HOW LONG DOES IT TAKE? LEARNING TASK:

Before sending astronauts to investigate the new planet of Exponentia, NASA decided to run a number of tests on the astronauts.

1. A specific multi-vitamin is eliminated from an adult male's bloodstream at a rate of about 20% per hour. The vitamin reaches peak level in the bloodstream of 300 milligrams.

a. How much of the vitamin remains 2 hours after the peak level? 5 hours after the peak level? Make a table of values to record your answers. Write expressions for how you obtain your answers.

Time (hours) since peak	0	1	2	3	4	5
Vitamin concentration in bloodstream (mg)	300	240	192	153.6	122.88	98.304

b. Using your work from (a), write expressions for each computed value using the original numbers in the problem (300 and 20%). For example, after 2 hours, the amount of vitamin left is [300 * (1 - .2)]* (1 - .2).

After 1 hour: $300*(1-.2)=240$; After 2 hours: $300*(1-.2)^2=192$; After 3 hours:

$300*(1-.2)^3 153.6$; After 4 hours: $300*(1-.2)^4=122.88$; After 5 hours: $300*(1-.2)^5=98.304$

c. Using part (b), write a function for the vitamin level with respect to the number of hours after the peak level, x.

$f(x)=300(1-.2)^x$ or $f(x)=300(.8)^x$

d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a).

$f(0)=300(.8)^0= 300$; $f(1)=300(.8)^1=240$; $f(2)=300(.8)^2=192$; $f(3)=300(.8)^3=153.6$;

$f(4)=300(.8)^4=122.88$; $f(5)=300(.8)^5=98.304$

e. After how many hours will there be less than 10 mg of the vitamin remaining in the bloodstream? Explain how you would determine this answer using both a table feature and a graph.

Between 15 and 16 hours after peak, the concentration dips below 10 mg.

f. Write an equation that you could solve to determine when the vitamin concentration is exactly 10 mg. Could you use the table feature to solve this equation? Could you use the graph feature? How could you use the intersection feature of your calculator? Solve the problem using one of these methods.

$300(.8)^x=10$; you could use the table feature of a graphing calculator to approximate the solution by making smaller and smaller table steps.

g. How would you solve the equation you wrote in (f) algebraically? What is the first step?

The first step would be to divide by 300. → $.8^x= 1/30$.

To finish solving the problem algebraically, we must know how to find inverses of exponential functions. This topic will be explored later in this unit.

2. A can of Instant Energy, a 16-ounce energy drink, contains 80 mg of caffeine. Suppose the caffeine in the bloodstream peaks at 80 mg. If ½ of the caffeine has been eliminated from the bloodstream after 5 hours (the half-life of caffeine in the bloodstream is 5 hours), complete the following:

a. How much caffeine will remain in the bloodstream after 5 hours? 10 hours? 1 hour? 2 hours? Make a table to organize your answers. Explain how you came up with your answers. (Make a conjecture. You can return to your answers later to make any corrections.)

Time (hours) since peak	0	1	2	5	10
Caffeine in bloodstream (mg)	80	69.644	60.629	40	20

b. Unlike problem (1), in this problem in which 80% remained after each hour, in this problem, 50% remains after each 5 hours.

i. In problem (1), what did the exponent in your equation represent?

The exponent represented the number of hours that had passed.

ii. In this problem, our exponent needs to represent the number of 5-hour time periods that elapsed. If you represent 1 hour as 1/5 of a 5-hour time period, how do you represent 2 hours? 3 hours? 10 hours? x hours? 2 hours is 2/5 of a time period; 3 hours is 3/5; 10 hours is 2 5-hour time periods; x hours is x/5 of a time period.

c. Using your last answer in part (b) as your exponent, write an exponential function to model the amount of caffeine remaining in the blood stream x hours after the peak level.

$F(x)=80(.5)^{x/5}$

d. How would use the function you wrote in (c) to answer the questions in (a)? Use your function and check to see that you get the same answers as you did in part (a). (Be careful with your fractional exponents when entering in the calculator. Use parentheses.) If you need to, draw a line through your original answers in part (a) and list your new answers.

$F(0)=80(.5)^{0/5}=80$; $f(1)=80(.5)^{1/5}=69/644$; $f(2)=80(.5)=60.629$; $f(5)=80(.5)^{1}=40$; $f(10)=80(.5)^{2}=20$

e. Determine the amount of caffeine remaining in the bloodstream 3 hours after the peak level? What about 8 hours after peak level? 20 hours? (Think about how many 5-hour intervals are in the number of hours you're interested in.)

$F(3)=80(.5)^{3/5}=53.78$; $f(8)=80(.5)^{8/5}$; $f(20)=80(.5)^{4}=5$

f. Suppose the half-life of caffeine in the bloodstream was 3 hours instead of 5.

i. Write a function for this new half-life time.

$F(x)=80(.5)^{x/3}$

ii. Determine the amount of caffeine in the bloodstream after 1 hour, 2 hours, 5 hours, and 10 hours. (You need to consider how many 3-hour time intervals are used in each time value.)
F(0)=80(.5)$^{0/3}$=80; f(1)=80(.5)$^{1/3}$=63.496; f(2)=80(.5)$^{2/3}$=50.397; f(5)=80(.5)$^{5/3}$=25.198; f(10)=80(.5)$^{10/3}$=7.937

iii. Which half-life time results in the caffeine being eliminated faster? Explain why this makes sense. Since the half-life in part f is less than the half-life in part c, caffeine is eliminated faster. Because the caffeine is eliminated faster in part f, there is less remaining in the bloodstream after the same number of hours than in part c.

g. Graph both equations (from d and f) on graph paper. How are the graphs similar? Different? What are the intercepts? When are they increasing/decreasing? Are there asymptotes? If so, what are they? Do the graphs intersect? Where?

(The graphs are not included here.) The graphs have the same shape; the same y-intercept (0, 80); they are both always decreasing, never increasing; they are both asymptotic to the x-axis. They only intersect at the y-intercept. After that, because the second (half-life of 3) decreases faster thatn the first, they will not intersect again.

Note that if we could only use integer exponents; e.g. 1, 2, 3, etc; our graphs would be discontinuous. We would have points (see right), rather than the smooth, continuous curve you graphed above. It makes sense, in thinking about time, that we need all rational time values, e.g. 1/3 hour, 5/8 hour, etc. This raises the idea of rational exponents, that is, computing values such as 3 ^3/4 or (1/2)^7/3.

3. Rational Exponents. In previous courses, you learned about different types of numbers and lots of rules of exponents.

a. What are integers? Rational numbers? Which set of numbers is a subset of the other? Explain why this is true. Integers are whole numbers and their opposites. Rational numbers are quotients of integers, where the denominator is not equal to zero. Integers are a subset of rational numbers.

b. Based on (a), what is the difference between integer exponents and rational exponents?

An integer exponent means that the exponent is an integer value; a rational exponent means that the exponent can be a rational number, including fractional or decimal values.

c. Complete the following exponent rules. (If you don't remember the rules from your previous classes, try some examples to help you.)

For a > 0 and b > 0, and all values of m and n,

A^0 = 1_ a^1 = a a^n = a^n

$(a^m)(a^n)$ = a^{m+n} $(a^m)/(a^n)$ = a^{m-n} a^{-n} = $1/a^n$

$(a^m)^n$ = a^{mn} $(ab)^m$ = $a^m b^m$ $(a/b)^m$ = a^m/b^m

If $a^m = a^n$, then m = n.

The same rules you use for integer exponents also apply to rational exponents.

d. You have previously learned that the nth root of a number x can be represented as

$x^{1/n}$.

i.Using your rules of exponents, write another expression for (x1/n)m.
$x^{m/n}$

ii. Using your rules of exponents, write another expression for (xm)1/n.

$x^{m/n}$

iii. What do you notice about the answers in (ii) and (iii)? What does this tell you about rational exponents? They are the same.

This leads us to the definition of rational exponents. For a > 0, and integers m and n, with $n > 0$, $a^{m/n} = \sqrt{a^m} = (\sqrt{a})^m$; $a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$.

e. Rewrite the following using simplified rational exponents.
   1. $X^{3/7}$ 2. X^5 3. $X^{6/2}=x^3$

f. Simplify each of the following. Show your steps. For example, $27^{2/3}= (27^{1/3})^2= 3^2=9$.
   1. 8 2. 216 3. 2187

Problems involving roots and rational exponents can sometimes be solved by rewriting expressions or by using inverses.

g. Consider the caffeine situation from above. The half-life of caffeine in an adult's bloodstream is 5 hours. How much caffeine remains in the bloodstream each hour? Our original equation was $f(x) = 80(.5)^{x/5}$. Use the rules of rational exponents to rewrite the equation so that the answer to the above question is given in the equation. What percent of the caffeine remains in the bloodstream each hour?
   $F(x)=80(.5)^{x/5}=80(.5^{1/5})^x=80(.870550633)^x$; approximately 87% remains after an hour.

   To solve equations such as $x^3 = 27$, we take the cube root of both sides. Alternately, we can raise both sides of the equation to the 1/3 power. That is , we raise both sides of the equation to the power that is the inverse (or reciprocal) of the power in the problem. To solve $x^{2/3} = 27$, we can either square both sides and then take the cube root, we can take the cube root of both sides and then square them, or we can raise both sides to the 2/3 power.
   $X^{3/2=27} \rightarrow (x^{3/2})^{2/3}=27^{2/3} \rightarrow x=(27^{1/3})^2 \rightarrow x=3^2=9$

h. Rewrite each of the following using rational exponents and use inverses to solve for the variable. (You may need to use a calculator for some of them. Be careful!)
   1. B=32 2. C=10.93 3. D=625

Let's look at some more problems that require the use of rational exponents.

4. Let's use a calculator to model bacteria growth. Begin with 25 bacteria.

a. If the number of bacteria doubles each hour, how many bacteria are alive after 1 hour? 2 hours?

b. Complete the chart below.

Time (hours)	0	1	2	3	4	5	6
Population							

c. Write a function that represents the population of bacteria after x hours. (Check that your function gives you the same answers you determined above. Think about what it means if the base number is 1. What type of base number is needed if the population is increasing?)

d. Use this expression to find the number of bacteria present after 7 ½ and 15 hours.

e. Suppose the initial population was 60 instead of 25. Write a function that represents the population of bacteria after x hours. Find the population after 7 ½ hours and 15 hours.

f. Graph the functions in part c and e. How are the graphs similar or different? What are the intercepts? What do the intercepts indicate? When are they increasing/decreasing? Are there asymptotes? If so, what are they? Do the graphs intersect? Where?

g. Revisit the graphs in problem 2. Compare with the graphs above. How are they similar and different? How do the equations indicate if the graphs will be increasing or decreasing?

h. Consider the following: Begin with 25 bacteria. The number of bacteria doubles every 4 hours. Write a function, using a rational exponent, for the number of bacteria present after x hours.

i. Rewrite the function in g, using the properties of exponents, so that the exponent is an integer. What is the rate of growth of the bacteria each hour?

j. If there are originally 25 bacteria, at what rate are they growing if the population of the bacteria doubles in 5 hours? (Hint: Solve $50 = 25(1 + r)^5$.) What about if the population triples in 5 hours? (Write equations and solve.)

k. If there are originally 25 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Solve the problem algebraically.

1. If there are originally 60 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Explain how you solved the problem. (Solving the problem algebraically will be addressed later in the unit.)