Logarithmic and Exponential Equations

11.5 OBJECTIVES

1. Solve a logarithmic equation
2. Solve an exponential equation
3. Solve an application involving an exponential equation

Much of the importance of the properties of logarithms developed in the previous section lies in the application of those properties to the solution of equations involving logarithms and exponentials. Our work in this section will consider solution techniques for both types of equations. Let's start with a definition.

> **Definitions:** Logarithmic Equation
>
> A **logarithmic equation** is an equation that contains a logarithmic expression.

We solved some simple examples in Section 11.3. Let's review for a moment. To solve $\log_3 x = 4$ for x, recall that we simply convert the logarithmic equation to exponential form. Here,

$$x = 3^4$$

so

$$x = 81$$

and 81 is the solution to the given equation.

Now, what if the logarithmic equation involves more than one logarithmic term? Example 1 illustrates how the properties of logarithms must then be applied.

Example 1

Solving a Logarithmic Equation

Solve each logarithmic equation.

(a) $\log_5 x + \log_5 3 = 2$

The original equation can be written as

NOTE We apply the product rule for logarithms:

$\log_b M + \log_b N = \log_b MN$

$$\log_5 3x = 2$$

Now, because only a single logarithm is involved, we can write the equation in the equivalent exponential form:

$$3x = 5^2$$

$$3x = 25$$

$$x = \frac{25}{3}$$

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(b) $\log x + \log (x - 3) = 1$

Write the equation as

$\log x(x - 3) = 1$

or

$x(x - 3) = 10^1$

We now have

$$x^2 - 3x = 10$$

$$x^2 - 3x - 10 = 0$$

$$(x - 5)(x + 2) = 0$$

Possible solutions are $x = 5$ or $x = -2$.
    Note that substitution of -2 into the original equation gives

$\log (-2) + \log (-5) = 1$

Because logarithms of negative numbers are *not* defined, -2 is an extraneous solution and we must reject it. The only solution for the original equation is 5.

CHECK YOURSELF 1

Solve $\log_2 x + \log_2 (x + 2) = 3$ *for x.*

The quotient property is used in a similar fashion for solving logarithmic equations. Consider Example 2.

Example 2

Solving a Logarithmic Equation

Solve each equation for x.

(a) $\log_5 x - \log_5 2 = 2$

Rewrite the original equation as

$\log_5 \dfrac{x}{2} = 2$

Now,

$$\frac{x}{2} = 5^2$$

$$\frac{x}{2} = 25$$

$$x = 50$$

(b) $\log_3 (x + 1) - \log_3 x = 3$

$$\log_3 \frac{x + 1}{x} = 3$$

$$\frac{x + 1}{x} = 27$$

NOTE Again, you should verify that substituting $\frac{1}{26}$ for x leads to a positive value in each of the original logarithms.

$$x + 1 = 27x$$

$$1 = 26x$$

$$x = \frac{1}{26}$$

CHECK YOURSELF 2

Solve $\log_5 (x + 3) - \log_5 x = 2$ for x.

The solution of certain types of logarithmic equations calls for the one-to-one property of the logarithmic function.

Rules and Properties: One-To-One Property of Logarithmic Functions

If $\log_b M = \log_b N$

then $M = N$

Example 3

Solving a Logarithmic Equation

Solve the following equation for x.

$$\log (x + 2) - \log 2 = \log x$$

Again, we rewrite the left-hand side of the equation. So

$$\log \frac{x + 2}{2} = \log x$$

Because the logarithmic function is one-to-one, this is equivalent to

$$\frac{x + 2}{2} = x$$

or

$$x = 2$$

CHECK YOURSELF 3

Solve for x.

$$\log (x + 3) - \log 3 = \log x$$

The following algorithm summarizes our work in solving logarithmic equations.

Step by Step: Solving Logarithmic Equations

Step 1 Use the properties of logarithms to combine terms containing logarithmic expressions into a single term.

Step 2 Write the equation formed in step 1 in exponential form.

Step 3 Solve for the indicated variable.

Step 4 Check your solutions to make sure that possible solutions do not result in the logarithms of negative numbers or zero.

Let's look now at **exponential equations,** which are equations in which the variable appears as an exponent.

We solved some particular exponential equations in Section 11.2. In solving an equation such as

$$3^x = 81$$

we wrote the right-hand member as a power of 3, so that

NOTE Again, we want to write both sides as a power of the same base, here 3.

$$3^x = 3^4$$

or

$$x = 4$$

The technique here will work only when both sides of the equation can be conveniently expressed as powers of the same base. If that is not the case, we must use logarithms for the solution of the equation, as illustrated in Example 4.

Example 4

Solving an Exponential Equation

Solve $3^x = 5$ for x.

We begin by taking the common logarithm of both sides of the original equation.

NOTE Again:

if $M = N$, then

$\log_b M = \log_b N$

$$\log 3^x = \log 5$$

Now we apply the power property so that the variable becomes a coefficient on the left.

$$x \log 3 = \log 5$$

Dividing both sides of the equation by $\log 3$ will isolate x, and we have

CAUTION

This is *not* log 5 − log 3, a common error.

$$x = \frac{\log 5}{\log 3}$$

$$= 1.465 \qquad \text{(to three decimal places)}$$

Note: You can verify the approximate solution by using the $\boxed{y^x}$ key on your calculator. Raise 3 to power 1.465.

CHECK YOURSELF 4

Solve $2^x = 10$ for x.

Example 5 shows how to solve an equation with a more complicated exponent.

Example 5

Solving an Exponential Equation

Solve $5^{2x+1} = 8$ for x.

The solution begins as in Example 4.

$$\log 5^{2x+1} = \log 8$$

NOTE On the left, we apply $\log_b M^p = p \log_b M$

$$(2x + 1) \log 5 = \log 8$$

$$2x + 1 = \frac{\log 8}{\log 5}$$

NOTE On a graphing calculator, the sequence would be

$$2x = \frac{\log 8}{\log 5} - 1$$

$\boxed{(}$ $\boxed{\log}$ 8 $\div$ $\boxed{\log}$ 5 $-$ 1 $\boxed{)}$ $\boxed{\div}$ 2

$$x = \frac{1}{2}\left(\frac{\log 8}{\log 5} - 1\right)$$

$$x \approx 0.146$$

✓ CHECK YOURSELF 5

Solve $3^{2x-1} = 7$ for x.

The procedure is similar if the variable appears as an exponent in more than one term of the equation.

Example 6

Solving an Exponential Equation

Solve $3^x = 2^{x+1}$ for x.

$$\log 3^x = \log 2^{x+1}$$

NOTE Use the power property to write the variables as coefficients.

$$x \log 3 = (x + 1) \log 2$$

$$x \log 3 = x \log 2 + \log 2$$

NOTE We now isolate x on the left.

$$x \log 3 - x \log 2 = \log 2$$

$$x(\log 3 - \log 2) = \log 2$$

NOTE To check the reasonableness of this result, use your calculator to verify that

$$x = \frac{\log 2}{\log 3 - \log 2}$$

$3^{1.710} = 2^{2.710}$

$$\approx 1.710$$

✓ CHECK YOURSELF 6

Solve $5^{x+1} = 3^{x+2}$ for x.

The following algorithm summarizes our work with solving exponential equations.

Step by Step: Solving Exponential Equations

Step 1 Try to write each side of the equation as a power of the same base. Then equate the exponents to form an equation.

Step 2 If the above procedure is not applicable, take the common logarithm of both sides of the original equation.

Step 3 Use the power rule for logarithms to write an equivalent equation with the variables as coefficients.

Step 4 Solve the resulting equation.

There are many applications of our work with exponential equations. Consider the following.

Example 7

An Interest Application

If an investment of P dollars earns interest at an annual interest rate r and the interest is compounded n times per year, then the amount in the account after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \tag{1}$$

If $1000 is placed in an account with an annual interest rate of 6%, find out how long it will take the money to double when interest is compounded annually and quarterly.

(a) Compounding interest annually.

NOTE Because the interest is compounded *once* a year, $n = 1$.

Using equation (1) with $A = 2000$ (we want the original 1000 to double). $P = 1000$, $r = 0.06$, and $n = 1$, we have

$$2000 = 1000(1 + 0.06)^t$$

Dividing both sides by 1000 yields

$$2 = (1.06)^t$$

We now have an exponential equation that can be solved by our earlier techniques.

$$\log 2 = \log (1.06)^t$$

$$= t \log 1.06$$

NOTE From accounting, we have the **rule of 72**, which states that the doubling time is approximately 72 divided by the interest rate as a percentage. Here $\frac{72}{6} = 12$ years.

or

$$t = \frac{\log 2}{\log 1.06}$$

$$\approx 11.9 \text{ years}$$

It takes just a little less than 12 years for the money to double.

(b) Compounding interest quarterly.

Now $n = 4$ in equation (1), so

$$2000 = 1000\left(1 + \frac{0.06}{4}\right)^{4t}$$

$$2 = (1.015)^{4t}$$

$$\log 2 = \log (1.015)^{4t}$$

$$\log 2 = 4t \log 1.015$$

$$\frac{\log 2}{4 \log 1.015} = t$$

$$t \approx 11.6 \text{ years}$$

Note that the doubling time is reduced by approximately 3 months by the more frequent compounding.

CHECK YOURSELF 7

Find the doubling time in Example 7 if the interest is compounded monthly.

Problems involving rates of growth or decay can also be solved by using exponential equations.

Example 8

A Population Application

A town's population is presently 10,000. Given a projected growth rate of 7% per year, t years from now the population P will be given by

$$P = 10,000e^{0.07t}$$

In how many years will the town's population double?
    We want the time t when P will be 20,000 (doubled in size). So

$$20,000 = 10,000e^{0.07t}$$

or

NOTE Divide both sides by 10,000.

$$2 = e^{0.07t}$$

In this case, we take the *natural logarithm* of both sides of the equation. This is because e is involved in the equation.

$$\ln 2 = \ln e^{0.07t}$$

NOTE Apply the power property.

$\ln e = 1$

$$\ln 2 = 0.07t \ln e$$

$$\ln 2 = 0.07t$$

$$\frac{\ln 2}{0.07} = t$$

$$t \approx 9.9 \text{ years}$$

The population will double in approximately 9.9 years.

CHECK YOURSELF 8

If $1000 is invested in an account with an annual interest rate of 6%, compounded continuously, the amount A in the account after t years is given by

$$A = 1000e^{0.06t}$$

Find the time t that it will take for the amount to double (A = 2000). Compare this time with the result of the Check Yourself 7 exercise. Which is shorter? Why?

CHECK YOURSELF ANSWERS

1. $\{2\}$ **2.** $\left\{\dfrac{1}{8}\right\}$ **3.** $\left\{\dfrac{3}{2}\right\}$ **4.** $\{3.322\}$ **5.** $\{1.386\}$ **6.** $\{1.151\}$ **7.** 11.58 years

8. 11.55 years; the doubling time is shorter, because interest is compounded more frequently

11.5 Exercises

In exercises 1 to 20, solve each logarithmic equation for x.

1. $\log_4 x = 3$

2. $\log_3 x = -2$

3. $\log (x + 1) = 2$

4. $\log_5 (2x - 1) = 2$

5. $\log_2 x + \log_2 8 = 6$

6. $\log 5 + \log x = 2$

7. $\log_3 x - \log_3 6 = 3$

8. $\log_4 x - \log_4 8 = 3$

9. $\log_2 x + \log_2 (x + 2) = 3$

10. $\log_3 x + \log_3 (2x + 3) = 2$

11. $\log_7 (x + 1) + \log_7 (x - 5) = 1$

12. $\log_2 (x + 2) + \log_2 (x - 5) = 3$

13. $\log x - \log (x - 2) = 1$

14. $\log_5 (x + 5) - \log_5 x = 2$

15. $\log_3 (x + 1) - \log_3 (x - 2) = 2$

16. $\log (x + 2) - \log (2x - 1) = 1$

17. $\log (x + 5) - \log (x - 2) = \log 5$

18. $\log_3 (x + 12) - \log_3 (x - 3) = \log_3 6$

19. $\log_2 (x^2 - 1) - \log_2 (x - 2) = 3$

20. $\log (x^2 + 1) - \log (x - 2) = 1$

In exercises 21 to 38, solve each exponential equation for x. Give your solutions in decimal form, correct to three decimal places.

21. $5^x = 625$

22. $4^x = 64$

23. $2^{x+1} = \dfrac{1}{8}$

24. $9^x = 3$

25. $8^x = 2$

26. $3^{2x-1} = 27$

27. $3^x = 7$

28. $5^x = 30$

29. $4^{x+1} = 12$

30. $3^{2x} = 5$

31. $7^{3x} = 50$

32. $6^{x-3} = 21$

33. $5^{3x-1} = 15$

34. $8^{2x+1} = 20$

35. $4^x = 3^{x+1}$

36. $5^x = 2^{x+2}$

37. $2^{x+1} = 3^{x-1}$

38. $3^{2x+1} = 5^{x+1}$

ANSWERS

39. _____

40. _____

41. _____

42. _____

43. _____

44. _____

45. _____

46. _____

47. _____

48. _____

49. _____

Use the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

to solve exercises 39 to 42.

39. **Interest.** If $5000 is placed in an account with an annual interest rate of 9%, how long will it take the amount to double if the interest is compounded annually?

40. Repeat exercise 39 if the interest is compounded semiannually.

41. Repeat exercise 39 if the interest is compounded quarterly.

42. Repeat exercise 39 if the interest is compounded monthly.

Suppose the number of bacteria present in a culture after t hours is given by $N(t) = N_0 \cdot 2^{t/2}$, in which N_0 is the initial number of bacteria. Use the formula to solve exercises 43 to 46.

43. How long will it take the bacteria to increase from 12,000 to 20,000?

44. How long will it take the bacteria to increase from 12,000 to 50,000?

45. How long will it take the bacteria to triple? *Hint:* Let $N(t) = 3N_0$.

46. How long will it take the culture to increase to 5 times its original size? *Hint:* Let $N(t) = 5N_0$.

The radioactive element strontium 90 has a half-life of approximately 28 years. That is, in a 28-year period, one-half of the initial amount will have decayed into another substance. If A_0 is the initial amount of the element, then the amount A remaining after t years is given by

$$A(t) = A_0\left(\frac{1}{2}\right)^{t/28}$$

Use the formula to solve exercises 47 to 50.

47. If the initial amount of the element is 100 g, in how many years will 60 g remain?

48. If the initial amount of the element is 100 g, in how many years will 20 g remain?

49. In how many years will 75% of the original amount remain? *Hint:* Let $A(t) = 0.75A_0$.

50. In how many years will 10% of the original amount remain? *Hint:* Let $A(t) = 0.1A_0$.

Given projected growth, t years from now a city's population P can be approximated by $P(t) = 25{,}000e^{0.045t}$. Use the formula to solve exercises 51 and 52.

51. How long will it take the city's population to reach 35,000?

52. How long will it take the population to double?

The number of bacteria in a culture after t hours can be given by $N(t) = N_0e^{0.03t}$, in which N_0 is the initial number of bacteria in the culture. Use the formula to solve exercises 53 and 54.

53. In how many hours will the size of the culture double?

54. In how many hours will the culture grow to four times its original population?

The atmospheric pressure P, in inches of mercury (in. Hg), at an altitude h feet above sea level is approximated by $P(t) = 30e^{-0.00004h}$. Use the formula to solve exercises 55 and 56.

55. Find the altitude if the pressure at that altitude is 25 in. Hg.

56. Find the altitude if the pressure at that altitude is 20 in. Hg.

Carbon 14 dating is used to measure the age of specimens and is based on the radioactive decay of the element carbon 14. If A_0 is the initial amount of carbon 14, then the amount remaining after t years is $A(t) = A_0e^{-0.000124t}$. Use the formula to solve exercises 57 and 58.

57. Estimate the age of a specimen if 70% of the original amount of carbon 14 remains.

58. Estimate the age of a specimen if 20% of the original amount of carbon 14 remains.

59. In some of the earlier exercises, we talked about bacteria cultures that double in size every few minutes. Can this go on forever? Explain.

60. The population of the United States has been doubling every 45 years. Is it reasonable to assume that this rate will continue? What factors will start to limit that growth?

ANSWERS

50. _____

51. _____

52. _____

53. _____

54. _____

55. _____

56. _____
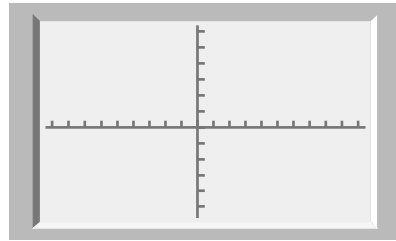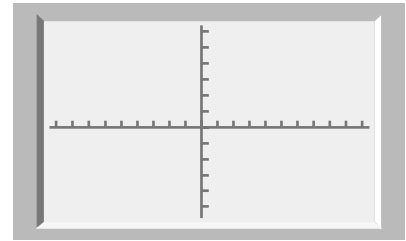
57. _____

58. _____

59. _____

60. _____

In exercises 61 to 64, use your calculator to find the graph for each equation, then explain the result.

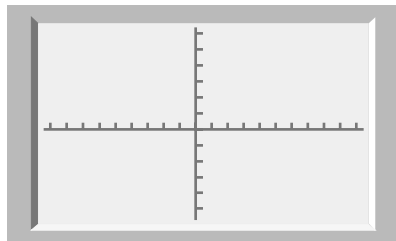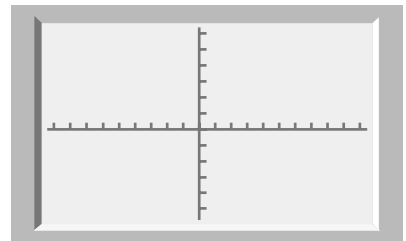**61.** $y = \log 10^x$

62. $y = 10^{\log x}$

63. $y = \ln e^x$

64. $y = e^{\ln x}$

Answers

1. $\{64\}$ **3.** $\{99\}$ **5.** $\{8\}$ **7.** $\{162\}$ **9.** $\{2\}$ **11.** $\{6\}$ **13.** $\left\{\dfrac{20}{9}\right\}$

15. $\left\{\dfrac{19}{8}\right\}$ **17.** $\left\{\dfrac{15}{4}\right\}$ **19.** $\{5, 3\}$ **21.** $\{4\}$ **23.** $\{-4\}$ **25.** $\left\{\dfrac{1}{3}\right\}$

27. $\{1.771\}$ **29.** $\{0.792\}$ **31.** $\{0.670\}$ **33.** $\{0.894\}$ **35.** $\{3.819\}$

37. $\{4.419\}$ **39.** 8.04 y   **41.** 7.79 y   **43.** 1.47 h   **45.** 3.17 h

47. 20.6 y   **49.** 11.6 y   **51.** 7.5 y   **53.** 23.1 h   **55.** 4558 ft

57. 2876 y   **59.**   **61.**

63.