935° |

Angles as a Part of the Unit Circle

4. The circle below is called the *unit circle*. Why do you believe this is so?

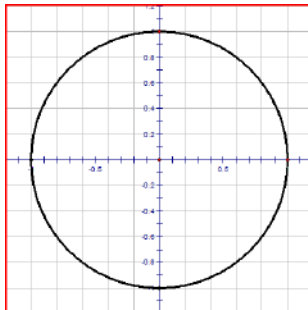


5. Duplicate this graph and circle on a piece of your own graph paper. Make the radius of your circle 10 squares long (10 squares = 1 unit).
6. Fold in the angle bisectors of the right angles formed at the origin. What angles result from these folds?
7. Use a protractor to mark angles at all multiples of 30° on the circle. Why didn't we use paper folding for these angles?
8. Which of the angles from #6 have reference angles of 30° ?
9. Which of the angles from #6 have reference angles of 60° ?

10. What is the angle measure when the terminal side of the angle lies on the negative side of the x-axis?
11. What is the angle measure when the terminal side of the angle lies on the negative side of the y-axis?
12. What is the angle measure when the terminal side of the angle lies on the positive side of the y-axis?
13. For what angle measures can the initial side and terminal side overlap?

Right Triangles and Coordinates on the Unit Circle Learning Task:

1. The circle below is referred to as a “unit circle.” Why is this the circle’s name?

**Part I**

2. Using a protractor, measure a 30° angle with vertex at the origin, initial side on the positive x-axis and terminal side above the x-axis. Label the point where the terminal side intersects the circle as “A”. Approximate the coordinates of point A using the grid.
3. Now, drop a perpendicular segment from the point you just put on the circle to the x-axis. You should notice that you have formed a right triangle. How long is the hypotenuse of your triangle? Using trigonometric ratios, specifically sine and cosine, determine the lengths of the two legs of the triangle. How do these lengths relate the coordinates of point A? How should these lengths relate to the coordinates of point A?
4. Using a Mira or paper folding, reflect this triangle across the y-axis. Label the resulting image point as point B. What are the coordinates of point B? How do these coordinates relate to the coordinates of point A? What obtuse angle was formed with the positive x-axis (the initial side) as a result of this reflection? What is the reference angle for this angle?

5. Which of your two triangles can be reflected to create the angle in the third quadrant with a 30° reference angle? What is this angle measure? Complete this reflection. Mark the lengths of the three legs of the third quadrant triangle on your graph. What are the coordinates of the new point on circle? Label the point C.
6. Reflect the triangle in the first quadrant over the x-axis. What is this angle measure? Complete this reflection. Mark the lengths of the three legs of the triangle formed in quadrant four on your graph. What are the coordinates of the new point on circle? Label this point D.
7. Let's look at what you know so far about coordinates on the unit circle. Complete the table.

θ	x-coordinate	y-coordinate

Notice that all of your angles so far have a reference angle of 30° .

Part II

8. Now, let's look at the angles on the unit circle that have 45° reference angles. What are these angle measures?
9. Mark the first quadrant angle from #8 on the unit circle. Draw the corresponding right triangle as you did in Part I. What type of triangle is this? Use the Pythagorean Theorem to determine the lengths of the legs of the triangle. Confirm that these lengths match the coordinates of the point where the terminal side of the 45° angle intersects the unit circle using the grid on your graph of the unit circle.
10. Using the process from Part I, draw the right triangle for each of the angles you listed in #8. Determine the lengths of each leg and match each length to the corresponding x- or y-coordinate on the unit circle. List the coordinates on the circle for each of these angles in the table.

θ	x-coordinate	y-coordinate

Part III

11. At this point, you should notice a pattern between the length of the horizontal leg of each triangle and one of the coordinates on the unit circle. Which coordinate on the unit circle is given by the length of the horizontal leg of the right triangles?
12. Which coordinate on the unit circle is given by the length of the vertical leg of the right triangles?
13. Is it necessary to draw all four of the triangles with the same reference angle to determine the coordinates on the unit circle? What relationship(s) can you use to determine the coordinates instead?
14. Use your method from #13 to determine the (x, y) coordinates where each angle with a 60° reference angle intersects the unit circle. Sketch each angle on the unit circle and clearly label the coordinates. Record your answers in the table.

θ	x-coordinate	y-coordinate

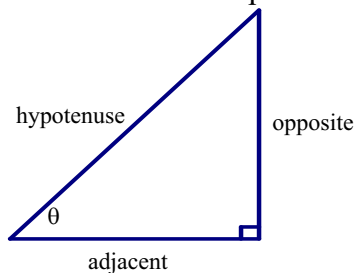
Part IV

15. There are a few angles for which we do not draw right triangles even though they are very important to the study of the unit circle. These are the angles with terminal sides on the axes. What are these angles? What are their coordinates on the unit circle?

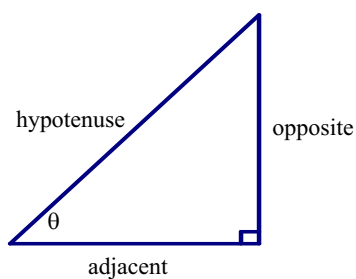
θ	x-coordinate	y-coordinate

More Relationships in the Unit Circle Learning Task:

1. In Mathematics II, you learned three trigonometric ratios in relation to right triangles. What are these relationships?



2. There are three additional trigonometric ratios that you will use in this unit: secant, cosecant and cotangent.



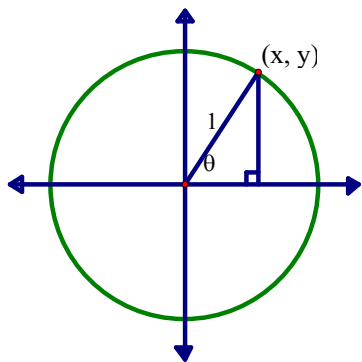
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

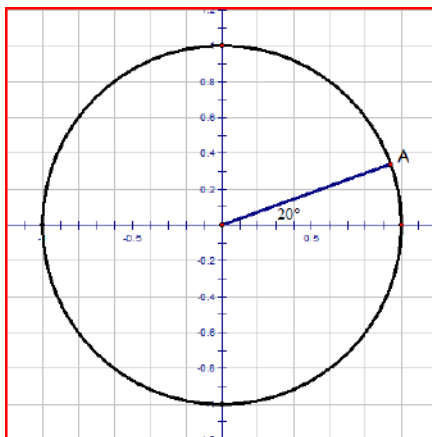
How do these ratios relate to the trigonometric ratios from #1?

3. Moving the triangle onto the unit circle allows us to represent these six trigonometric relationships in terms of x and y. Express each of the six ratios in terms of x and y.



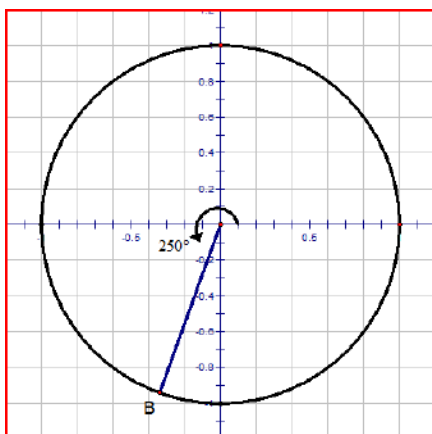
4. Based on these relationships $x = \underline{\hspace{2cm}}$ and $y = \underline{\hspace{2cm}}$. This is a special case of the general trigonometric coefficients ($r\cos\theta$, $r\sin\theta$) where $r = 1$.

5.



- Use this relationship to determine the coordinates of A. Both coordinates are positive. Why is this true?
- What angle would have coordinates $(-0.9397, -0.3420)$ on the unit circle? Why?
- What angle would have $(0.9397, -0.3420)$ as its coordinates? Why?

6.



- What is the reference angle for 250° ?
- What are the coordinates of this angle on the unit circle?
- What 2^{nd} quadrant angle has the same reference angle? What are the coordinates of this angle on the unit circle?

7. Using a scientific or graphing calculator, you can quite easily find the sine, cosine and tangent of a given angle. This is not true for secant, cosecant, or cotangent. Remember from

$\frac{1}{\sin \theta}$

Mathematics II, that \sin^{-1} is not the same as $\frac{1}{\sin \theta}$. Since the three new trigonometric ratios are not on a calculator, how can you use the definitions of the ratios from #2 to calculate the values?

8. A student entered $\sin 30$ in her calculator and got -0.98803. What went wrong?
9. Based on the graph of the unit circle on the grid, estimate each of the values. Do not use the trig keys on the calculator for this problem. You will need to use a protractor to mark each angle and then estimate the coordinates where the terminal side of the angle intersects the unit circle.

- | | |
|---------------------|---------------------|
| a. $\sec 60^\circ$ | d. $\sec -75^\circ$ |
| b. $\csc 180^\circ$ | e. $\csc 490^\circ$ |
| c. $\cot 235^\circ$ | f. $\cot 920^\circ$ |

10. Use a calculator to find each of the following values.

- | | |
|---------------------|----------------------|
| a. $\sin 40^\circ$ | e. $\tan 300^\circ$ |
| b. $\csc 40^\circ$ | f. $\cot 300^\circ$ |
| c. $\cos 165^\circ$ | g. $\csc 90^\circ$ |
| d. $\sec 165^\circ$ | h. $\sec -140^\circ$ |

UnWrapping the Unit Circle – Graphs from the Unit Circle Learning Task:

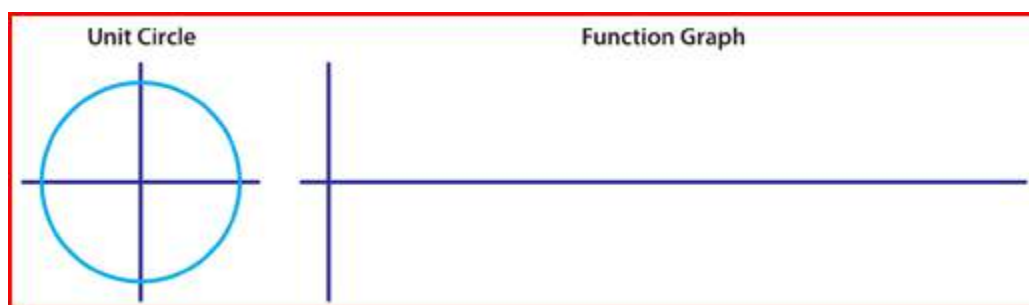
From Illuminations: Resources for Teaching Math, National Council of Teachers of Mathematics

Materials:

- Bulletin Board paper or butcher paper (approximately 8 feet long)
- Uncooked spaghetti
- Masking tape
- Protractor
- Meter stick
- Colored marker
- Yarn (about 7 feet long)

Part I: Unwrapping the Sine Curve

Tape the paper to the floor, and construct the diagram below. The circle's radius should be about the length of one piece of uncooked spaghetti. If your radius is smaller, break the spaghetti to the length of the radius. This is a unit circle with the spaghetti equal to one unit.



Using a protractor, make marks every 15° around the unit circle. Place a string on the unit circle at 0° , which is the point $(1, 0)$, and wrap it counterclockwise around the circle. Transfer the marks from the circle to the string.

Transfer the marks on the string onto the x -axis of the function graph. The end of the string that was at 0° must be placed at the origin of the function graph. Label these marks on the x -axis with the related angle measures from the unit circle (e.g., 0° , 15° , 30° , etc.).

1. What component from the unit circle do the x -values on the function graph represent?

x -values = _____

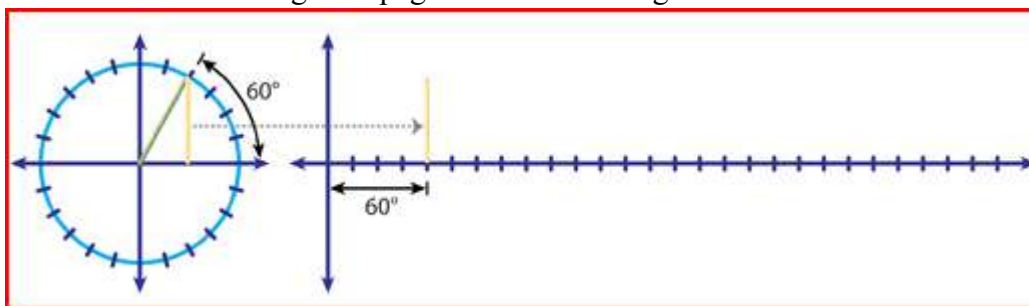
Use the length of your spaghetti to mark one unit above and below the origin on the y -axis of the function graph. Label these marks 1 and -1 , respectively.

Draw a right triangle in the unit circle where the hypotenuse is the radius of the circle

to the 15° mark and the legs lie along and perpendicular to the x -axis.

Break a piece of spaghetti to the length of the vertical leg of this triangle, from the 15° mark on the circle to the x -axis. Let this piece of spaghetti represent the y -value for the point on the function graph where $x = 15^\circ$. Place the spaghetti piece appropriately on the function graph and make a dot at the top of it. **Note:** Since this point is above the x -axis in the unit circle, the corresponding point on the function graph should also be above the x -axis.

Transferring the Spaghetti for the Triangle Drawn to the 60° Mark



Continue constructing triangles and transferring lengths for all marks on the unit circle. After you have constructed all the triangles, transferred the lengths of the vertical legs to the function graph, and added the dots, draw a smooth curve to connect the dots.

- The vertical leg of a triangle in the unit circle, which is the y -value on the function graph, represents what function of the related angle measure?

y -values = _____

Label the function graph you just created on your butcher paper $y = \sin x$.

- What is the period of the sine curve? That is, what is the wavelength? After how many radians does the graph start to repeat? How do you know it repeats after this point?
- What are the zeroes of this function? (Remember: The x -values are measuring angles and zeroes are the x -intercepts.)
- What are the x -values at the maxima and minima of this function?
- What are the y -values at the maxima and minima?

7. Imagine this function as it continues in both directions. Explain how you can predict the value of the sine of 390° .
8. Explain why $\sin 30^\circ = \sin 150^\circ$. Refer to both the unit circle and the graph of the sine curve.

Part II: Unwrapping the Cosine Curve

You used the length of the vertical leg of a triangle in the unit circle to find the related y -value in the sine curve. Determine what length from the unit circle will give you the y -value for a cosine curve. Using a different color, create the graph on your butcher paper and label it $y = \cos x$.

9. In what ways are the sine and cosine graphs similar? Be sure to include a discussion of intercepts, maxima, minima, and period.
10. In what ways are the sine and cosine graphs different? Again, be sure to include a discussion of intercepts, maxima, minima, and period.
11. Will sine graphs continue infinitely in either direction? How do you know? Identify the domain and range of $y = \sin x$.
12. Will cosine graphs continue infinitely in either direction? How do you know? Identify the domain and range of $y = \cos x$.

What is a Radian? Learning Task:

On a separate sheet of blank paper, use a compass to draw a circle of any size. Make sure the center of the circle is clearly marked. Use a straightedge to draw a radius of the circle.

Take a piece of string and “measure” the radius of the circle. Cut the string to exactly the length of the radius.

1. Beginning at the end of the radius, wrap the cut string around the edge of the circle. Mark where the string ends on the circle. Move the string to this new point and wrap it to the circle again. Continue this process until you have gone completely around the circle. How many radius lengths did it take to complete the distance around the circle? What geometric concept does this reflect?
2. Remember from your study of circles in Mathematics 2 that arcs can be measured in degrees or by length. In Trigonometry, we can measure arcs by degrees or radians. Based on your process in #1, what do you think a radian is?
3. Let’s consider the unit circle. We know the radius is equal to 1 unit, thus the circumference is 2π . How does this value relate to the work you did in #1?
4. So far, we know that the complete circle measures 2π radians and 360° . Can we simplify this relationship?
5. Let’s convert several common angles from degrees to radians:
 - a. 180° is half of a circle, so it is how many radians?
 - b. 90° is a quarter of a circle, so it is how many radians?
 - c. 270° is three-quarters of a circle, so it is how many radians?
 - d. 45° is _____ of a circle, so it is _____ radians.

e. 120° is _____ of a circle so it is _____ radians.

6. Other angles can also be converted using the relationship between the degree measure of the angle and the associated arc length, or radian measure.

Degrees to Radians

Radians to Degrees

a. $32^\circ =$ _____

f. $\frac{7\pi}{8} =$ _____

b. $200^\circ =$ _____

g. $\frac{3\pi}{4} =$ _____

c. $140^\circ =$ _____

h. $8\pi =$ _____

d. $920^\circ =$ _____

i. $-\frac{12\pi}{5} =$ _____

e. $-40^\circ =$ _____

j. $2 =$ _____

7. Just as you have found the values of the six trigonometric functions for specific degree measures, you will also need to find the values of these functions for radian measures. Use your knowledge of the unit circle to determine each of the following values.

a. $\sin \frac{\pi}{4} =$ _____

d. $\csc \frac{7\pi}{3} =$ _____

b. $\cos \frac{2\pi}{3} =$ _____

e. $\sec \frac{7\pi}{4} =$ _____

c. $\tan 2\pi =$ _____

f. $\cot \frac{7\pi}{6} =$ _____

8. The values of the trigonometric functions are not readily found from the unit circle. For these values, you will use a scientific or graphing calculator. Be sure your calculator is in radian mode before proceeding.

a. $\csc \frac{\pi}{5} =$ _____

c. $\cot \frac{13\pi}{12} =$ _____

b. $\sec \frac{8\pi}{9} =$ _____

d. $\cos \frac{19\pi}{10} =$ _____