Completing square of a quadratic function

Warm Up

Write each expression as a trinomial.

1. $(x - 5)^2$ $x^2 - 10x + 25$

2. $(3x + 5)^2$ $9x^2 + 30x + 25$

Factor each expression.

3. $x^2 - 18 + 81$ $(x - 9)^2$

4. $16x^2 + 24x + 9$ $(4x + 3)^2$

Objectives

Solve quadratic equations by completing the square.

Write quadratic equations in vertex form.

Vocabulary

completing the square

Many quadratic equations contain expressions that cannot be easily factored. For equations containing these types of expressions, you can use square roots to find roots.

Square-Root Property

WORDS	NUMBERS	ALGEBRA
To solve a quadratic equation, you can take the square root of both sides. Be sure to consider the positive and negative square roots.	$x^2 = 15$ $\sqrt{x^2} = \pm\sqrt{15}$ $x = \pm\sqrt{15}$	If $x^2 = a$ and a is a nonnegative real number, then $x = \pm\sqrt{a}$.

Reading Math

Read $\pm\sqrt{a}$ as "plus or minus square root of *a*."

Example 1A: Solving Equations by Using the Square Root Property

Solve the equation.

$$\mathbf{4x^2 + 11 = 59}$$

$4x^2 = 48$ *Subtract 11 from both sides.*

$x^2 = 12$ *Divide both sides by 4 to isolate the square term.*

$x = \pm\sqrt{12}$ *Take the square root of both sides.*

$x = \pm 2\sqrt{3}$ *Simplify.*

Example 1A Continued

Check Use a graphing calculator.

```
4*(2√(3))²+11
                    59
4*(⁻2√(3))²+11
                    59
```

Example 1B: Solving Equations by Using the Square Root Property

Solve the equation.

$$x^2 + 12x + 36 = 28$$

$(x + 6)^2 = 28$ *Factor the perfect square trinomial*

$x + 6 = \pm\sqrt{28}$ *Take the square root of both sides.*

$x = -6 \pm \sqrt{28}$ *Subtract 6 from both sides.*

$x = -6 \pm 2\sqrt{7}$ *Simplify.*

Example 1B Continued

Check Use a graphing calculator.

$$(^{-}6+2\sqrt{(7)})^2+12(^{-}6+2\sqrt{(7)})+36 = 28$$

$$(^{-}6-2\sqrt{(7)})^2+12(^{-}6-2\sqrt{(7)})+36 = 28$$

Check It Out! Example 1a

Solve the equation.

$$\mathbf{4x^2 - 20 = 5}$$

$$4x^2 = 25$$

Add 20 to both sides.

$$x^2 = \frac{25}{4}$$

Divide both sides by 4 to isolate the square term.

$$x = \frac{\pm\sqrt{25}}{\pm\sqrt{4}}$$

Take the square root of both sides.

$$x = \pm\frac{5}{2}$$

Simplify.

Check It Out! Example 1a Continued

Check Use a graphing calculator.

```
4*(-5/2)²-20
                    5
4*(5/2)²-20
                    5
```

Check It Out! Example 1b

Solve the equation.

$$\boldsymbol{x^2 + 8x + 16 = 49}$$

$(x + 4)^2 = 49$ *Factor the perfect square trinomial.*

$x + 4 = \pm\sqrt{49}$ *Take the square root of both sides.*

$x = -4 \pm \sqrt{49}$ *Subtract 4 from both sides.*

$x = -11, 3$ *Simplify.*

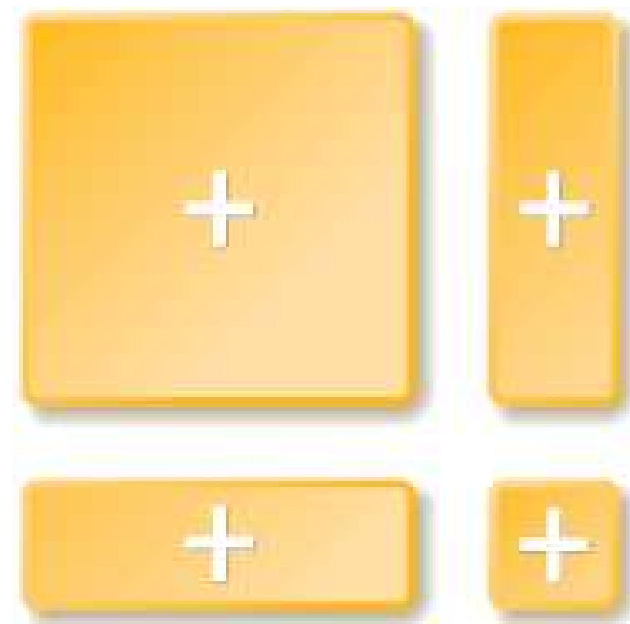
Check It Out! Example 1b Continued

Check Use a graphing calculator.

```
(-11)²+8(-11)+16
                  49
(3)²+8(3)+16
                  49
```

The methods in the previous examples can be used only for expressions that are perfect squares. However, you can use algebra to rewrite any quadratic expression as a perfect square.

You can use algebra tiles to model a perfect square trinomial as a perfect square. The area of the square at right is $x^2 + 2x + 1$. Because each side of the square measures $x + 1$ units, the area is also $(x + 1)(x + 1)$, or $(x + 1)^2$. This shows that $(x + 1)^2 = x^2 + 2x + 1$.

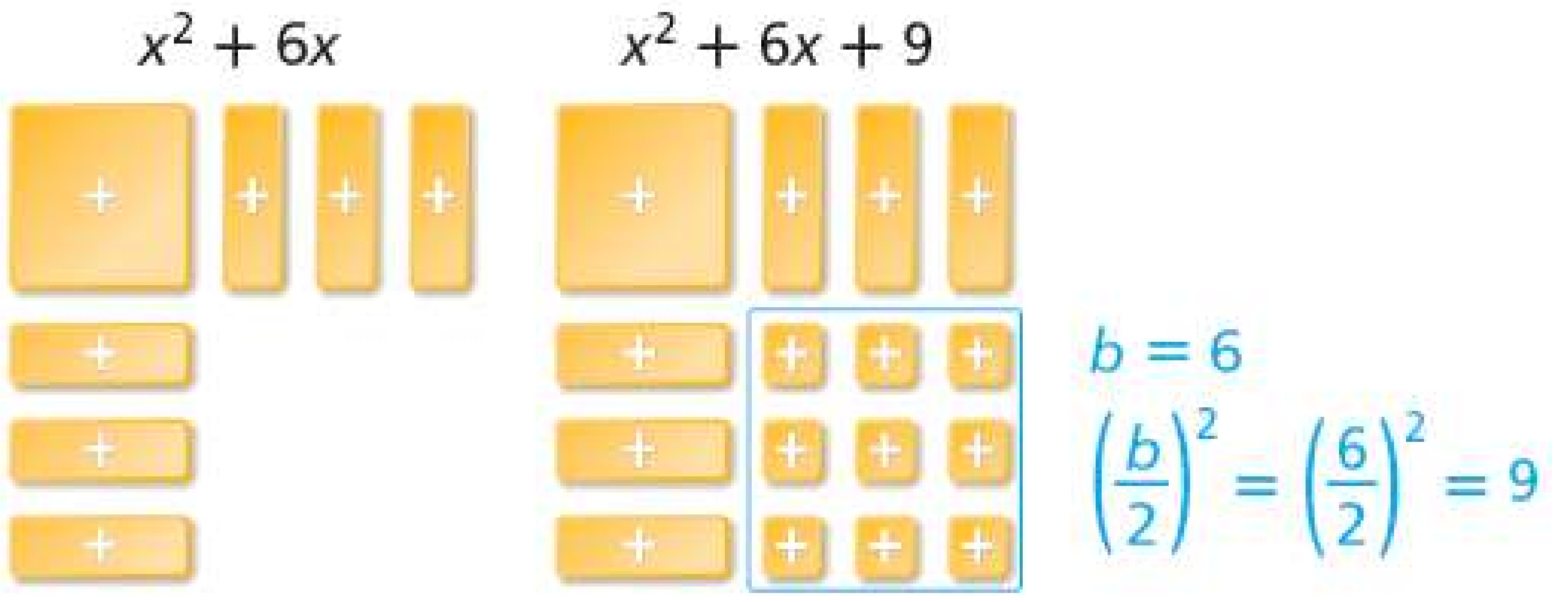


If a quadratic expression of the form $x^2 + bx$ *cannot* model a square, you can add a term to form a perfect square trinomial. This is called **completing the square**.

Completing the Square

WORDS	NUMBERS	ALGEBRA
To complete the square of $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.	$x^2 + 6x + \blacksquare$	$x^2 + bx + \blacksquare$
	$x^2 + 6x + \left(\frac{6}{2}\right)^2$	$x^2 + bx + \left(\frac{b}{2}\right)^2$
	$x^2 + 6x + 9$	$\left(x + \frac{b}{2}\right)^2$
	$(x + 3)^2$	

The model shows completing the square for $x^2 + 6x$ by adding 9 unit tiles. The resulting perfect square trinomial is $x^2 + 6x + 9$. Note that completing the square does not produce an equivalent expression.



Example 2A: Completing the Square

Complete the square for the expression. Write the resulting expression as a binomial squared.

$\boldsymbol{x^2 - 14x + \square}$

$$\left(\frac{-14}{2}\right)^2 = (-7)^2 = 49$$ *Find* $\left(\frac{b}{2}\right)^2$.

$x^2 - 14x + 49$ *Add.*

$(x - 7)^2$ *Factor.*

Check Find the square of the binomial.

$$(x - 7)^2 = (x - 7)(x - 7)$$
$$= x^2 - 14x + 49$$

Example 2B: Completing the Square

Complete the square for the expression. Write the resulting expression as a binomial squared.

$x^2 + 9x + \square$

$$\left(\frac{9}{2}\right)^2 = \frac{81}{4}$$

Find $\left(\frac{b}{2}\right)^2$.

$$x^2 + 9x + \frac{81}{4}$$

Add.

$$\left(x + \frac{9}{2}\right)^2$$

Factor.

Check Find the square of the binomial.

$$\left(x + \frac{9}{2}\right)^2 = \left(x + \frac{9}{2}\right)\left(x + \frac{9}{2}\right)$$

$$= x^2 + 9x + \frac{81}{4}$$

Check It Out! Example 2a

Complete the square for the expression. Write the resulting expression as a binomial squared.

$\boldsymbol{x^2 + 4x +}$ $\square$

$\left(\frac{4}{2}\right)^2 = (2)^2 = 4$ *Find* $\left(\frac{b}{2}\right)^2$.

$x^2 + 4x + 4$ *Add.*

$(x + 2)^2$ *Factor.*

Check Find the square of the binomial.

$$(x + 2)^2 = (x + 2)(x + 2)$$
$$= x^2 + 4x + 4$$

Check It Out! Example 2b

Complete the square for the expression. Write the resulting expression as a binomial squared.

$x^2 - 4x + \square$

$\left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$ *Find* $\left(\frac{b}{2}\right)^2$.

$x^2 - 4x + 4$ *Add.*

$(x - 2)^2$ *Factor.*

Check Find the square of the binomial.

$$(x - 2)^2 = (x - 2)(x - 2)$$

$$= x^2 - 4x + 4$$

Check It Out! Example 2c

Complete the square for the expression. Write the resulting expression as a binomial squared.

$x^2 + 3x +$ ☐

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$$ *Find* $\left(\frac{b}{2}\right)^2$.

$$x^2 + 3x + \frac{9}{4}$$ *Add.*

$$\left(x + \frac{3}{2}\right)^2$$ *Factor.*

Check Find the square of the binomial.

$$\left(x+\frac{3}{2}\right)^2 = \left(x+\frac{3}{2}\right)\left(x+\frac{3}{2}\right)$$

$$= x^2 + 3x + \frac{3}{2}$$

You can complete the square to solve quadratic equations.

Solving Quadratic Equations $ax^2 + bx + c = 0$ by Completing the Square
1. Collect variable terms on one side of the equation and constants on the other.
2. As needed, divide both sides by a to make the coefficient of the x^2-term 1.
3. Complete the square by adding $\left(\frac{b}{2}\right)^2$ to both sides of the equation.
4. Factor the variable expression as a perfect square.
5. Take the square root of both sides of the equation.
6. Solve for the values of the variable.

Example 3A: Solving a Quadratic Equation by Completing the Square

Solve the equation by completing the square.

$$\boldsymbol{x^2 = 12x - 20}$$

$$x^2 - 12x = -20$$

Collect variable terms on one side.

$$x^2 - 12x + \square = -20 + \square$$

Set up to complete the square.

$$x^2 - 12x + \left(-\frac{12}{2}\right)^2 = -20 + \left(-\frac{12}{2}\right)^2$$

Add $\left(\frac{b}{2}\right)^2$ to both sides.

$$x^2 - 12x + 36 = -20 + 36$$

Simplify.

Example 3A Continued

$(x - 6)^2 = 16$ *Factor.*

$x - 6 = \pm\sqrt{16}$ *Take the square root of both sides.*

$x - 6 = \pm 4$ *Simplify.*

$x - 6 = 4$ or $x - 6 = -4$ *Solve for x.*

$x = 10$ or $x = 2$

Example 3B: Solving a Quadratic Equation by Completing the Square

Solve the equation by completing the square.

$\mathbf{18x + 3x^2 = 45}$

$x^2 + 6x = 15$ — *Divide both sides by 3.*

$x^2 + 6x + \square = 15 + \square$ — *Set up to complete the square.*

$x^2 + 6x + \left(\frac{6}{2}\right)^2 = 15 + \left(\frac{6}{2}\right)^2$ — *Add $\left(\frac{b}{2}\right)^2$ to both sides.*

$x^2 + 6x + 9 = 15 + 9$ — *Simplify.*

Example 3B Continued

$(x + 3)^2 = 24$ *Factor.*

$$x + 3 = \pm\sqrt{24}$$ *Take the square root of both sides.*

$$x = -3 \pm 2\sqrt{6}$$ *Simplify.*

Check It Out! Example 3a

Solve the equation by completing the square.

$\boldsymbol{x^2 - 2 = 9x}$

$x^2 - 9x = 2$

Collect variable terms on one side.

$x^2 - 9x + \square = 2 + \square$

Set up to complete the square.

$$x^2 - 9x + \left(\frac{9}{2}\right)^2 = 2 + \left(\frac{9}{2}\right)^2$$

Add $\left(\frac{b}{2}\right)^2$ to both sides.

$$x^2 - 9x + \frac{81}{4} = 2 + \frac{81}{4}$$

Simplify.

Check It Out! Example 3a Continued

$$\left(x - \frac{9}{2}\right)^2 = \frac{89}{4}$$

Factor.

$$x - \frac{9}{2} = \pm\sqrt{\frac{89}{4}}$$

Take the square root of both sides.

$$x = \frac{9 \pm \sqrt{89}}{2}$$

Simplify.

Check It Out! Example 3b

Solve the equation by completing the square.

$$3x^2 - 24x = 27$$

$$x^2 - 8x = 9$$

Divide both sides by 3.

$$x^2 - 8x + \square = 9 + \square$$

Set up to complete the square.

$$x^2 - 8x + \left(\frac{8}{2}\right)^2 = 9 + \left(\frac{8}{2}\right)^2$$

Add $\left(\frac{b}{2}\right)^2$ to both sides.

$$x^2 - 8x + 16 = 9 + 16$$

Simplify.

Check It Out! Example 3b Continued

Solve the equation by completing the square.

$(x-4)^2 = 25$ *Factor.*

$x-4 = \pm\sqrt{25}$ *Take the square root of both sides.*

$x-4 = \pm 25$ *Simplify.*

$x - 4 = -5$ or $x - 4 = 5$ *Solve for x.*

$x = -1$ or $x = 9$

Recall the vertex form of a quadratic function from lesson 5-1: $f(x) = a(x - h)^2 + k$, where the vertex is (h, k).

You can complete the square to rewrite any quadratic function in vertex form.

Helpful Hint

In Example 3, the equation was balanced by adding $\left(\frac{b}{2}\right)^2$ to *both* sides. Here, the equation is balanced by adding and subtracting $\left(\frac{b}{2}\right)^2$ on *one* side.

Example 4A: Writing a Quadratic Function in Vertex Form

Write the function in vertex form, and identify its vertex.

$$\boldsymbol{f(x) = x^2 + 16x - 12}$$

$f(x)=(x^2 + 16x + \boxed{}\,) - 12 - \boxed{}$ *Set up to complete the square.*

$f(x) = \left[x^2 + 16x + \left(\frac{16}{2}\right)^2\right] - 12 - \left(\frac{16}{2}\right)^2$ *Add and subtract* $\left(\frac{b}{2}\right)^2$.

$f(x) = (x + 8)^2 - 76$ *Simplify and factor.*

Because $h = -8$ and $k = -76$, the vertex is $(-8, -76)$.

Example 4A Continued

Check Use the axis of symmetry formula to confirm vertex.

$$x = -\frac{b}{2a} = -\frac{16}{2(1)} = -8$$

$$y = f(-8) = (-8)^2 + 16(-8) - 12 = -76 \checkmark$$

Example 4B: Writing a Quadratic Function in Vertex Form

Write the function in vertex form, and identify its vertex

$\boldsymbol{g(x) = 3x^2 - 18x + 7}$

$g(x) = 3(x^2 - 6x) + 7$

Factor so the coefficient of x^2 is 1.

$g(x) = 3(x^2 - 6x + \square\ ) + 7 - \square$

Set up to complete the square.

$$g(x) = 3\left[x^2 - 6x + \left(-\frac{6}{2}\right)\right]^2 + 7 - 3\left(-\frac{6}{2}\right)^2$$

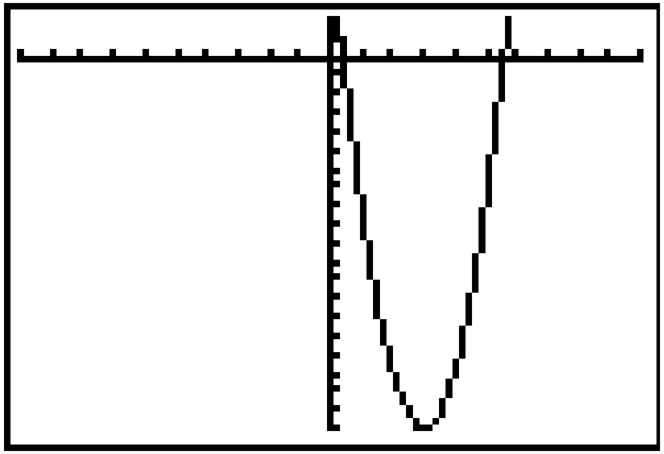
Add $\left(\frac{b}{2}\right)^2$. Because $\left(\frac{b}{2}\right)^2$ is multiplied by 3, you must subtract 3 $\left(\frac{b}{2}\right)^2$.

Example 4B Continued

$g(x) = 3(x - 3)^2 - 20$ *Simplify and factor.*

Because $h = 3$ and $k = -20$, the vertex is $(3, -20)$.

Check A graph of the function on a graphing calculator supports your answer.



Check It Out! Example 4a

Write the function in vertex form, and identify its vertex

$$\boldsymbol{f(x) = x^2 + 24x + 145}$$

$f(x) = (x^2 + 24x + \square\ ) + 145 - \square$ *Set up to complete the square.*

$$f(x) = \left[x^2 + 24x + \left(\frac{24}{2}\right)^2\right] + 145 - \left(\frac{24}{2}\right)^2$$ *Add and subtract* $\left(\frac{b}{2}\right)^2$.

$f(x) = (x + 12)^2 + 1$ *Simplify and factor.*

Because $h = -12$ and $k = 1$, the vertex is $(-12, 1)$.

Check It Out! Example 4a Continued

Check Use the axis of symmetry formula to confirm vertex.

$$x = -\frac{b}{2a} = -\frac{24}{2(1)} = -12$$

$$y = f(-12) = (-12)^2 + 24(-12) + 145 = 1 \checkmark$$

Check It Out! Example 4b

Write the function in vertex form, and identify its vertex

$\boldsymbol{g(x) = 5x^2 – 50x + 128}$

$g(x) = 5(x^2 – 10x) + 128$

Factor so the coefficient of x^2 is 1.

$g(x) = 5(x^2 – 10x + \square) + 128 – \square$

Set up to complete the square.

$$g(x) = 5\left[x^2 - 10x + \left(-\frac{10}{2}\right)^2\right] + 128 - 5\left(-\frac{10}{2}\right)^2$$

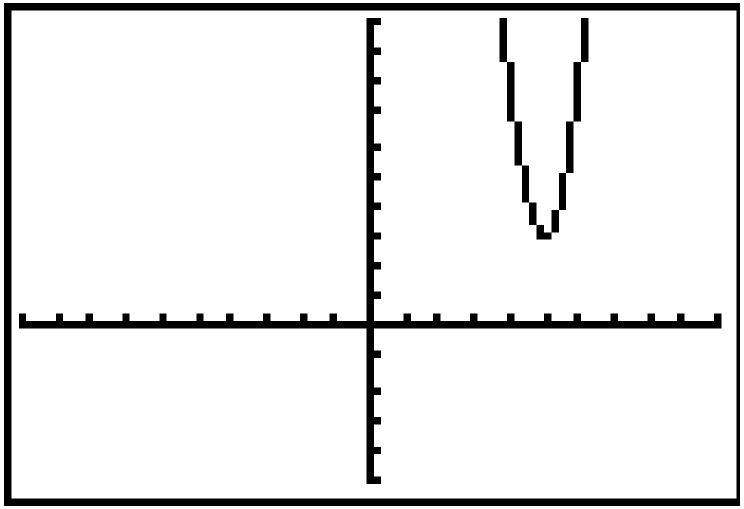
Add $\left(\frac{b}{2}\right)^2$. Because $\left(\frac{b}{2}\right)^2$ is multiplied by 5, you must subtract $5\left(\frac{b}{2}\right)^2$.

Check It Out! Example 4b Continued

$g(x) = 5(x - 5)^2 + 3$ *Simplify and factor.*

Because $h = 5$ and $k = 3$, the vertex is (5, 3).

Check A graph of the function on a graphing calculator supports your answer.



Lesson Quiz

1. Complete the square for the expression $x^2 - 15x +$ ☐ . Write the resulting expression as a binomial squared.

$$x^2 - 15x + \frac{225}{4} = \left(x - \frac{15}{2}\right)^2$$

Solve each equation.

2. $x^2 - 16x + 64 = 20$ $\quad 8 \pm 2\sqrt{5}$

3. $x^2 - 27 = 4x$ $\quad 2 \pm \sqrt{31}$

Write each function in vertex form and identify its vertex.

4. $f(x) = x^2 + 6x - 7$

$f(x) = (x + 3)^2 - 16$; $(-3, -16)$

5. $f(x) = 2x^2 - 12x - 27$

$f(x) = 2(x - 3)^2 - 45$; $(3, -45)$