

# **Mathematics IV Frameworks Student Edition**

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## **Unit 2 Sequences and Series**

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**1<sup>st</sup> Edition  
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Georgia Department of Education**

Georgia Department of Education  
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**Unit 2: Page 1 of 34**

## Table of Contents

INTRODUCTION: .....3

Renaissance Festival Learning Task.....8

Fascinating Fractals Learning Task .....18

Diving into Diversions Learning Task .....29

## Mathematics IV – Unit 2

### Sequences and Series

### Student Edition

#### INTRODUCTION:

In 8<sup>th</sup> and 9<sup>th</sup> grades, students learned about arithmetic and geometric sequences and their relationships to linear and exponential functions, respectively. This unit builds on students' understandings of those sequences and extends students' knowledge to include arithmetic and geometric series, both finite and infinite. Summation notation and properties of sums are also introduced. Additionally, students will examine other types of sequences and, if appropriate, proof by induction. They will use their knowledge of the characteristics of the types of sequences and the corresponding functions to compare scenarios involving different sequences.

#### ENDURING UNDERSTANDINGS:

- All arithmetic and geometric sequences can be expressed recursively and explicitly. Some other sequences also can be expressed in both ways but others cannot.
- Arithmetic sequences are identifiable by a common difference and can be modeled by linear functions. Infinite arithmetic series always diverge.
- Geometric sequences are identifiable by a common ratio and can be modeled by exponential functions. Infinite geometric series diverge if  $|r| \geq 1$  and converge if  $|r| < 1$ .
- The sums of finite arithmetic and geometric series can be computed with easily derivable formulas.
- Identifiable sequences and series are found in many naturally occurring objects.
- Repeating decimals can be expressed as fractions by summing appropriate infinite geometric series.

#### KEY STANDARDS ADDRESSED:

##### **MM4A9. Students will use sequences and series**

- a. Use and find recursive and explicit formulae for the terms of sequences.
- b. Recognize and use simple arithmetic and geometric sequences.
- c. Find and apply the sums of finite and, where appropriate, infinite arithmetic and geometric series.
- d. Use summation notation to explore finite series.

**RELATED STANDARDS ADDRESSED:****MM4A1. Students will explore rational function.**

- a. Investigate and explore characteristics of rational functions, including domain, range, zeros, points of discontinuity, intervals of increase and decrease, rates of change, local and absolute extrema, symmetry, asymptotes, and end behavior.

**MM4A4. Students will investigate functions.**

- a. Compare and contrast properties of functions within and across the following types: linear, quadratic, polynomial, power, rational, exponential, logarithmic, trigonometric, and piecewise.
- b. Investigate transformations of functions.

**MM4P1. Students will solve problems (using appropriate technology).**

- a. Build new mathematical knowledge through problem solving.
- b. Solve problems that arise in mathematics and in other contexts.
- c. Apply and adapt a variety of appropriate strategies to solve problems.
- d. Monitor and reflect on the process of mathematical problem solving.

**MM4P2. Students will reason and evaluate mathematical arguments.**

- a. Recognize reasoning and proof as fundamental aspects of mathematics.
- b. Make and investigate mathematical conjectures.
- c. Develop and evaluate mathematical arguments and proofs.
- d. Select and use various types of reasoning and methods of proof.

**MM4P3. Students will communicate mathematically.**

- a. Organize and consolidate their mathematical thinking through communication.
- b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
- c. Analyze and evaluate the mathematical thinking and strategies of others.
- d. Use the language of mathematics to express mathematical ideas precisely.

**MM4P4. Students will make connections among mathematical ideas and to other disciplines.**

- a. Recognize and use connections among mathematical ideas.
- b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
- c. Recognize and apply mathematics in contexts outside of mathematics.

**MM4P5. Students will represent mathematics in multiple ways.**

- a. Create and use representations to organize, record, and communicate mathematical ideas.
- b. Select, apply, and translate among mathematical representations to solve problems.
- c. Use representations to model and interpret physical, social, and mathematical phenomena.

## UNIT OVERVIEW:

The launching activity begins by revisiting ideas of arithmetic sequences studied in eighth and ninth grades. Definitions, as well as the explicit and recursive forms of arithmetic sequences are reviewed. The task then introduces summations, including notation and operations with summations, and summing arithmetic series.

The second set of tasks reviews geometric sequences and investigates sums, including infinite and finite geometric series, in the context of exploring fractals. It is assumed that students have some level of familiarity with geometric sequences and the relationship between geometric sequences and exponential functions.

The third group addresses some common sequences and series, including the Fibonacci sequence, sequences with factorials, and repeating decimals.

The culminating task is set in the context of applying for a job at an interior design agency. In each task, students will need to determine which type of sequence is called for, justify their choice, and occasionally prove they are correct. Students will complete the handshake problem, the salary/retirement plan problem, and some open-ended design problems that require the use of various sequences and series.

## VOCABULARY AND FORMULAS

**Arithmetic sequence:** A sequence of terms  $a_1, a_2, a_3, \dots$  with  $d = a_n - a_{n-1}$ . The explicit formula is given by  $a_n = a_1 + (n-1)d$  and the recursive form is  $a_1 =$  value of the first term and  $a_n = a_{n-1} + d$ .

**Arithmetic series:** The sum of a set of terms in arithmetic progression  $a_1 + a_2 + a_3 + \dots$  with  $d = a_n - a_{n-1}$ .

**Common difference:** In an arithmetic sequence or series, the difference between two consecutive terms is  $d$ ,  $d = a_n - a_{n-1}$ .

**Common ratio:** In a geometric sequence or series, the ratio between two consecutive terms is  $r$ ,  $r = \frac{a_n}{a_{n-1}}$ .

**Explicit formula:** A formula for a sequence that gives a direct method for determining the  $n$ th term of the sequence. It presents the relationship between two quantities, i.e. the term number and the value of the term.

**Factorial:** If  $n$  is a positive integer, the notation  $n!$  (read “ $n$  factorial”) is the product of all positive integers from  $n$  down through 1; that is,  $n! = n(n-1)(n-2)\dots(3)(2)(1)$ . Note that  $0!$ , by definition, is 1; i.e.  $0! = 1$ .

**Finite series:** A series consisting of a finite, or limited, number of terms.

**Infinite series:** A series consisting of an infinite number of terms.

**Geometric sequence:** A sequence of terms  $a_1, a_2, a_3, \dots$  with  $r = \frac{a_n}{a_{n-1}}$ . The explicit formula is given by  $a_n = a_1 r^{n-1}$  and the recursive form is  $a_1 =$  value of the first term and  $a_n = (a_{n-1})r$ .

**Geometric series:** The sum of a set of terms in geometric progression  $a_1 + a_2 + a_3 + \dots$  with  $r = \frac{a_n}{a_{n-1}}$ .

**Limit of a sequence:** The long-run value that the terms of a convergent sequence approach.

**Partial sum:** The sum of a finite number of terms of an infinite series.

**Recursive formula:** Formula for determining the terms of a sequence. In this type of formula, each term is dependent on the term or terms immediately before the term of interest. The recursive formula must specify at least one term preceding the general term.

**Sequence:** A sequence is an ordered list of numbers.

**Summation or sigma notation:**  $\sum_{i=1}^n a_i$ , where  $i$  is the index of summation,  $n$  is the upper limit of summation, and 1 is the lower limit of summation. This expression gives the partial sum, the sum

of the first  $n$  terms of a sequence. More generally, we can write  $\sum_{i=k}^n a_i$ , where  $k$  is the starting value.

**Sum of a finite arithmetic series:** The sum,  $S_n$ , of the first  $n$  terms of an arithmetic sequence is

given by  $S_n = \frac{n(a_1 + a_n)}{2}$ , where  $a_1 =$  value of the first term and  $a_n =$  value of the last term in the sequence.

**Sum of a finite geometric series:** The sum,  $S_n$ , of the first  $n$  terms of a geometric sequence is

given by  $S_n = \frac{a_1 - a_n r}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}$ , where  $a_1$  is the first term and  $r$  is the common ratio ( $r \neq 1$ ).

**Sum of an infinite geometric series:** The general formula for the sum  $S$  of an infinite geometric

series  $a_1 + a_2 + a_3 + \dots$  with common ratio  $r$  where  $|r| < 1$  is  $S = \frac{a_1}{1-r}$ . If an infinite geometric series has a sum, i.e. if  $|r| < 1$ , then the series is called a **convergent** geometric series. All other geometric (and arithmetic) series are **divergent**.

**Term of a sequence:** Each number in a sequence is a term of the sequence. The first term is generally noted as  $a_1$ , the second as  $a_2$ , ..., the  $n$ th term is noted as  $a_n$ .  $a_n$  is also referred to as the general term of a sequence.

**RENAISSANCE FESTIVAL LEARNING TASK:**

As part of a class project on the Renaissance, your class decided to plan a renaissance festival for the community. Specifically, you are a member of different groups in charge of planning two of the contests. You must help plan the archery and rock throwing contests. The following activities will guide you through the planning process.

**Group One: Archery Contest<sup>1</sup>**

Before planning the archery contest, your group decided to investigate the characteristics of the target. The target being used has a center, or bull's-eye, with a radius of 4 cm, and nine rings that are each 4 cm wide.

**1. The Target**

- a. Sketch a picture of the center and first 3 rings of the target.
- b. Write a sequence that gives the radius of each of the concentric circles that comprise the entire target.
- c. Write a recursive formula and an explicit formula for the terms of this sequence.
- d. What would be the radius of the target if it had 25 rings? Show how you completed this problem using the explicit formula.
- e. In the past, you have studied both arithmetic and geometric sequences. What is the difference between these two types of sequences? Is the sequence in (b) arithmetic, geometric, or neither? Explain.

One version of the explicit formula uses the first term, the common difference, and the number of terms in the sequence. For example, if we have the arithmetic sequence 2, 5, 8, 11, 14, ..., we see that the common difference is 3. If we want to know the value of the 20<sup>th</sup> term, or  $a_{20}$ , we could think of starting with  $a_1 = 2$  and adding the difference,  $d = 3$  a certain number of times. How many times would we need to add the common difference to get to the 20<sup>th</sup> term? \_\_\_\_\_ Because multiplication is repeated addition, instead of adding 3 that number of times, we could multiply the common difference, 3, by the number of times we would need to add it to 2.

This gives us the following explicit formula for an arithmetic sequence:  $a_n = a_1 + (n-1)d$ .

- f. Write this version of the explicit formula for the sequence in this problem. Show how this version is equivalent to the version above.

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<sup>1</sup> Elements of these problems were adapted from *Integrated Mathematics 3* by McDougal-Littell, 2002.)



- g. Can you come up with a reason for which you would want to add up the radii of the concentric circles that make up the target (for the purpose of the contest)? Explain.
- h. Plot the sequence from this problem on a coordinate grid. What should you use for the independent variable? For the dependent variable? What type of graph is this? How does the  $a_n$  equation of the recursive formula relate to the graph? How does the parameter  $d$  in the explicit form relate to the graph?
- i. Describe (using y-intercept and slope), but do not graph, the plots of the arithmetic sequences defined explicitly or recursively as follows:

$$1. \quad a_n = 3 + \frac{4}{3}(n-1)$$

$$3. \quad a_n = 4.5 - 3.2(n-1)$$

$$2. \quad \begin{cases} a_1 = -2 \\ a_n = a_{n-1} + \frac{1}{2} \end{cases}$$

$$4. \quad \begin{cases} a_1 = 10 \\ a_n = a_{n-1} - \frac{2}{5} \end{cases}$$

2. The Area of the Target: To decide on prizes for the archery contest, your group decided to use the areas of the center and rings. You decided that rings with smaller areas should be worth more points. But how much more? Complete the following investigation to help you decide.

- a. Find the sequence of the areas of the rings, including the center. (Be careful.)
- b. Write a recursive formula and an explicit formula for this sequence.
- c. If the target was larger, what would be the area of the 25<sup>th</sup> ring?
- d. Find the total area of the bull's eye by adding up the areas in the sequence.
- e. Consider the following sum:  $S_n = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_{n-2} + a_n$ . Explain why that equation is equivalent to  $S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \dots + (a_n - 2d) + (a_n - d) + a_n$ .

Rewrite this latter equation and then write it out backwards. Add the two resulting equations. Use this to finish deriving the formula for the sum of the terms in an arithmetic sequence. Try it out on a few different short sequences.

- f. Use the formula for the sum of a finite arithmetic sequence in part (e) to verify the sum of the areas in the target from part (d).
- g. Sometimes, we do not have all the terms of the sequence but we still want to find a specific sum. For example, we might want to find the sum of the first 15 multiples of 4. Write an explicit formula that would represent this sequence. Is this an arithmetic

sequence? If so, how could we use what we know about arithmetic sequences and the sum formula in (e) to find this sum? Find the sum.

- h. What happens to the sum of the arithmetic series we've been looking at as the number of terms we sum gets larger? How could you find the sum of the first 200 multiples of 4? How could you find the sum of all the multiples of 4? Explain using a graph and using mathematical reasoning.
- i. Let's practice a few arithmetic sum problems.
1. Find the sum of the first 50 terms of 15, 9, 3, -3, ...
  2. Find the sum of the first 100 natural numbers
  3. Find the sum of the first 75 positive even numbers
  4. Come up with your own arithmetic sequence and challenge a classmate to find the sum.
- j. Summarize what you learned / reviewed about arithmetic sequences and series during this task.

3. Point Values: Assume that each participant's arrow hits the surface of the target.

- a. Determine the probability of hitting each ring and the bull's-eye.

Target Piece	Area of Piece (in $\text{cm}^2$ )	Probability of Hitting this Area
Bull's Eye	$16\pi$	
Ring 1	$48\pi$	
Ring 2	$80\pi$	
Ring 3	$112\pi$	
Ring 4	$144\pi$	
Ring 5	$176\pi$	
Ring 6	$208\pi$	
Ring 7	$240\pi$	
Ring 8	$272\pi$	
Ring 9	$304\pi$	

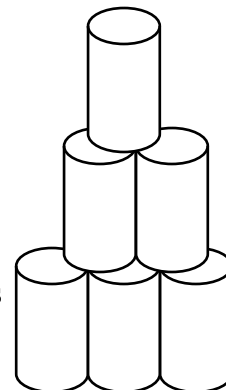
- b. Assign point values for hitting each part of the target, justifying the amounts based on the probabilities just determined.

- c. Use your answer to (b) to determine the expected number of points one would receive after shooting a single arrow.
- d. Using your answers to part (c), determine how much you should charge for participating in the contest OR for what point values participants would win a prize. Justify your decisions.

## Group Two: Rock Throwing Contest<sup>2</sup>

For the rock throwing contest, your group decided to provide three different arrangements of cans for participants to knock down.

1. For the first arrangement, the tin cans were set up in a triangular pattern, only one can deep. (See picture.)



- If the top row is considered to be row 1, how many cans would be on row 10?
  - Is this an arithmetic or a geometric sequence (or neither)? Write explicit and recursive formulas for the sequence that describes the number of cans in the  $n$ th row of this arrangement.
- c. It is important to have enough cans to use in the contest, so your group needs to determine how many cans are needed to make this arrangement. Make a table of the number of rows included and the total number of cans.

Rows Included	Total Cans
1	1
2	3
3	
4	
5	
6	
7	
8	

- d. One of your group members decides that it would be fun to have a “mega-pyramid” 20 rows high. You need to determine how many cans would be needed for this pyramid, but you don’t want to add all the numbers together. One way to find the sum is to use the summation formula you found in the Archery Contest. How do you find the sum in an arithmetic sequence? \_\_\_\_\_ Find the sum of a pyramid arrangement 20 rows high using this formula.

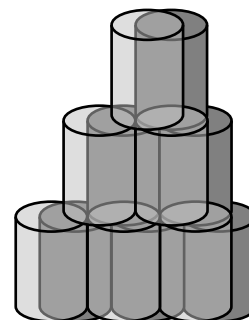
We can also write this problem using summation notation:  $\sum_{i=1}^n a_i$ , where  $i$  is the index of summation,  $n$  is the upper limit of summation, and 1 is the lower limit of summation. We can

<sup>2</sup> Adapted from Manouchehri, A. (2007). Inquiry-discourse mathematics instruction. *Mathematics Teacher*, 101, 290–300.

think of  $a_i$  as the explicit formula for the sequence. In this pyramid problem, we have  $\sum_{i=1}^{20} i$  because we are summing the numbers from 1 to 20. We also know what this sum is equal to.

$\sum_{i=1}^{20} i = \frac{n}{2}(a_1 + a_n) = \frac{20}{2}(1 + 20)$ . What if we did not know the value of  $n$ , the upper limit but we did know that the first number is 1 and that we were counting up by 1s? We would then have  $\sum_{i=1}^n i = \frac{n}{2}(1 + n) = \frac{n(n+1)}{2}$ . This is a very common, important formula in sequences. We will use it again later.

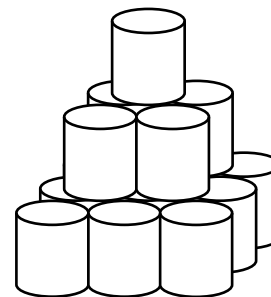
- e. Propose and justify a specific number of cans that could be used in this triangular arrangement. Remember, it must be realistic for your fellow students to stand or sit and throw a rock to knock down the cans. It must also be reasonable that the cans could be set back up rather quickly. Consider restricting yourself to less than 50 cans for each pyramid. Describe the set-up and exactly how many cans you need.
2. For the second arrangement, the group decided to make another triangular arrangement; however, this time, they decided to make the pyramid 2 or 3 cans deep. (The picture shows the 2-deep arrangement.)



- a. This arrangement is quite similar to the first arrangement. Write an explicit formula for the sequence describing the number of cans in the  $n$ th row if there are 2 cans in the top row, as pictured.
- b. Determine the number of cans needed for the 20<sup>th</sup> row.
- c. Similar to above, we need to know how many cans are needed for this arrangement. How will this sum be related to the sum you found in problem 1?
- d. The formula given above in summation notation only applies when we are counting by ones. What are we counting by to determine the number of cans in each row? What if the cans were three deep? What would we be counting by? In this latter case, how would the sum of the cans needed be related to the sum of the cans needed in the arrangement in problem 1?

- e. This leads us to an extremely important property of sums:  $\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$ , where  $c$  is a constant. What does this property mean? Why is it useful?

- f. Suppose you wanted to make an arrangement that is 8 rows high and 4 cans deep. Use the property in 2e to help you determine the number of cans you would need for this arrangement.
- g. Propose and justify a specific number of cans that could be used in this triangular arrangement. You may decide how many cans deep ( $>1$ ) to make the pyramid. Consider restricting yourself to less than 50 cans for each pyramid. Describe the set-up and exactly how many cans you need. Show any calculations.
3. For the third arrangement, you had the idea to make the pyramid of cans resemble a true pyramid. The model you proposed to the group had 9 cans on bottom, 4 cans on the second row, and 1 can on the top row.



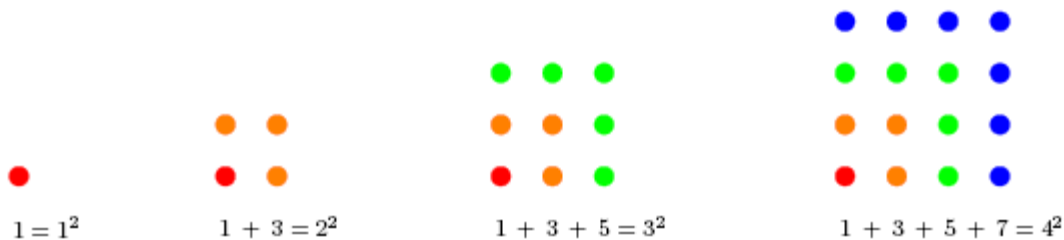
- a. Complete the following table.

Row	Number of Cans	Change from Previous Row
1	1	1
2	4	3
3	9	5
4		
5		
6		
7		
8		
9		
10		

- b. How many cans are needed for the  $n$ th row of this arrangement?
- c. What do you notice about the numbers in the third column above? Write an equation that relates column two to column three. Then try to write the equation using summation notation.
- d. How could you prove the relationship you identified in 3c?
- e. Let's look at a couple of ways to prove this relationship. Consider a visual approach to a proof.<sup>3</sup> Explain how you could use this approach to prove the relationship.

<sup>3</sup> A similar approach can be found in the August 2006 issue of the *Mathematics Teacher*: Activities for Students: Visualizing Summation Formulas by Gunhan Caglayan.

We can represent the sum of the first  $n$  odd natural numbers as the number of dots contained in the square arrays drawn in the figures below:



- f. We know that the sum of the first  $n$  natural number is  $\frac{n(n+1)}{2}$ , so  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ . If we multiply both sides of the equation by 2, we get the sum of the first  $n$  EVEN numbers. How can we use this new equation to help us find the sum of the first  $n$  ODD numbers?
- g. Consider another approach. We have  $S(n) = 1 + 3 + 5 + \dots + (2n-1)$ . If we reverse the ordering in this equation, we get  $S(n) = (2n-1) + \dots + 5 + 3 + 1$ . What happens if we add the corresponding terms of these two equations? How will that help us prove the relationship we found earlier?
- h. In a future task, you will learn another way to prove this relationship. You will also look at the sum of rows in this can arrangement. Can you conjecture a formula for the sum of the first  $n$  square numbers? Try it out a few times.
4. Throughout this task, you learned a number of facts and properties about summation notation and sums. You learned what summation notation is and how to compute some sums using the notation. There is another important property to learn that is helpful in computing sums. We'll look at that here, along with practicing using summation notation.
- a. Write out the terms of these series.

i.  $\sum_{i=1}^4 5i$

ii.  $\sum_{i=1}^5 i + 6$

iii.  $\sum_{i=1}^5 i^2 + 3$

- b. You have already seen one sum property. Here are the important properties you need to know. Explain why the two new properties make mathematical sense to you.

**Properties of sums** ( $c$  represents a constant)

1.  $\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$

2.  $\sum_{i=1}^n c = cn$

3.  $\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$

- c. Express each series using summation notation. Then find the sum.

i.  $2 + 4 + 6 + \dots + 24$

ii.  $5 + 8 + 11 + 14 + \dots + 41$

- d. Compute each sum using the properties of sums.

i.  $\sum_{i=1}^{20} (-3i - 4)$

ii.  $\sum_{i=1}^{20} 4i$

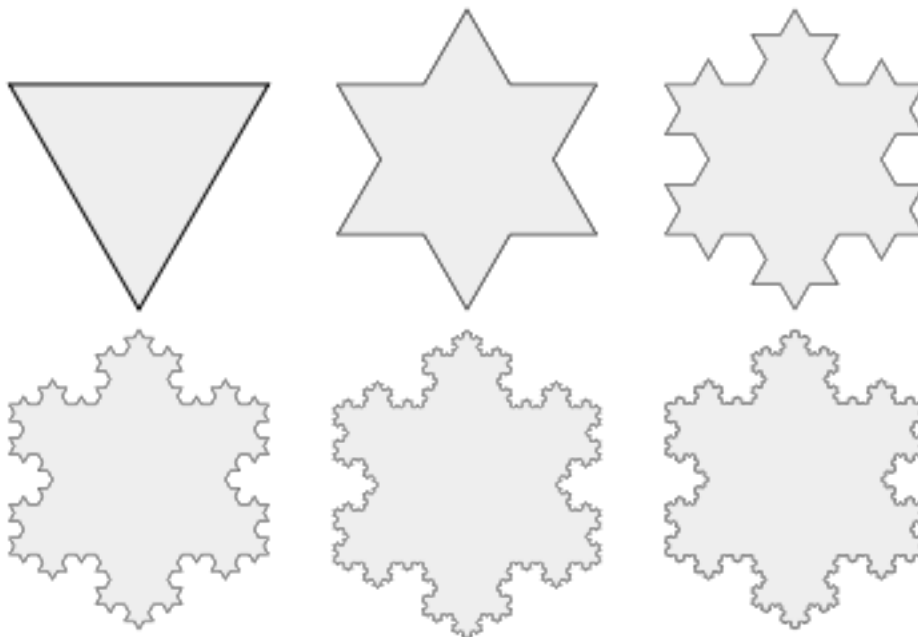
iii.  $\sum_{i=1}^{20} (-4i)$



### **FASCINATING FRACTALS LEARNING TASK:**

Sequences and series arise in many classical mathematics problems as well as in more recently investigated mathematics, such as fractals. The task below investigates some of the interesting patterns that arise when investigating manipulating different figures.

#### **Part One: Koch Snowflake<sup>4</sup>**



(Images obtained from Wikimedia Commons at [http://commons.wikimedia.org/wiki/Koch\\_snowflake](http://commons.wikimedia.org/wiki/Koch_snowflake))

This shape is called a fractal. Fractals are geometric patterns that are repeated at ever smaller increments. The fractal in this problem is called the Koch snowflake. At each stage, the middle third of each side is replaced with an equilateral triangle. (See the diagram.)

To better understand how this fractal is formed, let's create one!

On a large piece of paper, construct an equilateral triangle with side lengths of 9 inches.

Now, on each side, locate the middle third. (How many inches will this be?) Construct a new equilateral triangle in that spot and erase the original part of the triangle that now forms the base of the new, smaller equilateral triangle.

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<sup>4</sup> Often the first picture is called stage 0. For this problem, it is called stage 1. The Sierpinski Triangle, the next problem, presents the initial picture as Stage 0.

A number of excellent applets are available on the web for viewing iterations of fractals.

How many sides are there to the snowflake at this point? (Double-check with a partner before continuing.)

Now consider each of the sides of the snowflake. How long is each side? Locate the middle third of each of these sides. How long would one-third of the side be? Construct new equilateral triangles at the middle of each of the sides.

How many sides are there to the snowflake now? Note that every side should be the same length.

Continue the process a few more times, if time permits.

1. Now complete the first three columns of the following chart.

	<b>Number of Segments</b>	<b>Length of each Segment (in)</b>	<b>Perimeter (in)</b>
<b>Stage 1</b>	3	9	27
<b>Stage 2</b>			
<b>Stage 3</b>			

2. Consider the **number of segments** in the successive stages.

- Does the sequence of number of segments in each successive stage represent an arithmetic or a geometric sequence (or neither)? Explain.
- What type of graph does this sequence produce? Make a plot of the stage number and number of segments in the figure to help you determine what type of function you will use to model this situation.
- Write a recursive and explicit formula for the number of segments at each stage.
- Find the 7<sup>th</sup> term of the sequence. Find 12<sup>th</sup> term of the sequence. Now find the 16<sup>th</sup>. Do the numbers surprise you? Why or why not?

3. Consider the **length of each segment** in the successive stages.

- Does this sequence of lengths represent an arithmetic or a geometric sequence (or neither)? Explain.

- b. Write a recursive and explicit formula for the length of each segment at each stage.
- c. Find the 7<sup>th</sup> term of the sequence. Find the 12<sup>th</sup> term of the sequence. Now find the 16<sup>th</sup>. How is what is happening to these numbers similar or different to what happened to the sequence of the number of segments at each stage? Why are these similarities or differences occurring?
4. Consider the **perimeter** of the Koch snowflake.
- a. How did you determine the perimeter for each of the stages in the table?
- b. Using this idea and your answers in the last two problems, find the approximate perimeters for the Koch snowflake at the 7<sup>th</sup>, 12<sup>th</sup>, and 16<sup>th</sup> stages.
- c. What do you notice about how the perimeter changes as the stage increases?
- d. **Extension:** B. B. Mandelbrot used the ideas above, i.e. the length of segments and the associated perimeters, in his discussion of fractal dimension and to answer the question, “How long is the coast of Britain?” Research Mandelbrot’s argument and explain why some might argue that the coast of Britain is infinitely long.
5. Up to this point, we have not considered the **area** of the Koch snowflake.
- a. Using whatever method you know, determine the exact area of the original triangle.
- b. How do you think we might find the area of the second stage of the snowflake? What about the third stage? The 7<sup>th</sup> stage? Are we adding area or subtracting area?
- c. To help us determine the area of the snowflake, complete the first two columns of the following chart. Note: The sequence of the number of “new” triangles is represented by a geometric sequence. Consider how the number of segments might help you determine how many new triangles are created at each stage.

Stage	Number of Segments	“New” triangles created	Area of each of the “new” triangles	Total Area of the New Triangles
<b>1</b>	3	--	--	--
<b>2</b>		3		
<b>3</b>				
<b>4</b>				
<b>5</b>				
...				
<b><i>n</i></b>				

- d. Determine the exact areas of the “new” triangles and the total area added by their creation for Stages 1 – 4. Fill in the chart above. (You may need to refer back to problem 1 for the segment lengths.)
- e. Because we are primarily interested in the total area of the snowflake, let’s look at the last column of the table. The values form a sequence. Determine if it is arithmetic or geometric. Then write the recursive and explicit formulas for the total area added by the new triangles at the  $n$ th stage.
- f. Determine how much area would be added at the 10<sup>th</sup> stage.
6. Rather than looking at the area at a specific stage, we are more interested in the **TOTAL area** of the snowflake. So we need to sum the areas. However, these are not necessarily numbers that we want to try to add up. Instead, we can use our rules of exponents and properties of summations to help us find the sum.
- a. Write an expression using summation notation for the sum of the areas in the snowflake.
- b. Explain how the expression you wrote in part a is equivalent to

$$\frac{81\sqrt{3}}{4} + \left(\frac{9}{4}\right)^4 \left(\frac{4\sqrt{3}}{3}\right) \sum_{i=2}^n \left(\frac{4}{9}\right)^i$$

Now, the only part left to determine is how to find the **sum of a finite geometric series**. Let’s take a step back and think about how we form a finite geometric series:

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \dots + a_1r^{n-1}$$

Multiplying both sides by  $r$ , we get  $rS_n = a_1r + a_1r^2 + a_1r^3 + \dots + a_1r^n$

Subtracting these two equations:  $S_n - rS_n = a_1 - a_1r^n$

Factoring:  $S_n(1 - r) = a_1(1 - r^n)$

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

And finally,

- c. Let’s use this formula to find the total area of only the new additions through the 5<sup>th</sup> stage. (What is  $a_1$  in this case? What is  $n$ ?) Check your answer by summing the values in your table.
- d. Now, add in the area of the original triangle. What is the total area of the Koch snowflake at the fifth stage?

Do you think it's possible to find the area of the snowflake for a value of  $n$  equal to infinity? This is equivalent to finding the **sum of an infinite geometric series**. You've already learned that we cannot find the sum of an infinite arithmetic series, but what about a geometric one?

- e. Let's look at an easier series:  $1 + \frac{1}{2} + \frac{1}{4} + \dots$ . Make a table of the first 10 sums of this series. What do you notice?

<b>Terms</b>	1	2	3	4	5	6	7	8	9	10
<b>Sum</b>	1	$\frac{3}{2}$	$\frac{7}{4}$							

- f. Now, let's look at a similar series:  $1 + 2 + 4 + \dots$ . Again, make a table. How is this table similar or different from the one above? Why do think this is so?

<b>Terms</b>	1	2	3	4	5	6	7	8	9	10
<b>Sum</b>	1	3	7							

Recall that any real number  $-1 < r < 1$  gets smaller when it is raised to a positive power; whereas numbers less than  $-1$  and greater than  $1$ , i.e.  $|r| > 1$ , get larger when they are raised to a positive power.

Thinking back to our sum formula,  $S_n = \frac{a_1(1-r^n)}{1-r}$ , this means that if  $|r| < 1$ , as  $n$  gets larger,  $r^n$  approaches 0. If we want the sum of an infinite geometric series, we now have

$S_\infty = \frac{a_1(1-0)}{1-r} = \frac{a_1}{1-r}$ . We say that if sum of an infinite series exists—in this case, the sum of an infinite geometric series only exists if  $|r| < 1$ —then the series **converges to a sum**. If an infinite series does not have a sum, we say that it **diverges**. All arithmetic series diverge.

- g. Of the series in parts e and f, which would have an infinite sum? Explain. Find, using the formula above, the sum of the infinite geometric series.
- h. Write out the formula for the sum of the first  $n$  terms of the sequence you summed in part g. Graph the corresponding function. What do you notice about the graph and the sum you found?
- i. **Graphs and infinite series.**

Write each of the following series using sigma notation. Then find the sum of the first 20 terms of the series; write out the formula. Finally, graph the function corresponding to the

sum formula for the first  $n$ th terms. What do you notice about the numbers in the series, the function, the sum, and the graph?

1.  $2 + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 + \dots$

2.  $4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \dots$

- j. Let's return to the area of the Koch Snowflake. If we continued the process of creating new triangles infinitely, could we find the area of the entire snowflake? Explain.
- k. If it is possible, find the total area of the snowflake if the iterations were carried out an infinite number of times.

This problem is quite interesting: We have a finite area but an infinite perimeter!

## Part Two: The Sierpinski Triangle



(Images taken from Wikimedia Commons at [http://en.wikipedia.org/wiki/File:Sierpinski\\_triangle\\_evolution.svg](http://en.wikipedia.org/wiki/File:Sierpinski_triangle_evolution.svg).)

Another example of a fractal is the Sierpinski triangle. Start with a triangle of side length 1. This time, we will consider the original picture as Stage 0. In Stage 1, divide the triangle into 4 congruent triangles by connecting the midpoints of the sides, and remove the center triangle. In Stage 2, repeat Stage 1 with the three remaining triangles, removing the centers in each case. This process repeats at each stage.

1. Mathematical Questions: Make a list of questions you have about this fractal, the Sierpinski triangle. What types of things might you want to investigate?
2. Number of Triangles in the Evolution of the Sierpinski Fractal
  - a. How many shaded triangles are there at each stage of the evolution? How many removed triangles are there? Use the table to help organize your answers.

Stage	Number of Shaded Triangles	Number of Newly Removed Triangles
0	1	0
1		
2		
3		
4		
5		

- b. Are the sequences above—the number of shaded triangles and the number of newly removed triangles—arithmetic or geometric sequences? How do you know?
- c. How many shaded triangles would there be in the  $n$ th stage? Write both the recursive and explicit formulas for the number of shaded triangles at the  $n$ th stage.

- d. How many newly removed triangles would there be in the  $n$ th stage? Write both the recursive and explicit formulas for the number of newly removed triangles at the  $n$ th stage.
- e. We can also find out how many removed triangles are in each evolution of the fractal. Write an expression for the total number of removed triangles at the  $n$ th stage. Try a few examples to make sure that your expression is correct.
- f. Find the total number of removed triangles at the 10<sup>th</sup> stage.
- g. If we were to continue iterating the Sierpinski triangle infinitely, could we find the total number of removed triangles? Why or why not. If it is possible, find the sum.

### 3. Perimeters of the Triangles in the Sierpinski Fractal

- a. Assume that the sides in the original triangle are one unit long. Find the perimeters of the shaded triangles. Complete the table below.

Stage	Length of a Side of a Shaded Triangle	Perimeter of each Shaded Triangle	Number of Shaded Triangles	Total Perimeter of the Shaded Triangles
0	1	3	1	3
1			3	
2				
3				
4				
5				
n				

- b. Find the perimeter of the shaded triangles in the 10<sup>th</sup> stage.
- c. Is the sequence of values for the total perimeter arithmetic, geometric, or neither? Explain how you know.
- d. Write recursive and explicit formulas for this sequence. In both forms, the common ratio should be clear.

### 4. Areas in the Sierpinski Fractal

- a. Assume that the length of the side of the original triangle is 1. Determine the exact area of each shaded triangle at each stage. Use this to determine the total area of the shaded triangles at each stage. (Hint: How are the shaded triangles at stage alike or different?)



Stage	Length of a Side of a Shaded Triangle (in)	Area of each Shaded Triangle (in <sup>2</sup> )	Number of Shaded Triangles	Total Area of the Shaded Triangles (in <sup>2</sup> )
0	1		1	
1			3	
2				
3				
4				
5				
n				

- b. Explain why both the sequence of the area of each shaded triangle and the sequence of the total area of the shaded triangles are geometric sequences. What is the common ratio in each? Explain why the common ratio makes sense in each case.
- c. Write the recursive and explicit formulas for the sequence of the area of each shaded triangle. Make sure that the common ratio is clear in each form.
- d. Write the recursive and explicit formulas for the sequence of the total area of the shaded triangles at each stage. Make sure that the common ratio is clear in each form.
- e. Propose one way to find the sum of the areas of the removed triangles using the results above. Find the sum of the areas of the removed triangles in stage 5.
- f. Another way to find the sum of the areas of the removed triangles is to find the areas of the *newly removed* triangles at each stage and sum them. Use the following table to help you organize your work.

Stage	Length of a Side of a Removed Triangle (in)	Area of each Newly Removed Triangle (in <sup>2</sup> )	Number of Newly Removed Triangles	Total Area of the Newly Removed Triangles (in <sup>2</sup> )
0	0	0	0	0
1				
2				
3				
4				
5				
n				

- g. Write the explicit and recursive formulas for the area of the removed triangles at stage  $n$ .

- h. Write an expression using summation notation for the sum of the areas of the removed triangles at each stage. Then use this formula to find the sum of the areas of the removed triangles in stage 5.
- i. Find the sum of the areas of the removed triangles at stage 20. What does this tell you about the area of the shaded triangles at stage 20? (Hint: What is the area of the original triangle?)
- j. If we were to continue iterating the fractal, would the sum of the areas of the removed triangles converge or diverge? How do you know? If it converges, to what value does it converge? Explain in at least two ways.

### Part Three: More with Geometric Sequences and Series

Up to this point, we have only investigated geometric sequences and series with a positive common ratio. We will look at some additional sequences and series to better understand how the common ratio impacts the terms of the sequence and the sum.

- For each of the following sequences, determine if the sequence is arithmetic, geometric, or neither. If arithmetic, determine the common difference  $d$ . If geometric, determine the common ratio  $r$ . If neither, explain why not.
  - 2, 4, 6, 8, 10, ...
  - 2, -4, 6, -8, 10, ...
  - 2, -4, -6, -8, -10, ...
  - 2, 4, 8, 16, 32, ...
  - 2, -4, 8, -16, 32, ...
  - 2, -4, -8, -16, -32, ...
- Can the signs of the terms of an arithmetic or geometric sequence alternate between positive and negative? If not, explain. If so, explain when.

- Write out the first 6 terms of the series  $\sum_{i=1}^n (-2)^i$ ,  $\sum_{i=0}^n (-2)^i$ , and  $\sum_{i=1}^n (-1)^{i-1} (2^i)$ . What do you notice?

- Write each series in summation notation and find sum of first 10 terms.

a.  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$

b.  $3 + \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \dots$

c.  $-4 + 12 - 36 + 108 - \dots$

- Which of the series above would converge? Which would diverge? How do you know? For the series that will converge, find the sum of the infinite series.

**DIVING INTO DIVERSIONS LEARNING TASK:**

Sequences and series often help us solve mathematical problems more efficiently than without their help. Often, however, one must make conjectures, test out the conjectures, and then prove the statements before the sequences and series can be generalized for specific situations. This task explores some of the usefulness of sequences and series. So, let's dive into some mathematical diversions on sequences and series!

**Part One: Fibonacci Sequence<sup>5</sup>****1. Honeybees and Family Trees**

The honeybee is an interesting insect. Honeybees live in colonies called hives and they have unusual family trees. Each hive has a special female called the queen. The other females are worker bees; these females produce no eggs and therefore do not reproduce. The males are called drones.

Here is what is really unusual about honeybees: Male honeybees hatch from unfertilized eggs laid by the queen and therefore have a female parent but no male parent. Female honeybees hatch from the queen's fertilized eggs. Therefore, males have only 1 parent, a mother, whereas females have both a mother and a father.

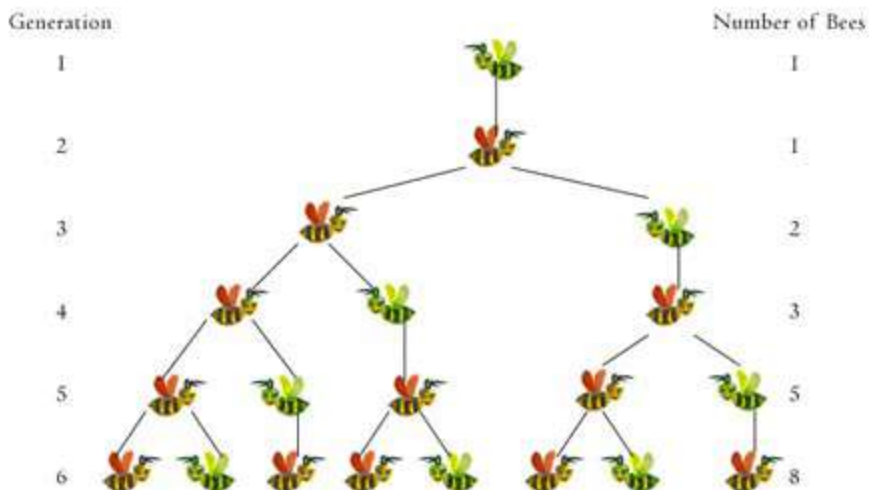
Consider the family tree of a male honeybee.

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<sup>5</sup> The traditional problem for examining the Fibonacci sequence is the rabbit problem. The honeybee problem, however, is more realistic. A number of additional Fibonacci activities are available on Dr. Knott's webpage at [www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibpuzzles.html](http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibpuzzles.html)

Lessons on the Rabbit problem can be found on Dr. Knott's page and on NCTM Illuminations.

(Male =  ; Female = )



(Picture obtained from <http://www.chabad.org/media/images/112/TilE1126142.jpg> on the webpage [www.chabad.org](http://www.chabad.org).)

- a. In the first generation, there is 1 male honeybee. This male honeybee has only one parent, a mother, at generation 2. The third generation consists of the male honeybee’s grandparents, the mother and father of his mother. How many great-grandparents and great-great grandparents does the male honeybee have? Explain why this makes sense.
- b. How many ancestors does the male honeybee have at each previous generation? Complete the following table. Find a way to determine the number of ancestors without drawing out or counting all the honeybees.

Term #	1	2	3	4	5	6	7	8	9	10
Value	1	1	2							

- c. The sequence of the number of bees in each generation is known as the *Fibonacci sequence*. Write a recursive formula for the Fibonacci sequence. (Hint: The recursion formula involves two previous terms.)

$F_1 = \underline{\hspace{2cm}}$      $F_2 = \underline{\hspace{2cm}}$      $F_n = \underline{\hspace{2cm}}$ ,  $n \geq 2$

- d. The Fibonacci sequence is found in a wide variety of objects that occur in nature: plants, mollusk shells, etc. Find a picture of an item that exhibits the Fibonacci sequence and explain how the sequence can be seen.

## 2. Golden Ratio

The sequence of Fibonacci is not the only interesting thing to arise from examining the Honeybee problem (or other similar phenomena). This next problem investigates an amazing fact relating the Fibonacci sequence to the *golden ratio*.

a. Using a spreadsheet or the list capabilities on your graphing calculator to create a list of the first 20 terms of the Fibonacci sequence and their ratios.

- In the first column, list the term number.
- In the second column, record the value of that term. (Spreadsheets can do this quickly.)
- In the third column, make a list, beginning in the second row, of each term and its preceding term. For example, in the row with term 2, the calculate  $F_2/F_1$ . (Again, a spreadsheet can do this quite quickly.)

Record the ratios in the table below.

What do you notice about the ratios?

Term	Ratio
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Term	Ratio
12	
13	
14	
15	
16	
17	
18	
19	
20	

b. Create a plot of the term number and the ratio. What do you notice?

c. When a sequence approaches a specific value in the long run (as the term number approaches infinity), we say that the sequence has a *limit*. What appears to be the limit of the sequence of ratios of the terms in the Fibonacci sequence?

- d. It would be nice to know the exact value of this limit. Take a step back. Find the solutions to the quadratic equation  $x^2 - x - 1 = 0$ . Find the decimal approximation of the solutions. Does anything look familiar?
- e. This number is called the *golden ratio* and is often written as  $\phi$  (phi). Just like the Fibonacci sequence, the golden ratio is present in many naturally occurring objects, including the proportions of our bodies. Find a picture of an item that exhibits the golden ratio and explain how the ratio is manifested.
- f. Return to your spreadsheet and create a new column of ratios. This time, find the ratio of a term and its following term, e.g.  $F_3/F_4$ . What appears to be the limit of this sequence of ratios?
- g. How do you think the limit in part f might be related to the golden ratio? Test out your conjectures and discuss with your classmates.

**Part Two: Does  $0.9999\dots = 1$ ?**

1. In previous courses, you may have learned a variety of ways to express decimals as fractions. For terminating decimals, this was relatively simple. The repeating decimals, however, made for a much more interesting problem. How might you change  $0.44444444\dots$  into a fraction?
2. One way you may have expressed repeating decimals as fractions is through the use of algebra.
  - a. For instance, let  $x = 0.44444\dots$ . What does  $10x$  equal? You now have two equations. Subtract the two equations and solve for  $x$ . Here's your fraction!
  - b. Use the algebra method to express each of the following repeating decimals as fractions.  
 $0.7777\dots$   
 $0.454545\dots$   
 $0.255555\dots$
3. We can also use geometric series to express repeating decimals as fractions. Consider  $0.444444\dots$  again.
  - a. How can we write this repeating decimal as an infinite series?
  - b. Now find the sum of this infinite series.
4. Express each of the following repeating decimals as infinite geometric series. Then find the sum of each infinite series.
  - a.  $0.\overline{5}$
  - b.  $0.\overline{47}$
  - c.  $0.1\overline{6}$
5. Let's return to the original question: does  $0.9999\dots = 1$ ? Use at least two different methods to answer this question.



**Part Three: Factorial Fun:** Lots of useful sequences and series involve factorials. You first learned about factorials in Math 1 when you discussed the binomial theorem, combinations and permutations.

1. As a brief refresher, calculate each of the following. Show how each can be calculated without the use of a calculator.

a.  $0!$       b.  $4!$       c.  $\frac{5!}{3!}$       d.  $\frac{10!}{7!3!}$

2. Write out the first five terms of each of the following sequences.

a.  $a_n = \frac{2^n}{(n-1)!}$       b.  $a_n = \frac{n^2}{n!}$       c.  $a_n = \frac{(-1)^{n+1}}{(n+1)!}$

3. Consider the sequences in number 2. Do you think each will diverge or converge? Explain. (Hint: It may be helpful to plot the sequences in a graphing calculator using “Seq” mode.)

4. Write an explicit formula, using factorials, for each of the following sequences.

a. 2, 4, 12, 40, 240, ...      b.  $1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \dots$       c.  $\frac{1}{2}, \frac{1}{3}, \frac{3}{10}, \frac{1}{5}, \dots$

### 5. The Exciting Natural Base $e$

In Math 2, we saw that we could approximate the transcendental number  $e$  by considering the compound interest formula.

- What is the formula for compounded interest and what value of  $e$  does your calculator provide?
- Find the value of a \$1 investment at 100% for 1 year at different values of  $n$ .

N	Value of Investment
1	
2	
4	
12	
52	
100	
1000	
10,000	
100,000	

- c. What investment value is being approached as  $n$  increased? So we say that as the value of  $n$  increased towards infinity, the expression approaches \_\_\_\_\_.
- d. We can also approximate  $e$  using a series with factorials. Write out and sum the first ten terms of  $\sum_{i=1}^n \frac{1}{i!}$ . What do you notice?
- e. Which approximation, the one employing the compound interest formula or the one using factorials, is more accurate with small values of  $n$ ?

In the future, perhaps in calculus, you will continue encountering sequences and values that can easily be expressed with factorials. Good luck!