1 Measuring space: accuracy and 2D geometry

Almost everything we do requires some understanding of our surroundings and the distance between objects. But how do we go about measuring the space around us?

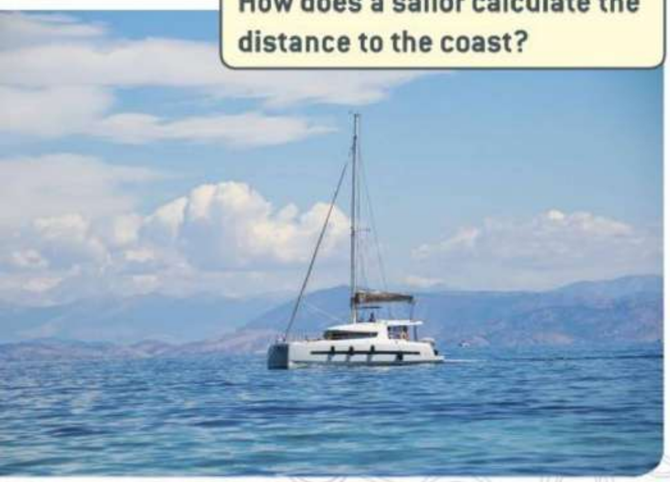
Concepts

- Quantity
- Space

Microconcepts

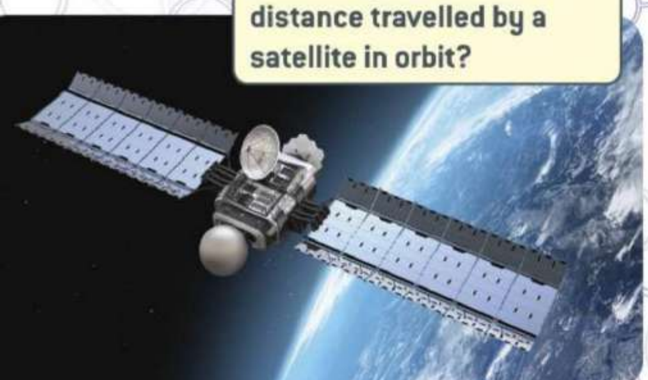
- Numbers
- Algebraic expressions
- Measurement
- Units of measure
- Approximation
- Estimation
- Upper/lower bound
- Error/percentage error
- Trigonometric ratios
- Angles of elevation and depression
- Length of arc

How does a sailor calculate the distance to the coast?



How can you measure the distance between two landmarks?

How can you calculate the distance travelled by a satellite in orbit?

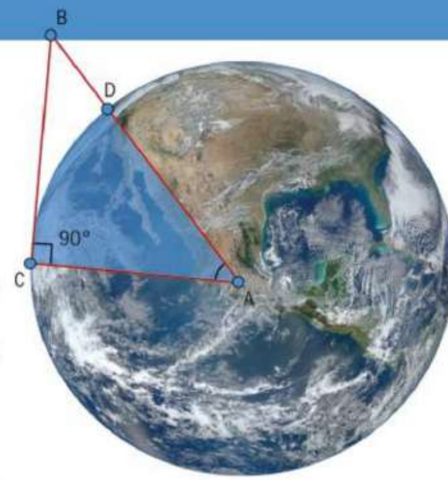


How do scientists measure the depths of lunar craters by measuring the length of the shadow cast by the edge of the crater?

How far can you see? If you stood somewhere higher or lower, how would that affect how much of the Earth you can see?

Think about the following:

- If you stood 10 metres above the ground, what would be the distance between you and the farthest object that you can see?
- One World Trade Center is the tallest building in New York City. If you stand on the Observatory floor, 382 m above the ground, how far can you see?
- How can you make a diagram to represent the distance to the farthest object that you can see? How do you think the distance you can see will change if you move the observation point higher?



Developing inquiry skills

Write down any similar inquiry questions that might be useful if you were asked to find how far you could see from a local landmark, or the top of the tallest building, in your city or country.

What type of questions would you need to ask to decide on the height of a control tower from which you could see the whole of an airfield? Write down any similar situations in which you could investigate how far you can see from a given point, and what to change so you could see more (or less).

Think about the questions in this opening problem and answer any you can. As you work through the chapter, you will gain mathematical knowledge and skills that will help you to answer them all.

Before you start

You should know how to:

1 Find the circumference of a circle with radius 2 cm.

eg $2\pi(2) = 12.6$ cm (12.5663…)

2 Find the area of a circle with radius 2 cm.

$\pi(2)^2 = 12.6$ cm (12.5663…)

3 Find the area of:

a a triangle with side 5 cm and height towards this side 8.2 cm

$$A = \frac{5 \times 8.2}{2} = 20.5\,\text{cm}^2$$

b a square with side 3 cm

$3^2 = 9\,\text{cm}^2$

c a trapezoid with bases 10 m and 7 m, and height 4.5 m.

$$\frac{10+7}{2} \times 4.5 = 8.5 \times 4.5 = 38.25\ \text{m}^2$$

Skills check

Click here for help with this skills check

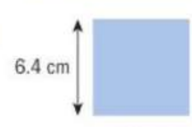
1 Find the circumference of a circle with radius $r = 5.3$ cm.

2 Find the area of a circle with radius 6.5 cm.

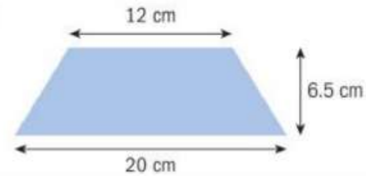
3 Find the areas of the following shapes.

a

b



c



1.1 Measurements and estimates

Investigation 1

A Measuring a potato

1 Make a list of all the physical properties of a potato. Which of these properties can you measure? How could you measure them? Are there any properties that you cannot measure? How do we determine what we can measure?

2 **Factual** What does it mean to measure a property of an object? How do we measure?

3 **Factual** Which properties of an object can we measure?

4 **Conceptual** Why do we use measurements and how do we use measuring to define properties of an object?

B Measuring length

5 Estimate the length of the potato.

6 Measure the length of the potato. How accurate do you think your measurement is?

C Measuring surface area

Recall that the area that encloses a 3D object is called the surface area.

7 Estimate the surface area of the potato and write down your estimate.

Use a piece of aluminium foil to wrap the potato and keep any overlapped areas to a minimum. Once the potato is entirely wrapped without any overlaps, unwrap the foil and place it over grid paper with 1 cm^2 units, trace around it and count the number of units that it covers.

8 Record your measurement. How accurate do you think it is?

9 Measure your potato again, this time using sheets of grid paper with units of 0.5 cm^2 and 0.25 cm^2. Again, superimpose the aluminium foil representing the surface area of your potato on each of the grids, trace around it on each sheet of grid paper, and estimate the surface area.

10 Compare your three measurements. What can you conclude?

11 **Factual** How accurate are your measurements? Could the use of different units affect your measurement?

D Measuring volume

You will measure the volume of a potato, which has an irregular shape, by using a technique that was used by the Ancient Greek mathematician Archimedes, called displacement. The potato is to be submerged in water and you will measure the distance the water level is raised.

12 Estimate the volume of the potato.

13 What units are you using to measure the volume of water? Can you use this unit to measure the volume of a solid?

Note the height of the water in the beaker before you insert the potato. Slowly and carefully place the potato in the water, and again note the height of the water. Determine the difference in water level.

International-mindedness

Where did numbers come from?

14 Record your measurement for the volume of the potato.

15 **Conceptual** If you used a beaker with smaller units, do you think that you would have a different measurement for the volume?

E Measuring weight

16 Estimate the weight of the potato and write down your estimate.

17 Use a balance scale to measure the weight of the potato. Which units will you use?

18 **Conceptual** Could the use of different units affect your measurement?

F Compare results

19 Compare your potato measurements with the measurements of another group. How would you decide which potato is larger? What measures can you use to decide?

20 **Conceptual** How do we describe the properties of an object?

Measurements help us compare objects and understand how they relate to each other.

Measuring requires approximating. If a smaller measuring unit is chosen then a more accurate measurement can be obtained.

When you measure, you first select a property of the object that you will measure. Then you choose an appropriate unit of measurement for that property. And finally, you determine the number of units.

Investigation 2

Margaret Hamilton worked for NASA as the lead developer for Apollo flight software. The photo here shows her in 1969, standing next to the books of navigation software code that she and her team produced for the Apollo mission that first sent humans to the Moon.

1 Estimate the height of the books of code stacked together, as shown in the image. What assumptions are you making?

2 Estimate the number of pages of code for the Apollo mission. How would you go about making this estimate? What assumptions are you making?

3 **Factual** What is an estimate? What is estimation? How would you go about estimating? How can comparing measures help you estimate?

4 **Conceptual** Why are estimations useful?

Estimation (or estimating) is finding an approximation as close as possible to the value of a measurement by sensible guessing. Often the estimate is used to check whether a calculation makes sense, or to avoid complicated calculations.

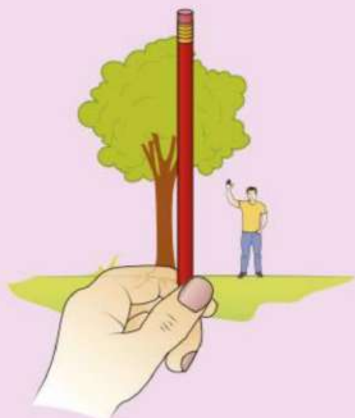
Estimation is often done by comparing the attribute that is measured to another one, or by sampling.

Did you know?

The idea of comparing and estimating goes way back. Some of the early methods of measurement are still in use today, and they require very little equipment!

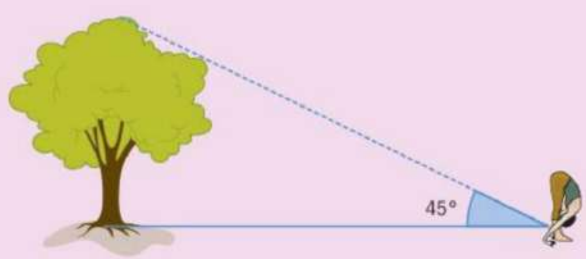
The logger method

Loggers often estimate tree heights by using simple objects, such as a pencil. An assistant stands at the base of the tree, and the logger moves a distance away from the tree and holds the pencil at arm's length, so that it matches the height of the assistant. The logger can then estimate the height of the tree in "pencil lengths" and multiply the estimate by the assistant's height.



The Native American method

Native Americans had a very unusual way of estimating the height of a tree. They would bend over and look through their legs!



This method is based on a simple reason: for a fit adult, the angle that is formed as they look through their legs is approximately 45 degrees. Can you explain how this method works?

1.2 Recording measurements, significant digits and rounding

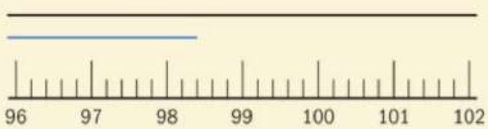
TOK

What might be the ethical implications of rounding numbers?

Investigation 3

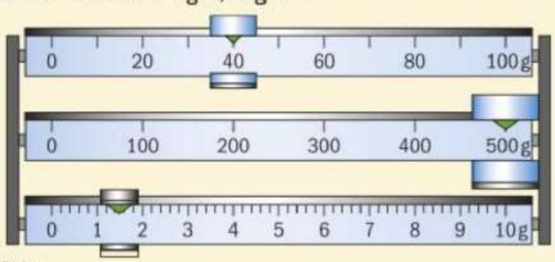
When using measuring instruments, we are able to determine only a certain number of digits accurately. In science, when measuring, the **significant figures** in a number are considered only those figures (digits) that are definitely known, plus one estimated figure (digit). This is summarized as "all of the digits that are known for certain, plus one that is a best estimate".

1 Read the temperature in degrees Fahrenheit from this scale.



What is the best reading of the temperature that you can do? How many digits are significant in your reading of this temperature?

2 A pack of coffee is placed on a triple-beam balance scale and weighed. The image below shows its weight, in grams.



Riders

Find the weight of the pack of coffee by carefully determining the reading of each of the three beam scales and adding these readings. How many digits are significant in your reading of this weight?

3 **Factual** What is the smallest unit to which the weight of the pack of coffee can be read on this scale?

4 A laboratory technician compares two samples that were measured as 95.270 grams and 23.63 grams. What is the number of significant figures for each measurement? Is 95.270 grams the same as 95.27 grams? If not, how are the two measurements different?

5 **Conceptual** What do the significant figures tell you about the values read from the instrument? What do the significant figures in a measurement tell you about the accuracy of the measuring instrument?

6 **Conceptual** How do the reading of the measuring instrument and the measuring units limit the accuracy of the measurement?

Decimal places

You may recall that in order to avoid long strings of digits it is often useful to give an answer to a number of decimal places (dp). For example, when giving a number to 2 decimal places, your answer would have exactly two digits after the decimal point. You round the final digit up if the digit after it is 5 or above, and you round the final digit down if the digit after it is below 5.

Significant figures

Measuring instruments have limitations. No instrument is advanced enough to determine an unlimited number of digits. For example, a scale can measure the mass of an object only up until a certain decimal place. Measuring instruments are able to determine only a certain number of digits precisely.

The digits that can be determined accurately or with some degree of reliability are called significant figures (sf). Thus, a scale that could register mass only up to hundredths of a gram until 99.99 g could only measure up to 4 digits with accuracy (4 significant figures).

Example 1

For each of the following, determine the number of significant figures.

21.35, 1.25, 305, 1009, 0.00300, 0.002

21.35 has 4 sf and 1.25 has 3 sf.	Non-zero digits are always significant.
305 has 3 sf and 1009 has 4 sf.	Any zeros between two significant digits are significant.
In 0.003**00** only the last two zeros are significant and the other zeros are not. It has 3 sf.	A final zero or trailing zeros in the decimal part **only** are significant.
0.002 has only 1 sf, and all zeros to the left of 2 are not sf.	

Rounding rules for significant figures

The rules for rounding to a given number of significant figures are similar to the ones for rounding to the nearest 10, 100, 1000, etc. or to a given number of decimal places.

EXAM HINT

In exams, give your answers as exact or accurate to 3 sf, unless otherwise specified in the problem.

Example 2

Round the following numbers to the required number of significant figures:

a 0.1235 to 2 sf **b** 0.2965 to 2 sf

c 415.25 to 3 sf **d** 3050 to 2 sf

a $0.\underline{12}35 = 0.12$ (2 sf)	Underline the 2 significant figures. The next digit is less than 5, so delete it and the digits to the right.
b $0.\underline{29}65 = 0.30$ (2 sf)	The next digit is greater than 5 so round up. Write the 0 after the 3, to give 2 sf.
c $\underline{415}.25 = 415$ (3 sf)	Do not write 415.0, as you only need to give 3 sf.
d $\underline{30}50 = 3100$ (2 sf)	Write the zeros to keep the place value.

Rounding rule for n significant figures

If the $(n + 1)$th figure is less than 5, keep the nth figure as it is and remove all figures following it.

If the $(n + 1)$th figure is 5 or higher, add 1 to the nth figure and remove all figures following it.

In either case, all figures after the nth one should be removed if they are to the right of the decimal point and replaced by zeros if they are to the left of the decimal point.

Exercise 1A

1 Round the following measurements to 3 significant figures.

a 9.478 m **b** 5.322 g

c 1.8055 cm **d** 6.999 in

e 4578 km **f** 13 178 kg

2 Round the numbers in question 1 parts **a** to **d** to 2 dp.

3 Determine the number of significant figures in the following measurements:

a 0.102 m **b** 1.002 dm

c 105 kg **d** 0.001020 km

e 1 000 000 µg

4 Find the value of the expression

$\frac{12.35 + 21.14 + 1.075}{\sqrt{3.5} - 1}$ and give your answer to 3 significant figures.

Example 3

A circle has radius 12.4 cm. Calculate:

a the circumference of the circle

b the area of the circle.

Write down your answers correct to 1 dp.

Continued on next page

a $C = 2 \times \pi \times 12.4$ $= 77.9\,\text{cm}$ (1 dp)	Use the formula for circumference of a circle, $C = 2\pi r$. The answer should be given correct to 1 dp, so you have to round to the nearest tenth.
b $A = \pi(12.4)^2$ $= 483.1\,\text{cm}^2$ (1 dp)	Use the formula for area of a circle, $A = \pi r^2$. The answer should be given correct to 1 dp, so you have to round to the nearest tenth.

Investigation 4

The numbers of visitors to the 10 most popular national parks in the United States in 2016 are shown in the table.

10 Most Visited National Parks (2016)	
Park	***Recreational Visits***
1. Great Smoky Mountains NP	11312786
2. Grand Canyon NP	5969811
3. Yosemite NP	5028868
4. Rocky Mountain NP	4517585
5. Zion NP	4295127
6. Yellowstone NP	4257177
7. Olympic NP	3390221
8. Acadia NP	3303393
9. Grand Teton NP	3270076
10. Glacier NP	2946681

1 Which park had the most visitors? How accurate are these figures likely to be?

2 Round the numbers of visitors given in the table to the nearest 10 000.

3 Are there parks with an equal number of visitors, when given correct to 10 000? If so, which are they?

4 Round the number of visitors, given in the table, to the nearest 100 000.

5 Are there parks with an equal number of visitors, when given correct to 100 000? If so, which are they?

6 Round the numbers of visitors given in the table to the nearest 1 000 000.

7 Are there parks with an equal number of visitors, when given correct to 1 000 000? If so, which are they?

8 Determine how many times the number of visitors of the most visited park is bigger than the number of visitors of the least visited park. Which parks are they?

9 Determine how many times the number of visitors of the most visited park is bigger than the number of visitors of the second most visited park.

10 **Factual** Determine how many times the number of visitors of the most visited park is bigger than the number of visitors of the third most visited park.

11 **Conceptual** How did rounding help you compare the numbers of park visitors?

12 **Conceptual** What are the limitations of a measurement reading in terms of accuracy?

13 **Conceptual** How is rounding useful?

Exercise 1B

1 Round each of the following numbers as stated:

- **a** 8888 to 3 sf
- **b** 3.749 to 3 sf
- **c** 27 318 to 1 sf
- **d** 0.00637 to 2 sf
- **e** $\sqrt{62}$ to 1 dp

2 Round the numbers in the table to the given accuracy.

	Number	Round to the nearest ten	Round to the nearest hundred	Round to the nearest thousand
a	2815			
b	75 391			
c	316 479			
d	932			
e	8253			

3 Round the following amounts to the given accuracy:

- **a** 502.13 EUR to the nearest EUR
- **b** 1002.50 USD to the nearest thousand USD
- **c** 12 BGN to the nearest 10 BGN
- **d** 1351.368 JPY to the nearest 100 JPY

4 A circle has radius 33 cm. Calculate the circumference of the circle. Write down your answer correct to 3 sf.

5 The area of a circle is 20 cm^2 correct to 2 sf. Calculate the radius of the circle correct to 2 sf.

6 Estimate the volume of a cube with side 4.82 cm. Write down your answer correct to 2 sf.

1.3 Measurements: exact or approximate?

Accuracy

The accuracy of a measurement often depends on the measuring units used. The smaller the measuring unit used, the greater the accuracy. If I use a balance scale that measures only to the nearest gram to weigh my silver earrings, I will get 11 g. But if I use an electronic scale that measures to the nearest hundredth of a gram, then I get 11.23 g.

TOK

To what extent do instinct and reason create knowledge?

Do different geometries (Euclidean and non-Euclidean) refer to or describe different worlds? Is a triangle always made up of straight lines? Is the angle sum of a triangle always 180°?

The accuracy would also depend on the precision of the measuring instrument. If I measured the weight of my earrings three times, the electronic scale might produce three different results: 11.23 g, 11.30 g and 11.28 g. Usually, the average of the available measures is considered to be the best measurement, but it is certainly not exact.

Each measuring device (metric ruler, thermometer, theodolite, protractor, etc.) has a different **degree of accuracy**, which can be determined by finding the smallest division on the instrument. Measuring the dimensions of a rug with half a centimetre accuracy could be acceptable, but a surgical incision with such precision most likely will not be good enough!

A value is **accurate** if it is close to the **exact value** of the quantity being measured. However, in most cases it is not possible to obtain the exact value of a measurement. For example, when measuring your weight, you can get a more accurate measurement if you use a balance scale that measures to a greater number of decimal places.

Investigation 5

The **yard** and the **foot** are units of length in both the British Imperial and US customary systems of measurement. Metal yard and foot sticks were the original physical standards from which other units of length were derived.

In the 19th and 20th centuries, differences in the prototype yards and feet were detected through improved technology, and as a result, in 1959, the lengths of a yard and a foot were defined in terms of the metre.

In an experiment conducted in class, several groups of students worked on measuring a standard yardstick and a foot-long string.

1. Group 1 used a ruler with centimetre and millimetre units and took two measurements: one of a yardstick and one of a foot-long string. Albena, the group note taker, rounded off the two measurements to the nearest centimetre and recorded the results for the yard length as 92 cm and for the foot length as 29 cm. Write down the possible values for the unrounded results that the group obtained. Give all possible unrounded values for each measure as intervals in the form $a \le x < b$.

2. Group 2 used a Vernier caliper, which is able to measure lengths to tenths of a millimetre. They also took measurements of a yardstick and of a piece of string a foot long. Velina, the group note taker, rounded off the two measurements to the nearest millimetre and recorded the results as 91.5 cm and 31.5 cm. Write down the possible values for the unrounded results that the group obtained. Give all possible unrounded values for each measure as intervals in the form $a \le x < b$.

International-mindedness

SI units

In 1960, the International System of Units, abbreviated SI from the French, "systeme International", was adopted as a practical system of units for international use and includes metres for distance, kilograms for mass and seconds for time.

3 **Conceptual** Can you explain why the intervals in parts **1** and **2** **include** the endpoint a but **exclude** the endpoint b?

4 **Conceptual** Based on your conclusions in parts **1** and **2**, make a conjecture about the interval in which the exact value should lie. How big is this interval in relation to the unit used? How would you determine the left and the right ends of the interval?

The left and the right ends of an interval in which an exact value of a measurement lies are respectively called the **lower bound** and the **upper bound**.

The lower bound and the upper bound are half a unit below and above a rounded value of a measurement. Thus the upper bound is calculated as the rounded measurement + 0.5 unit, and the lower bound is found as the rounded measurement − 0.5 unit.

Did you know?

The exact values of one yard and one foot are defined by an international agreement in 1959. A yard was defined as 0.9144 metres exactly, and a foot was defined as 0.3048 metres exactly.

Example 4

a Jane's weight is 68 kg to the nearest kg. Determine the upper and lower bounds of her weight.

b Rushdha's height is measured as 155 cm to the nearest cm. Write the interval within which her exact height lies.

a The upper and lower bounds are 68.5 and 67.5, respectively.	The range of possible values for Jane's weight is 68 ± 0.5 kg.
b $154.5 \le h < 155.5$	The range of possible values for Rushdha's height is 155 ± 0.5 cm.

Example 5

Majid ran 100 metres in 11.3 seconds. His time is measured to the nearest tenth of a second. Determine the upper and lower bounds of Majid's running time.

Lower bound = 11.3 − 0.05 = 11.25	Lower bound is 11.3 − 0.5 unit, and upper bound is 11.3 + 0.5 unit.
Upper bound = 11.3 + 0.05 = 11.35	Majid's time is given to the nearest tenth of a second. A unit is a tenth of a second or 0.1 sec, and 0.5 unit is 0.05 sec.

Example 6

A rectangular garden plot was measured as 172 m by 64 m. Determine the lower and upper bounds of its perimeter.

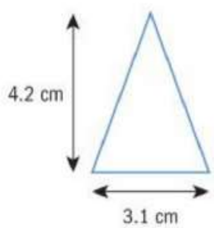
$171.5 \le L < 172.5$ $63.5 \le W < 64.5$ Then the lower bound of the perimeter is $2 \times (171.5 + 63.5) = 470$, and the upper bound of the perimeter is $2 \times (172.5 + 64.5) = 474$. Thus $470 \le P < 474$.	For the lower bound of the perimeter, use the shortest possible lengths of the sides: the measured values − 0.5 m; for the upper bound, use the longest possible lengths: the measured values + 0.5 m.

Exercise 1C

1 Find the range of possible values for the following measurements, which were rounded to the nearest mm, tenth of m, and hundredth of kg respectively:

a 24 mm **b** 3.2 m **c** 1.75 kg

2 A triangle has a base length of 3.1 cm and corresponding height 4.2 cm, correct to 1 decimal place. Calculate the upper and lower bounds for the area of the triangle as accurately as possible.



3 With 72 million bicycles, correct to the nearest million, Japan is at the top of the list of countries with most bicycles per person. On average, Japanese people travel about 2 km by bicycle, correct to the nearest km, each day. Calculate the upper bound for the total distance travelled by all the bicycles in Japan.

4 To determine whether a business is making enough profit, the following formula is used:

$$P = \frac{S - C}{S}$$

where P is relative profit, S is sales and C is costs. If a company has \$340 000 worth of sales and \$230 000 of costs, correct to 2 significant figures, calculate the maximum and minimum relative profit to the appropriate accuracy.

Since measurements are approximate there is always error in the measurement results. A measurement error is the difference between the exact value (V_E) and the approximate value (V_A), ie

$$\text{Measurement error} = V_A - V_E$$

Investigation 6

Tomi and Massimo measured the length of a yardstick and the length of a foot-long string and obtained 92.44 cm for the length of a yard and 31.48 cm for the length of a foot.

TOK

Do the names that we give things impact how we understand them?

1 Given that the exact value of 1 yard is 91.44 cm and of 1 foot is 30.48 cm, find the measurement error in the results obtained by Tomi and Massimo.

Tomi thinks that the two measurements were equally inaccurate. Massimo thinks that one of the two measurements is more accurate than the other.

2 Who do you agree with: Tomi or Massimo? Why? Explain.

Massimo decides to find the magnitude of his measurement error as a percentage of the measured length.

3 Write down the error in measuring the length of 1 yard as a fraction of the exact length of 1 yard. Give your answer as a percentage.

4 Write down the error in measuring the length of 1 foot as a fraction of the exact length of 1 foot. Give your answer as a percentage.

5 **Conceptual** In what ways could expressing measurement errors as a percentage of the exact value be helpful?

6 **Conceptual** How could measurement errors be compared?

The percentage error formula calculates the error as a percentage of the measured quantity. For example, a weight measured as 102 kg when the exact value is 100 kg gives a measurement error of 2 kg and percentage error of $\frac{2}{100} = 2\%$. A weight measured as 27 kg when the exact value is 25 kg gives the same measurement error of 2 kg but a percentage error of $\frac{2}{25} = 8\%$.

Percentage error $= \left|\frac{V_A - V_E}{V_E}\right| \times 100\%$, where V_A is the approximate value and V_E is the exact value.

Example 7

The fraction $\frac{22}{7}$ is often used as an approximation of π.

a How close (to how many decimal places) does $\frac{22}{7}$ approximate π?

b Find the percentage error of this approximation, giving your answer to 2 dp.

a $\frac{22}{7} - \pi = 0.001264...$ Thus $\frac{22}{7}$ approximates π to 2 dp.	Measurement error $= V_A - V_E$, where V_A is the approximate value and V_E is the exact value.

Continued on next page

b Percentage error $= \left\lvert \dfrac{\frac{22}{7} - \pi}{\pi} \right\rvert \times 100\%$ $= 0.04\%$	Percentage error $= \left\lvert \dfrac{V_A - V_E}{V_E} \right\rvert \times 100\%$ Be careful to take the absolute value of the fraction as the percentage error is always a positive number!

In multistep calculations, you must be careful not to round figures until you have your final answer. Premature rounding of initial or intermediate results may lead to an incorrect answer.

TOK

How does the perception of the language being used distort our understanding?

Example 8

Calculate the density of a cube of sugar weighing 2.45 grams, where the side of the cube is 1.2 cm, correct to 1 dp.

Volume $= (1.2)^3 = 1.728\,\text{cm}^3$ Density $= \dfrac{2.45}{1.728} = 1.41782\ldots\ \text{g/cm}^3$ $= 1.4\ \text{g/cm}^3$ (correct to 1 dp)	If you first rounded the mass to 1 dp, then calculated the volume to 1 dp and then divided, you would obtain: $\dfrac{2.5}{1.7} = 1.47058\ldots\ \text{g/cm}^3$, $= 1.5\ \text{g/cm}^3$ (correct to 1 dp) This gives a percentage error of $\left\lvert \dfrac{1.5 - 1.41782\ldots}{1.41782\ldots} \right\rvert \times 100\% = 5.80\%$ Make sure to avoid premature rounding!

Rounding of intermediate results during multistep calculations reduces the accuracy of the final answers. Thus, make sure to round your final answer only.

Exercise 1D

1. Find the percentage error in using 3.14 instead of π.

2. In 1856, Andrew Waugh announced Mount Everest to be 8840 m high, after several years of calculations based on observations made by the Great Trigonometric Survey. More recent surveys confirm the height as 8848 m. Calculate the percentage error made in the earlier survey.

3. Eratosthenes estimated the circumference of the Earth as 250 000 stadia (the length of an athletic stadium). If we assume he used the most common length of stadia of his time, his estimate of the circumference of the Earth would be 46 620 km. Currently, the accepted average circumference of the Earth is 40 030.2 km. Find the percentage error of Eratosthenes' estimate of the circumference of the Earth.

4 The temperature today in Chicago is 50°F. Instead of using the standard conversion formula $°C = \frac{5}{9} \times (°F - 32)$, Tommaso uses his grandmother's rule, which is easier but gives an approximate value: "*Subtract* 32 *from the value in* °F *and multiply the result by* 0.5."

a Calculate the actual and an approximate temperature in °C, using the standard formula and Tommaso's grandmother's rule.

b Calculate the percentage error of the approximate temperature value in °C.

1.4 Speaking scientifically

Investigation 7

1 a Complete the table, following the pattern:

Number							
Written as powers of 10	10^3	10^2	10^1	10^0		10^{-2}	10^{-3}
Written as decimals	1000						
Written as fractions	$\frac{1000}{1}$				$\frac{1}{10}$		

b When you move from left to right, from one column to the next, which operation would you use?

c How would you write 10^{-4} as a decimal and as a fraction?

d Write 10^{-n} as a fraction.

2 Complete the table, following the pattern:

Number							
Written as powers of 2	2^3	2^2	2^1	2^0		2^{-2}	2^{-3}
Written as decimals	8						
Written as fractions	$\frac{8}{1}$		$\frac{2}{1}$		$\frac{1}{2}$		

a Find 2^0.

b Find 10^0.

c Find x^0.

d How are algebraic expressions similar to and different from a numerical expression?

Numerical expressions consist only of numbers and symbols of operations (addition, subtraction, multiplication, division and exponentiation), whereas **algebraic expressions** contain numbers, variables and symbols of operations.

Exponentiation is a mathematical operation, written as a^n, where a is called the base and n the exponent or power.

If the exponent, n, is a **positive integer**, it determines **how many times** the base, a, is multiplied by itself. For example, 8^2 means 8×8.

If the exponent, n, is a **negative integer** it determines how many times **to divide** 1 by the base, a. For example, 8^{-2} means $\frac{1}{8^2}$ or $\frac{1}{8 \times 8}$.

The base, a, is the factor in the expression $a \times a \times a \times \ldots \times a$ if the exponent, n, is positive and is the factor in the expression $\frac{1}{a \times a \times a \times \ldots \times a}$ if the exponent, n, is negative.

International-mindedness

Indian mathematician Brahmagupta is credited with the first writings that included zero and negative numbers in the 7th century.

Investigation 8

1 a Complete the first and the third columns of the table. The middle column you can choose to finish or not.

Expression	Expanded expression	Written as power of 10
$10^2 \times 10^3$	$(10 \times 10) \times (10 \times 10 \times 10)$	10^5
$10^4 \times 10^5$		
	$(10 \times 10 \times 10 \times 10) \times (10 \times 10 \times 10 \times 10 \times 10)$	
		10^8
$10^5 \times 10^6$		
$10^1 \times 10^{10}$		
$(10^2)^3$	$(10 \times 10) \times (10 \times 10) \times (10 \times 10)$	

b Follow the pattern from the table and rewrite $10^m \times 10^n$ as a single power of 10.

c Rewrite $10^2 \times 10^0$ as a single power of 10.

d What can you conclude for the value of 10^0? Why?

e Follow the pattern and rewrite $x^m \times x^n$ as a single power.

f Use powers and multiplication to write three expressions whose value is 10^{11}.

g Rewrite $(10^2)^3$ as a single power.

h Follow the pattern and rewrite $(10^m)^n$ as a single power.

i Follow the pattern from part **h**, and rewrite $(x^m)^n$ as a single power.

j Write the expanded expression for $(2 \times 3)^5$. Then rewrite each term of the expanded expression as a product of two single powers.

k Follow the pattern in part **j**, and rewrite $(x \times y)^n$ as a product of two single powers.

2 **a** Complete the first and the third columns of the table. The middle column you can choose to finish or not.

Expression	Expanded expression	Written as power of 10
$\frac{10^3}{10^2}$	$\frac{10\times10\times10}{10\times10}$	10^1
$\frac{10^5}{10^3}$		
	$\frac{10\times10\times10\times10\times10\times10}{10\times10\times10}$	
		10^4
$\frac{10^6}{10^6}$		
$\frac{10^3}{10^0}$		

b Follow the pattern from the table and rewrite $\frac{10^m}{10^n}$ as a single power of 10.

c Rewrite $\frac{10^2}{10^0}$ as a single power of 10. What would that be?

d Follow the pattern and rewrite $\frac{x^m}{x^n}$ as a single power.

e Use powers and division to write three expressions whose value is 10^5.

f Write the expanded expression of $\left(\frac{2}{3}\right)^5$. Rewrite the expanded expression as a quotient of two powers.

g Write the expanded expression of $\left(\frac{x}{y}\right)^n$. Rewrite the expanded expression as a quotient of two powers.

Laws of exponents

Law	Example
$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^{-1} = \frac{1}{x}$	$4^{-1} = \frac{1}{4}$
$x^m x^n = x^{m+n}$	$x^2x^3 = x^{2+3} = x^5$

Continued on next page

International-mindedness

Archimedes discovered and proved the law of exponents, $10^x \times 10^y = 10^{x+y}$.

Law	Example
$\frac{x^m}{x^n} = x^{m-n}$	$\frac{x^6}{x^2} = x^{6-2} = x^4$
$(x^m)^n = x^{mn}$	$(x^2)^3 = x^{2\times 3} = x^6$
$(xy)^n = x^n y^n$	$(xy)^3 = x^3 y^3$
$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$
$x^{-n} = \frac{1}{x^n}$	$x^{-3} = \frac{1}{x^3}$

International-mindedness

Abu Kamil Shuja was called al-Hasib al-Misri, meaning "the calculator from Egypt", and was one of the first mathematicians to introduce symbols for indices in the 9th century.

Example 9

Use the laws of exponents to express each of the following in terms of powers of a single number:

a $\frac{2}{2^3} + (2^2)^3$ **b** $\frac{4^3 \times 4^{-7}}{4^2}$

a $2^{-2} + 2^6 = 64.25$	Use your GDC, or use $\frac{x^m}{x^n} = x^{m-n}$ for $\frac{2}{2^3}$ and $(x^m)^n = x^{mn}$ for $(2^2)^3$.
b $\frac{4^{-4}}{4^{-2}} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16} = 0.0625$	Use your GDC, or use $x^m x^n = x^{m+n}$ to simplify the numerator and then $\frac{x^m}{x^n} = x^{m-n}$ to simplify the quotient.

Example 10

Write the following expressions using a single power of x:

a $(x^2)^3$ **b** $\left(\frac{1}{x^2}\right)^4$ **c** $(x^3)^{-1}$ **d** $\frac{x^3 \times x^{-1}}{x^5}$

a x^6	Use $(x^m)^n = x^{mn}$.
b $\left(\frac{1}{x^2}\right)^4 = \left(x^{-2}\right)^4 = x^{-8}$	Use $x^{-n} = \frac{1}{x^n}$ to write $\frac{1}{x^2}$ as x^{-2}, and then apply $(x^m)^n = x^{mn}$.
c $(x^3)^{-1} = \frac{1}{x^3} = x^{-3}$	Use $(x^m)^n = x^{mn}$, or use $x^{-n} = \frac{1}{x^n}$ twice.
d $\frac{x^2}{x^5} = x^{-3}$	Use $\frac{x^m}{x^n} = x^{m-n}$.

Exercise 1E

1 Calculate the following numerical expressions, giving your answer as a single power of an integer.

a $2^3 \times 2^3$ b $5^2 \times 5^1$

c $\frac{6^7}{6^5}$ d $\frac{4^2}{5^{-2}}$

e $8^6 \times 8^{-3}$ f $(3^2)^4$

g $\frac{3^{-4}}{3^2 \times 3^9}$ h $\frac{2^7 \times 2^{-4}}{2^3}$

i $5^4 \times 3^4$ j $\frac{20^3}{4^3}$

2 Simplify the following algebraic expressions:

a $x^{-2} \times (x^2)^3 \times (x^2 \times x^6)$ b $\frac{x^0 \times (x^2)^3}{x^2 \times x^{-3}}$

Standard form

In standard form (also known as scientific notation), numbers are written in the form $a \times 10^n$, where a is called the **coefficient** or **mantissa**, with $1 \leq a < 10$, and n is an integer.

Scientific notation is certainly economical; a number such as *googol*, which in decimal notation is written as 1 followed by 100 zeros, is written simply as 10^{100} in scientific notation.

With scientific notation, **Avogadro's constant**, which is the number of particles (atoms or molecules) contained in 1 mole of a substance, is written as $6.022140857 \times 10^{23}$. If not for scientific notation it would take 24 digits to write!

You may have noticed that your graphic display calculator gives any results with lots of digits in scientific notation. However, instead of writing the results in the form $a \times 10^n$, it gives them in the form $a\text{E}n$, where n is an integer. For example, 3×10^5 will be given as 3E5, and 3.1×10^{-3} as 3.1E–3.

EXAM HINT

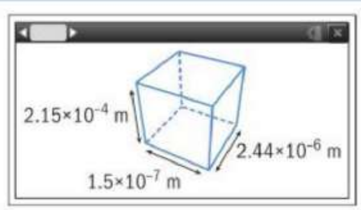
Note that in an exam you cannot write 3E5, as this is calculator notation and not the correct scientific notation. The correct notation is 3×10^5.

To convert from decimal to scientific notation on your GDC, change the Mode from Normal to Scientific. Thus, all of your number entries will be immediately converted to scientific calculator notation. To convert to correct scientific notation, you will need to replace E with 10, eg 5E2 should be written as 5×10^2.

Example 11

Find the volume of a computer chip (a cuboid) that is 2.44×10^{-6} m wide, 1.5×10^{-7} m long and 2.15×10^{-4} m high.

The volume of a cuboid = length × width × height



Make sure to give your answer in standard form.

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The volume is 7.869×10^{-17} m^3.	To check your answer is sensible, multiply the rounded values $2 \times 2 \times 2 = 8$, and check the powers of 10 come to 10^{-17}.

Exercise 1F

1 Find the measurements below in the form $a \times 10^n$, where $1 \le a < 10$ and n is an integer:

a the density of air at 27°C and 1 atm pressure, 0.00161 g/cm^3

b the radius of a calcium atom, 0.000 000 000 197 m

c one light-year, 9 460 000 000 000 km

d the mass of a neutron, 0.000 000 000 000 000 000 000 001 675 g.

2 Write down the following numbers found on a calculator display in scientific notation:

a 1.2E–1 b 5.04E7

c 4.005E–5 d 1E–3

3 The image of a speck of dust viewed through an electron microscope is 1.2×10^2 millimetres wide. The image is 5×10^2 times as large as the actual size. Determine the width, in millimetres, of the actual speck of dust.

4 a Convert the following from decimal to scientific notation:

age of the Earth = 4 600 000 000 years.

b Convert the following from scientific to decimal notation: 5×10^3.

5 One millilitre has about 15 drops. One drop of water has 1.67×10^{21} H_2O (water) molecules. Estimate the number of molecules in 1 litre of water. Write down your answer in standard form.

6 Scientists announced the discovery of a potential "Planet Nine" in our solar system in 2016.

The so-called "Planet Nine" is about 5000 times the mass of Pluto. Pluto's mass is 0.01303×10^{24} kg.

a Calculate the mass of Planet Nine. Write down your answer in standard form.

b The Earth's mass is 5.97×10^{24} kg. Find how many times Planet Nine is bigger or smaller compared to the Earth. Write down your answer correct to the nearest digit.

TOK

What do we mean when we say that one number is larger than another number?

7 The table below shows the population of different regions in 1985 and in 2005.

Place	Population in 1985	Population in 2005
Entire world	4.9×10^9	6.4×10^9
China	1.1×10^9	1.3×10^9
India	7.6×10^8	1.1×10^9
United States	2.4×10^8	3.0×10^8
Bulgaria	8.9×10^6	7.2×10^6

a Determine the ratio of the population of the entire world to that of China in 2005, giving your answer to the nearest integer.

b Find the increase in the world's population between 1985 and 2005.

c Calculate the percentage change in the population of each of the four countries, giving your answers accurate to 3 sf. List the countries according to their percentage change from highest to lowest.

d State whether or not you agree that it is always the case that the country with a bigger percentage change also has a bigger population increase between 1985 and 2005. Justify your answer.

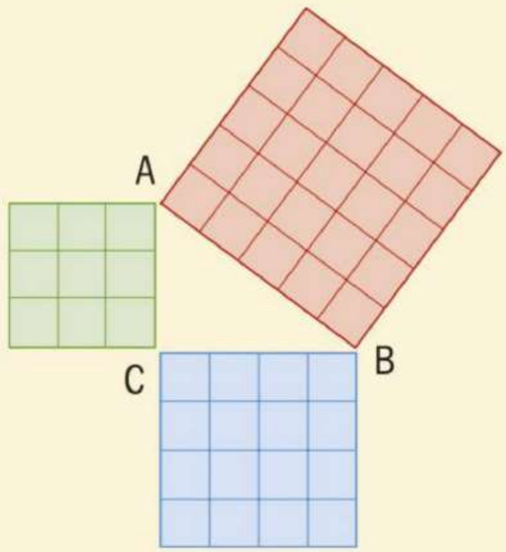
1.5 Trigonometry of right-angled triangles and indirect measurements

Investigation 9

1 Puzzle One:

What do you notice about the three squares built on the sides of the right-angled triangle ΔABC, where angle $\hat{C} = 90°$?

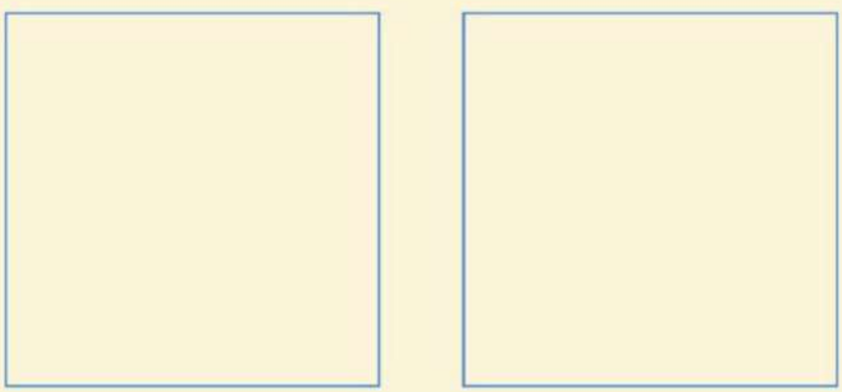
Can you state the relationship between the sides [AB], [BC] and [AC] of the triangle? If you cannot copy the shapes exactly, then use scissors to cut out copies of the red, blue and green squares, and see whether you can fit the green and blue squares exactly over the red one. You may cut up the blue and green squares along the internal lines to create smaller squares or rectangles.



2 Puzzle Two:

There are 11 puzzle pieces: 8 right-angled triangles and 3 squares of different sizes.
Cut out copies of the puzzle pieces and arrange them in the two frames.

Frames:



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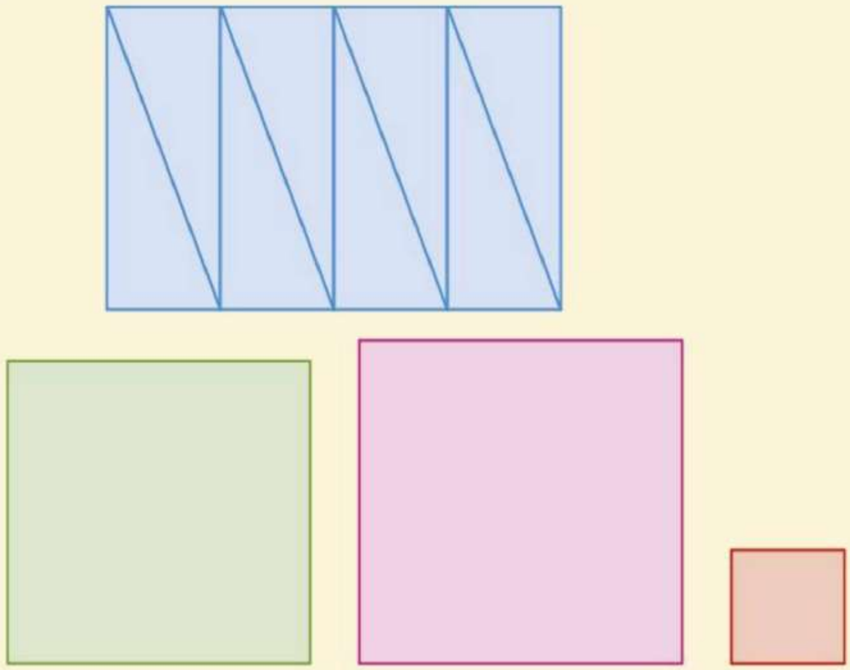
Pieces:



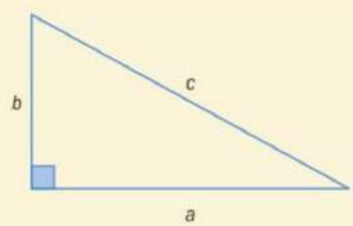
Is there more than one way to arrange the puzzle pieces into the two frames? What is the relationship between the areas of the frames?

What is the relationship between the areas of the three puzzle squares? What relationship between the sides of the right-angled triangles is revealed by the two puzzles?

3 **Conceptual** What is the relationship between the sides of any right-angled triangle?

The Pythagorean theorem

For every right-angled triangle $c^2 = a^2 + b^2$, where c is the hypotenuse, and a and b are the other two sides. The hypotenuse is the longest side and is opposite the right angle.



Did you know?

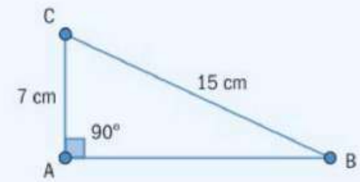
The Pythagorean theorem is named after the Greek mathematician Pythagoras, who lived in the 6th century BC and is credited with the first recorded proof of the theorem.

The theorem was known to some cultures well before Pythagoras; the ancient Egyptians knew about the 3-4-5 right-angled triangle (a triangle with sides 3, 4 and 5 units), and they knew how to measure a right angle by stretching a rope with equally spaced knots. The Egyptian surveyors (called "rope stretchers") used knotted cords to make measurements for building and restoring the boundaries of fields after the flooding of the River Nile.

If you know the lengths of two of the sides of a right-angled triangle you can always determine the length of the third side.

Example 12

Calculate the length of [AB] in the right-angled triangle ΔABC, where CB = 15 cm and AC = 7 cm.



Using the Pythagorean theorem:	
$AB^2 + 7^2 = 15^2$	
$AB = \sqrt{15^2 - 7^2}$	Rearrange to find AB.
$= 13.3$ cm (13.2666…)	Remember to give the final answer to 3 sf unless a different accuracy is specified in the question.

The converse of the Pythagorean theorem states: If for a triangle $c^2 = a^2 + b^2$, where a, b and c are its sides, then it must be a right-angled triangle.

The converse of the Pythagorean theorem is also useful, as if you know the three sides of a triangle you can determine whether the triangle has a right angle or not.

Example 13

Determine whether or not a triangle with sides 6 cm, 8 cm and 9 cm is a right-angled triangle.

Is $9^2 = 6^2 + 8^2$ true? Since $81 \neq 100$ we conclude that this triangle does not have a right angle.	To determine whether a triangle with sides 6 cm, 8 cm and 9 cm has a right angle, use the converse of the Pythagorean theorem. Check whether $9^2 = 6^2 + 8^2$.

Example 14

Sam's vegetable garden is a triangle. His plan shows that the plot is a right-angled triangle. Sam measures the sides of his vegetable garden as 17 m, 144 m and 145 m. Show that the plot is indeed a right-angled triangle.

Is $145^2 = 144^2 + 17^2$ true? Since $21\,025 = 20\,736 + 289$, Sam can conclude that his plot is a right-angled triangle.	To determine whether his plot is a right-angled triangle with sides 17 m, 144 m and 145 m, use the converse of the Pythagorean theorem. Check whether $145^2 = 144^2 + 17^2$.

Did you know?

The existence of irrational numbers is attributed to a Pythagorean, Hippasus (5th century BC). He proved that $\sqrt{2}$, the length of the hypotenuse of a right-angled triangle with sides 1 and 1, is an irrational number.

The **irrational numbers** are numbers that are not rational. And **rational numbers** are numbers that can be written as a quotient $\frac{p}{q}$ of two integers p and q, where $q \neq 0$. The legend claims that Hippasus was exiled for his discovery of the irrational numbers.

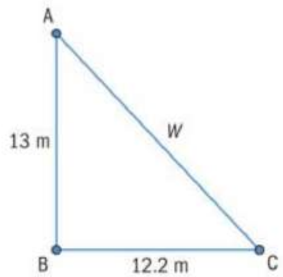
TOK

Is it ethical that Pythagoras gave his name to a theorem that may not have been his own creation?

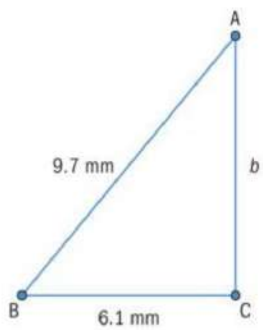
Exercise 1G

1 Find the length of the unknown side in each of the following right-angled triangles. Round your answers to 1 dp.

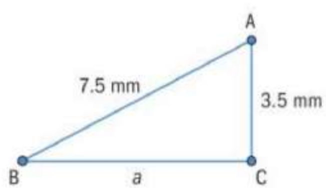
a



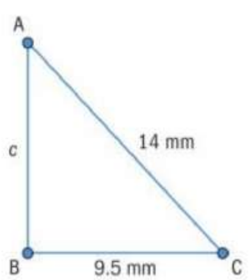
b



c



d

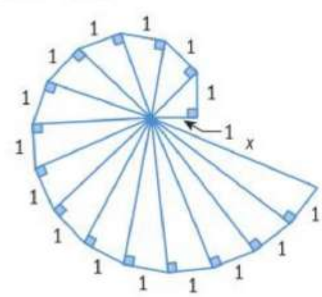


2 Determine whether or not the triangles with the following sides are right-angled triangles:

a 9 cm, 40 cm, 41 cm **b** 10 m, 24 m, 26 m

c 10, 10, $\sqrt{200}$ **d** 11.2, 7.5, 8.3

3 The spiral in the figure is made by starting with a right-angled triangle with both legs of length 1 unit.



a Find the hypotenuse of the first triangle.

Then, the second right-angled triangle is built with one leg measuring 1 unit and the other leg being the hypotenuse of the first triangle.

b Find the hypotenuse of this second triangle.

A third right-angled triangle is built on the second triangle's hypotenuse, again with the other leg measuring 1 unit.

c Find the hypotenuse of the third triangle.

The process is continued in the same fashion and the hypotenuse of the final triangle is denoted by x.

d Find the length of x.

4 A right-angled triangle has a hypotenuse of 8.2 cm and another side length of 4.3 cm. Draw a diagram. Calculate the area of the triangle. Give your answer correct to 1 dp.

Investigation 10

The right-angled triangles ΔABC and ΔADE shown in the diagram below are such that $[CB] \perp [AB]$, $[ED] \perp [AD]$ and $B\hat{A}C = 30°$. The lengths of the sides are $AC = 15.0$ cm, $CB = 7.5$ cm, $AB = 13.0$ cm, $AE = 10.0$ cm, $ED = 5.0$ cm and $AD = 8.7$ cm, given to 1 dp.

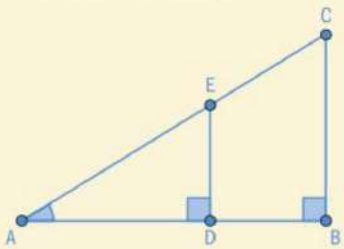


Diagram not to scale.

1 **Factual** For each side of ΔABC and ΔADE determine whether it is a hypotenuse, opposite to $B\hat{A}C$ or adjacent to $B\hat{A}C$. On your diagram, label each side of the two triangles with Opp, Adj or Hyp.

2 Place each measure of the sides of ΔABC and ΔADE in the appropriate cell in the table.

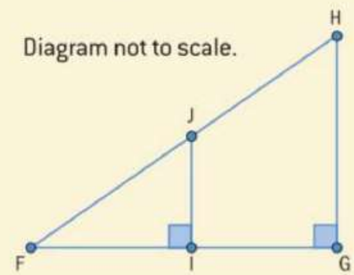
	Opp	Adj	Hyp	Opp/Hyp	Adj/Hyp	Opp/Adj
ΔABC						
ΔADE						

3 Calculate the ratios Opp/Hyp, Adj/Hyp and Opp/Adj for each triangle. Give your answers correct to 1 dp. Place each ratio in the appropriate cell in the table.

4 Compare the corresponding ratios for the two triangles. Write down your conclusions.

5 The right-angled triangles ΔFGH and ΔFIJ shown in the diagram on the right are such that $[HG] \perp [FG]$, $[JI] \perp [FI]$ and $G\hat{F}H = 60°$. The lengths of the sides are $FH = 15.0$ cm, $FG = 7.5$ cm, $FJ = 10.0$ cm and $FI = 5.0$ cm, given to 1 dp.

Diagram not to scale.



6 Find the lengths of $[HG]$ and $[JI]$.

7 **Factual** For each side of ΔFGH and ΔFIJ, determine whether it is a hypotenuse, opposite to $G\hat{F}H$ or adjacent to $G\hat{F}H$. On your diagram, label each side of the two triangles with Opp, Adj or Hyp.

8 Place each measure of the sides of ΔFGH and ΔFIJ in the appropriate cell of the table below.

9 Calculate the ratios Opp/Hyp, Adj/Hyp and Opp/Adj for each triangle. Give your answers correct to 1 dp. Place each ratio in the appropriate cell in the table.

	Opp	Adj	Hyp	Opp/Hyp	Adj/Hyp	Opp/Adj
ΔFGH						
ΔFIJ						

10 Compare the corresponding ratios for ΔFGH and ΔFIJ. Compare these ratios with the ratios that you calculated for ΔABC and ΔADE. Is the conclusion different from the one you arrived at for triangles ΔFGH and ΔFIJ? If so, how?

11 **Conceptual** What can you say about the ratios of corresponding pairs of sides in similar triangles? How does changing the acute angles in the triangle affect the ratios of the sides?

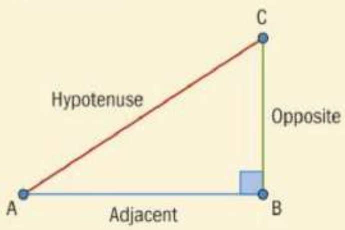
Trigonometric ratios

Trigonometry (from Greek *trigōnon*, "triangle", and *metron*, "to measure") studies relationships between sides and angles of triangles. Astronomers in the 3rd century noted for the first time what you saw in your investigation, that the ratios $\frac{\text{Adjacent}}{\text{Hypotenuse}}$, $\frac{\text{Opposite}}{\text{Hypotenuse}}$ and $\frac{\text{Opposite}}{\text{Adjacent}}$ are constant for all right-angled triangles with the same acute angles.

TOK

How certain is the shared knowledge of mathematics?

The ratios of the sides of a right-angled triangle are called **trigonometric ratios**. Three common trigonometric ratios are the **sine (sin), cosine (cos)** and **tangent (tan)**. These are defined for acute angle $\hat{A}$ in the right-angled triangle below:



$$\sin \hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos \hat{A} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan \hat{A} = \frac{\text{Opposite}}{\text{Adjacent}}$$

In the definitions above, "Opposite" refers to the length of the side opposite angle $\hat{A}$, "Adjacent" refers to the length of the side adjacent to angle $\hat{A}$, and "Hypotenuse" refers to the side opposite the right angle.

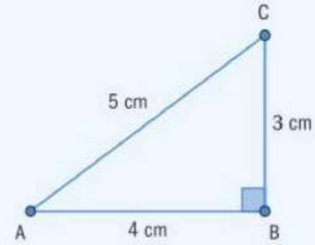
Some people use the mnemonic **SOH-CAH-TOA**, pronounced "soh-kuh-toh-uh", to help them remember the definitions of sine, cosine and tangent. The table below shows the origins of SOH-CAH-TOA.

Abbreviation	Verbal description	Definitions
SOH	Sine is **O**pposite over **H**ypotenuse	$\sin \hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$
CAH	Cosine is **A**djacent over **H**ypotenuse	$\cos \hat{A} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$
TOA	Tangent is **O**pposite over **A**djacent	$\tan \hat{A} = \frac{\text{Opposite}}{\text{Adjacent}}$

Reflect What are trigonometric ratios?

Example 15

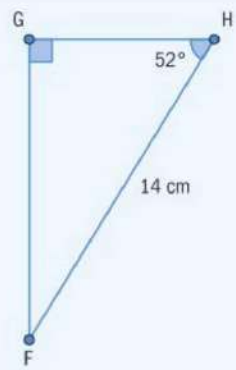
Find the size of $\hat{A}$ in triangle ΔABC, where angle $A\hat{B}C = 90°$.



$\sin\hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$ $\sin\hat{A} = \frac{3}{5}$ $\hat{A} = \sin^{-1}\left(\frac{3}{5}\right)$ $\hat{A} = 36.9°$, correct to 3 sf.	Use $\sin\hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$. Once we know the trigonometric ratio, we can calculate angle $\hat{A}$ by using the $\sin^{-1}$ function on the GDC and finding $\sin^{-1}\left(\frac{3}{5}\right)$. Remember to give your answer to 3 sf unless otherwise specified.

Example 16

Shown here is the right-angled triangle ΔFGH, where FH = 14 cm, $F\hat{H}G = 52°$ and angle $F\hat{G}H = 90°$. Find the length of side [GH].

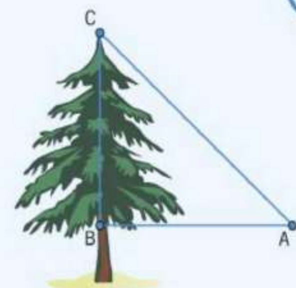


$\cos F\hat{H}G = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ $\cos 52° = \frac{GH}{14}$ $GH = 14\cos 52°$ $= 8.62$ cm (3 sf)	Since we want to find the length of side [GH] we use the cosine. We use $\cos F\hat{H}G = \frac{\text{Adjacent}}{\text{Hypotenuse}}$. We substitute the values for $F\hat{H}G$ and the hypotenuse in the cosine ratio and do the calculations.

If at least one side and one non-right angle are known in a right-angled triangle, then all other angles and side lengths can be determined.

Example 17

Emma is standing in front of a big tree. She measures her distance from the tree as AB = 15 m. She also measures $\hat{A} = 40^\circ$. Find the length BC. State what other information Emma needs in order to calculate the height of the tree.



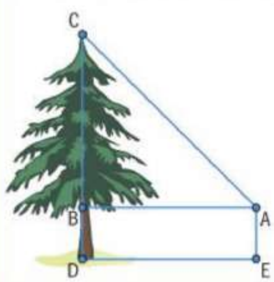
AB = 15 m and $\hat{A} = 40^\circ$.

$$\tan 40^\circ = \frac{CB}{15}$$

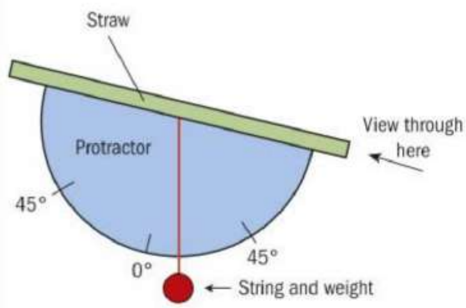
$$CB = 15 \tan 40^\circ$$

$$= 12.6\,\text{m (3 sf)}$$

Note that to obtain the height of the tree Emma will have to add to CB her height, which is represented by EA and DB.



$\hat{A}$ can be measured with an instrument called a clinometer, as shown in the image below.



Use $\tan \hat{A} = \dfrac{\text{Opposite}}{\text{Adjacent}}$.

Trigonometric ratios are very helpful in solving practical problems. Distances which are inaccessible can be indirectly measured through use of trigonometry.

Investigation 11

The distance between two landmarks A and B (trees or other) cannot be measured directly as there is a building standing between them. How could you find the distance AB?

Group work:

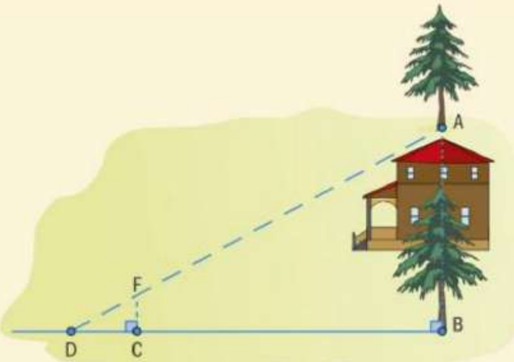
1 Find two landmarks A and B (trees or other) such that you cannot measure the distance between A and B directly. For example, there may be a building or a tall wall between them and only the top of A is visible from B.

2 Mark a line from B perpendicular to [AB] using a piece of string.

3 Mark two points C and D on this line. Stand at point D, with your partner standing at point C. Your partner now walks parallel to [AB] (or perpendicular to [BD]) until you decide that your partner is directly between yourself and landmark A. Mark this point F.

4 Measure the lengths of [DB], [DC] and [FC].

5 Calculate the length of [AB] from $\frac{DB}{DC} = \frac{AB}{FC}$.



Choose two different points for C and D, and repeat steps **3** to **5**, this time swapping roles with your partner.

Did you get the same result for the distance between A and B when you chose different points for C and D? Think of some sources of error in this experiment that may explain any differences you obtained. How could you check whether the person at C indeed walked perpendicularly to [DB] in step **3**?

6 Now, measure $A\hat{D}B$ and use trigonometric ratios to find the distance AB. Can you find AB by using different trigonometric ratios? If so, calculate AB twice. Did you get the same result for AB?

What might be some sources of error in finding AB using trigonometric ratios?

7 Compare the values for AB found in parts **5** and **6**. Which of the two methods do you think produces a more accurate result for the distance AB? Why?

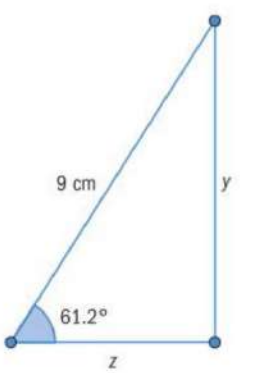
Could you think of other methods to find the distance AB indirectly?

8 What kind of real-life questions can be answered by using trigonometric ratios?

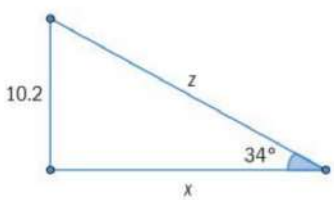
Exercise 1H

1 Determine the length of the unknown sides for each of the right-angled triangles below:

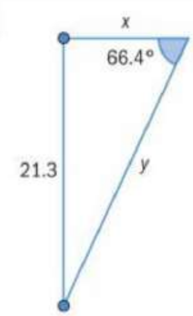
a



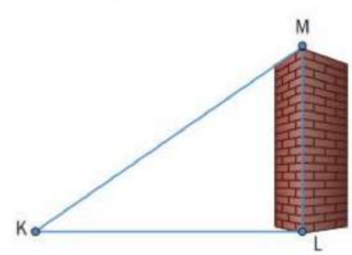
b



c



2 A ladder [KM] is 8.5 m long. It leans against a wall so that $L\hat{K}M = 30^\circ$ and ΔKLM is a right-angled triangle.



a Find the distance KL.

b Find how far up the wall the ladder reaches.

c The instructions for use of the ladder state that it should not lean against a wall at more than 55°. Find the maximum height up the wall that the ladder can reach.

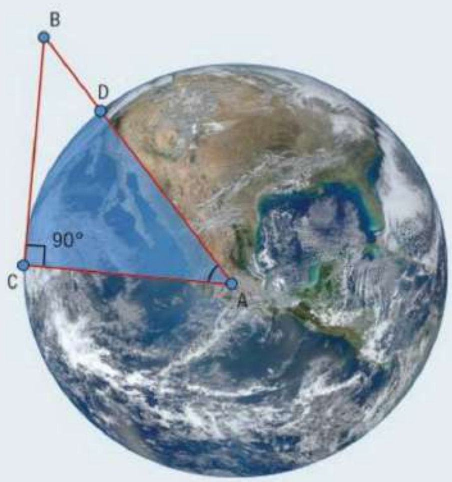
3 A hiker, whose eye is 1.6 metres above ground, stands 50 metres from the base of a vertical cliff. The angle between the line connecting her eye and the top of the cliff and a horizontal line is 58°.

a Draw a diagram representing the situation.

b Find the height of the cliff.

Developing inquiry skills

Imagine that you are standing at point B, which is the length of [BD] above the ground.



Which line segment in ΔABC do you think represents your line of sight to the horizon?

Why do you think ΔABC is a right-angled triangle?

Assume that the radius of the Earth is 6370 km.

How can you use what you have learned in this section to calculate the distance to the farthest object that you can see?

a If $BD = 10$ m, what would be the distance to the farthest object that you can see?

b If $BD = 382$ m, what would be the distance to the farthest object that you can see?

c Write an expression that represents the distance to the farthest object you can see, if $BD = h$ km.

What other assumptions are you making?

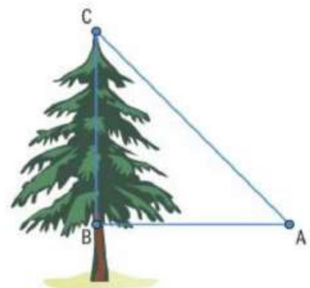
International-mindedness

The word "sine" started out as a totally different word and passed through Indian, Arabic and Latin before becoming the word that we use today.

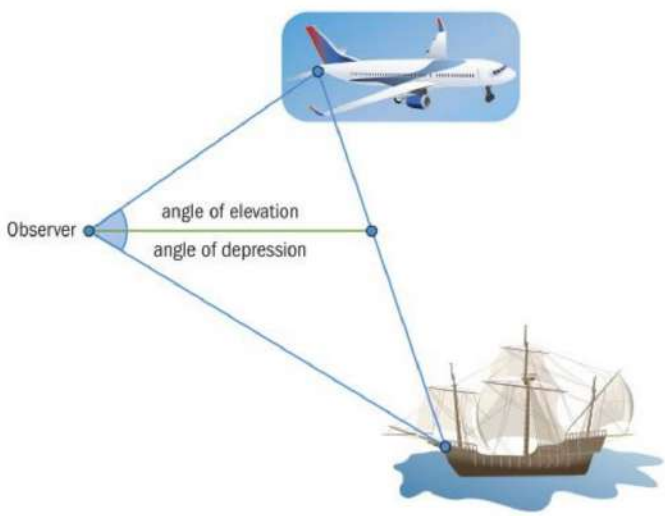
1.6 Angles of elevation and depression

Since ancient times, sailors and surveyors have measured angles to determine measurements that are inaccessible directly, such as the heights of trees, clouds or mountains, and distances to coasts or buildings.

In the problem about measuring the height of a tree from the previous section, angle BÂC is the angle between the line of sight of the observer to the top of the tree and a horizontal line [AB] from their eye level. This angle is called the **angle of elevation**.



Similarly, when the object is below the horizontal line at eye level, an **angle of depression** is formed, as shown on the diagram below.



The **angle of elevation** is the angle between the horizontal and the observer's line of sight to the object when the observer is looking upwards.

The **angle of depression** is the angle between the horizontal and the observer's line of sight to the object when the observer is looking downwards.

Reflect Why are angles of elevation and depression useful?

Investigation 12

In this investigation you will take measurements of accessible distances and angles, and then use them to calculate inaccessible heights. You will measure angles with a device called a clinometer and then perform the calculations. But first you have to make the clinometer yourself.

1 Making a clinometer

A clinometer is an instrument for measuring angles above or below a horizontal line (angles of elevation or depression). A metal tube allows the user to sight the top (or the base) of an object, and a protractor-like device on the side measures the angle.

To construct a clinometer you will need:

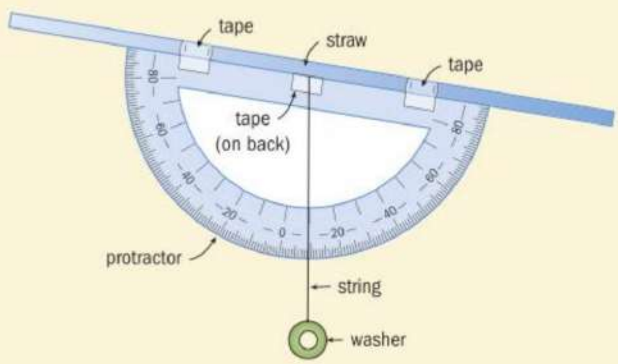
- drinking straw (instead of a metal tube)
- protractor
- string
- washer
- sticky tape
- tape measure.

2 Clinometer assembly

Attach the drinking straw to the top straight edge of the protractor with tape.

Attach the washer to one end of the string, and attach the string to the 0 marking on the top edge of the protractor and let it hang at least 5 cm below the curved edge of the protractor so that it can swing freely.

You are done! Now try it out.



3 Using your clinometer

Start by holding the clinometer so that the straw is parallel to the ground.

Slowly tilt the straw up until you can see the very top of the object that you are measuring through the straw.

Have your partner record the angle measure that the string passes through. Your measure should be between 0° and 90°.

- **a** Choose a tree or other tall object nearby whose height you are going to measure.
- **b** Move away from the object until you can see the top of the object through the straw of your clinometer.
- **c** One person should look through the clinometer at the top of the object you are measuring. The other person reads the angle that the string defines. Record this measurement. Remember that the number will be between 0° and 90°.

d Next, measure the distance to the object from the point on the ground at which the angle was measured. Record this measurement.

Measurement	Measure of angle of elevation	Measure of distance to the object	Calculated height of the object	Eye-level height	Total height of the object
1					
2					
3					

e Measure the height from the ground to the eye level of the person measuring the angle.

f Draw a clear and labelled diagram and write on it the information you have collected.

g Use the measurements you obtained to do the relevant calculations to find the height.

h After you have found out how high the object is from eye level, add the person's eye-level height to that number. This should give you the total height of the object.

i You can do a few sets of measurements and compare your results. What factors do you think could have affected the accuracy of your measurements and final result?

4 Suppose you are standing on the third floor of a building and want to find the distance to a building across the street. What information would you need and what measurements would you need to take in order to calculate this distance? Suppose you do have the necessary information and have obtained the measurements. Draw a diagram and explain how you would find the distance. How would you calculate the angle of depression if you know the side measurements?

5 **Factual** How are angles of elevation/depression calculated?

6 **Conceptual** Why are angles of elevation/depression useful?

Example 18

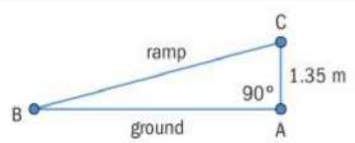
A ramp, [BC], is to be constructed to give wheelchair access to the door of a house 1.35 m above the ground. Safety regulations require that the angle of the ramp be less than 8°.

a Draw a diagram and label the vertices and the sides.

b Find the length of the ramp [BC], where $A\hat{B}C = 8°$.

c Find the distance AB, giving your answer to 1 dp.

a



b $\sin \hat{B} = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$

Use $\sin \hat{B} = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$ and solve for BC.

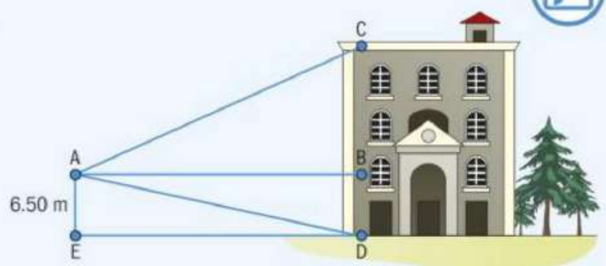
$\sin 8° = \dfrac{1.35}{BC}$

Continued on next page

$BC = \frac{1.35}{\sin 8°}$ $BC = 9.70$ m (3 sf)	
c $\tan \hat{B} = \frac{\text{Opposite}}{\text{Adjacent}}$ $\tan 8° = \frac{1.35}{AB}$ $AB = \frac{1.35}{\tan 8°}$ so $AB = 9.61$ m (3 sf)	Use $\tan \hat{B} = \frac{\text{Opposite}}{\text{Adjacent}}$ and solve for AB.

Example 19

From his studio window at point A, which is 6.50 m above the ground, Tomchik views the top and the base of a building on the opposite side of the street. The angles of elevation and depression of the top and the base of the building are 55° and 18° respectively. Find the height of the building he is viewing, giving your answer correct to 2 dp.



From ΔABD, $\tan B\hat{A}D = \frac{\text{Opposite}}{\text{Adjacent}}$ $\tan 18° = \frac{6.50}{AB}$ $AB = 20.00$ m (2 dp)	For ΔABD, we use $\tan B\hat{A}D = \frac{\text{Opposite}}{\text{Adjacent}}$, as we know that BD = AE = 6.50 m and $B\hat{A}D = 18°$.
From ΔABC, $\tan B\hat{A}C = \frac{\text{Opposite}}{\text{Adjacent}}$, $\tan 55° = \frac{CB}{20}$ $CB = 20 \tan 55°$ $CB = 28.56$ m (2 dp)	For ΔABC, we use also $\tan B\hat{A}D = \frac{\text{Opposite}}{\text{Adjacent}}$, as we now know that AB = 20 m and $B\hat{A}C = 55°$.
Then the height of the building is $28.56 + 6.50 = 35.06$ m	To find the height of the building you need to add the lengths of sides [CB] and [DB].

Exercise 1I

1 The angle of depression from the top of a cliff to a boat at sea is 17°. The boat is 450 m from the shore.

a Draw a diagram representing the situation.

b Find the height of the cliff, giving your answer rounded to the nearest metre.

2 Your family wants to buy an awning for a French window that will be long enough to keep out the sun when it is at its highest point in the sky. The height of the French window is 2.80 m. The angle of elevation of the sun from this point is 70°. Find how long the awning should be. Write down your answer correct to 2 dp.

3 Scientists measure the depths of lunar craters by measuring the length of the shadow cast by the edge of the crater using photos. In a photograph, the length of the shadow cast by the edge of the Moltke crater is about 606 metres, given to the nearest metre. The sun's angle of elevation is 65°. Find the depth of the crater, giving your answer rounded to the nearest metre.

4 The height of a building is 72 m. Find the angle of elevation from a point on ground level that is 55 m away from the base of the building.

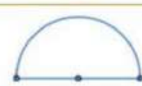
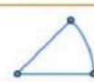
5 From a boat 160 m out at sea, the angle of elevation to the coast is 18°. The angle of elevation to the top of a lighthouse on the coast is 22°.

a Draw a diagram representing the situation.

b Find the height of the lighthouse.

Investigation 13

1 Complete the following table.

Angle	Section of a circle with this angle	Length of the corresponding arc
360°		$2\pi r$
270°		
180°		
90°		
45°		
1°		
$\alpha°$		

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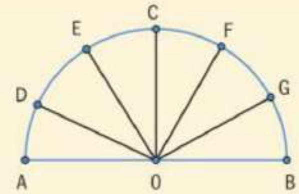
2 A window, which is a semicircle of radius 12 cm, is divided into six congruent sections, as shown in the diagram.

Find the length of the arc of the entire semicircular window.



3 Find the length of the arc of one of the six sections of the window.

4 If the window is divided into sections with a central angle of 1°, find the length of the arc of such a window section.

5 Find the length of the arc of a window section if its central angle were $\alpha°$.

6 Generate a formula for finding the length of the arc of a circle with central angle $\alpha°$ and radius r, based on your conclusions above.

7 **Conceptual** How are the arc length and circumference of a circle related?

Length of arc formula

The length of the arc of a sector of a circle with radius r and central angle $\alpha°$ is $\alpha \times \frac{2\pi r}{360}$ or $\alpha \times \frac{\pi r}{180}$.

TOK

What does it mean to say that mathematics is an axiomatic system?

Example 20

Find the length of the arc of a circle with radius 2 cm and central angle 120°.

The length of the arc can be found in two ways. **a** $\frac{1}{3} \times 2\pi(2)$ $= 4.19\text{ cm (3 sf)}$	One way would be to use the fact that 120° is $\frac{1}{3}$ of 360°, which means that the length of the arc will be $\frac{1}{3}$ of the circumference of a circle with radius 2 cm, thus $\frac{1}{3} \times 2\pi(2)$ or 4.19 cm (3 sf).
b $120 \times \frac{\pi r}{180} = 120 \times \frac{2\pi}{180}$ $= 4.19\text{ cm (3 sf)}$	Another way would be to directly use the formula: $120 \times \frac{\pi r}{180}$ or $120 \times \frac{2\pi}{180}$, which again results in 4.19 cm (3 sf).

Developing your toolkit

Now do the Modelling and investigation activity on page 46.

Exercise 1J

1 Determine the length of each arc with radius r and central angle α given below.

Give your answer correct to 2 dp.

a $r = 5\,\text{cm}, \alpha = 70°$

b $r = 4\,\text{cm}, \alpha = 45°$

c $r = 10.5\,\text{cm}, \alpha = 130°$

2 A clock is circular in shape with diameter 25 cm. Find the length of the arc between the markings 12 and 5, rounded to the nearest tenth of a centimetre.

3 The London Eye is a giant Ferris wheel in London. It is the tallest Ferris wheel in Europe, with a diameter of 120 m. The passenger capsules are attached to the circumference of the wheel, and the wheel rotates at 26 cm per second.

Find:

a the length that a passenger capsule would travel if the wheel makes a rotation of 200°

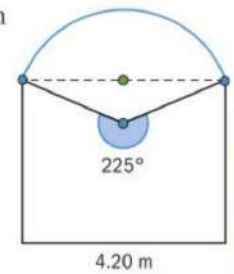
b the time, in minutes, that it would take for a passenger capsule to make a rotation of 200°

c the time, to the nearest whole minute, that it would take for a passenger capsule to make a full revolution.

4 A door with width 4.20 m has an arc as shown in the diagram. Find:

a the radius of the arc, to the nearest cm

b the length of the arc, to the nearest cm.



Chapter summary

Measurement

- Measurements help us to compare objects and understand how they relate to each other.
- Measuring requires approximating. If a smaller measuring unit is chosen then a more accurate measurement can be obtained.
- When you measure, you first select a property of the object that you will measure. Then you choose an appropriate unit of measurement for that property. And finally, you determine the number of units.

Estimation

- Estimation (or estimating) is finding an approximation as close as possible to the value of a measurement by sensible guessing. Often the estimate is used to check whether a calculation makes sense or to avoid complicated calculations.
- Estimation is often done by comparing the attribute that we measure to another one or by sampling.

Accuracy

- A value is **accurate** if it is close to the **exact value** of the quantity being measured. However, in most cases it is not possible to obtain the exact value of a measurement. For example, when measuring your weight, you can get a more accurate measurement if you have a balance scale that measures to a greater number of decimal places.

Continued on next page

- The left and the right ends of an interval in which an exact value of a measurement lies are respectively called the **lower bound** and the **upper bound**.
- The lower bound and the upper bound are half a unit below and above a rounded value of a measurement. Thus the upper bound is calculated as the rounded measurement + 0.5 unit and the lower bound is the rounded measurement − 0.5 unit.
- Since measurements are approximate there is always error in the measurement results. A measurement error is the difference between the exact value and the approximate value, ie

$$\text{Measurement error} = V_A - V_E$$

- **Percentage error** $= \left|\frac{V_A - V_E}{V_E}\right| \times 100\%$, where V_A is the approximate value and V_E is the exact value.
- Rounding of intermediate results during multistep calculations reduces the accuracy of the final answers. Thus, make sure to round your final answer only.

Speaking scientifically

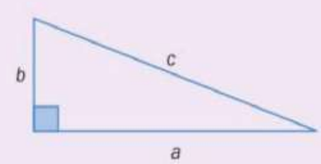
- **Numerical expressions** consist only of numbers and symbols of operations (addition, subtraction, multiplication, division and exponentiation), whereas **algebraic expressions** contain numbers, variables and symbols of operations.
- If the exponent, n, is a **positive integer**, it determines **how many times** the base, a, is multiplied by itself. For example, 8^2 means 8×8.
- If the exponent, n, is a **negative integer** it determines how many times **to divide** 1 by the base, a. For example, 8^{-2} means $\frac{1}{8^2}$ or $\frac{1}{8 \times 8}$.

Standard form

- In standard form (also known as scientific notation), numbers are written in the form $a \times 10^n$, where a is called the **coefficient** or **mantissa**, with $1 \le a < 10$, and n is an integer.
- You may have noticed that your graphic display calculator gives any results with lots of digits in scientific notation. However, instead of writing the results in the form $a \times 10^n$, it gives them in the form $a\text{E}n$, where n is an integer. For example, 3×10^5 will be written as 3E5, and 3.1×10^{-3} as 3.1E−3.
- To convert from decimal to scientific notation on your GDC, change the Mode from Normal to Scientific. Thus, all of your number entries will be immediately converted to scientific calculator notation. To convert to correct scientific notation, you will need to replace E with 10, eg 5E2 should be written as 5×10^2.

Trigonometry of right-angled triangles

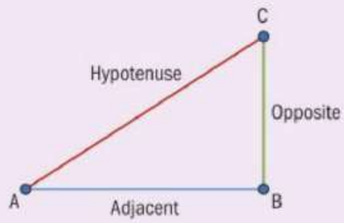
- **Pythagorean theorem**: For every right-angled triangle $c^2 = a^2 + b^2$, where c is the hypotenuse, and a and b are the other two sides. The hypotenuse is the longest side and is opposite the right angle.



- If you know two of the sides of a right-angled triangle you can always determine the third side.
- **The converse of the Pythagorean theorem**: If for a triangle $c^2 = a^2 + b^2$, where a, b and c are its sides, then it must be a right-angled triangle.
- The converse of the Pythagorean theorem is useful, as if you know the three sides of a triangle you can determine whether the triangle has a right angle or not.

Trigonometric ratios

- The ratios of the sides of right-angled triangles are called trigonometric ratios. Three common trigonometric ratios are the sine $(\sin)$, cosine $(\cos)$ and tangent $(\tan)$. These are defined for acute angle $\hat{A}$ in the right-angled triangle below:



$$\sin \hat{A} = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos \hat{A} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan \hat{A} = \frac{\text{Opposite}}{\text{Adjacent}}$$

- If at least one side and one non-right angle are known in a right-angled triangle, then all other angles and side lengths can be determined.
- Trigonometric ratios are very helpful in solving practical problems. Distances that are inaccessible can be indirectly measured through use of trigonometry.

Angles of elevation and depression

- The **angle of elevation** is the angle between the horizontal and the observer's line of sight to the object when the observer is looking upwards.
- The **angle of depression** is the angle between the horizontal and the observer's line of sight to the object when the observer is looking downwards.
- The **length of the arc** of a sector of a circle with radius r and with central angle $\alpha°$ is $\alpha \times \frac{2\pi r}{360}$.

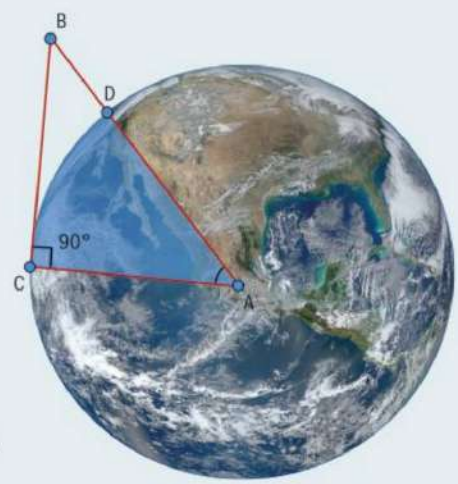
Developing inquiry skills

Does the measure of $\text{B}\hat{\text{A}}\text{C}$ depend on the length of $[\text{BD}]$ or how high the observer stands above the ground?

How can you use $\text{A}\hat{\text{B}}\text{C}$ to find the measure of $\text{B}\hat{\text{A}}\text{C}$?

Which is the arc of visibility for an observer at point B? Does the length of the arc of visibility depend on $\text{B}\hat{\text{A}}\text{C}$?

Assume that the radius of the Earth is 6370 km.



- **a** Calculate the measure of $\text{B}\hat{\text{A}}\text{C}$ if $\text{BD} = 10$ m.
- **b** Calculate the measure of $\text{B}\hat{\text{A}}\text{C}$ if $\text{BD} = 382$ m.
- **c** Express the measure of $\text{B}\hat{\text{A}}\text{C}$ if $\text{BD} = h$ km.
- **d** Calculate the length of the arc of visibility for $\text{BD} = 10$ m.
- **e** Calculate the length of the arc of visibility for $\text{BD} = 382$ m.
- **f** Calculate the length of the arc of visibility for $\text{BD} = h$ km.

What other assumptions are you making?