AP Calculus AB

Chapter 1

What is Calculus?

Calculus is the mathematics of change:

velocity, acceleration, tangent lines, slopes, areas, volumes, and many other concepts

that have enabled scientists, engineers, and economists to model real life situations.

See p. 43 of the book.

A moving body's average speed during an interval of time is found by dividing the distance covered by the elapsed time. The unit of measure is length per unit time.

Recall: Vertical Motion Formula on Earth

$$h(t) = -16t^2 + v_o(\sin\theta)\,t + h_o \text{ (in feet)}$$

or

$$h(t) = -4.9t^2 + v_o(\sin\theta)\,t + h_o \text{ (in meters)}$$

where $h(t)$ is the height of the object at time t in seconds, v_o is the initial vertical velocity, θ is the initial angle counterclockwise from the horizontal, and h_o is the initial height.

Example 1: A ball is dropped from a 10,000 foot building. What is the ball's average speed during the first 3 seconds?

$$h(t) = -16t^2 + v_o(\sin\theta)\,t + h_o \text{ (in feet)}$$

0 sec: $-16 \cdot 0^2 + 0 + 10{,}000' = 10{,}000'$

3 sec: $-16(3^2) + 0 + 10{,}000' = 9856'$

$$m = \frac{9856 - 10{,}000}{3 - 0 \text{ sec}} = \frac{-144'}{3 \text{ sec}}$$

$$= -48 \text{ ft/sec}$$

Now, this is actually velocity. It's negative because the ball's height is decreasing. Speed is the absolute value of velocity. So the answer is 48 ft/sec.

Example 2: Find the <u>speed</u> of the ball at time $t = 3$.

$$h(t) = -16t^2 + v_o(\sin\theta)\,t + h_o \text{ (in feet)}$$

Let's look at the height of the ball at time $t = 3$ and the height of the ball at some time a teensy weensy bit (b) later, say, $t = 3 + b$.

$$h(3) = -16 \cdot 3^2 + 0 \cdot 3 + 10{,}000$$

$$h(3+b) = -16(3+b)^2 + 0(3+b) + 10{,}000$$

Speed is the absolute value of velocity. The velocity is

$$\frac{\Delta h}{\Delta t} = \frac{h_2 - h_1}{t_2 - t_1}$$

$$= \frac{-16(9+6b+b^2) + 10{,}000 - (-16 \cdot 3^2 + 10{,}000)}{3 + b - 3}$$

$$= \frac{\cancel{-144} - 96b - 16b^2 + \cancel{10{,}000} + \cancel{144} - \cancel{10{,}000}}{b}$$

$$= \frac{-96b - 16b^2}{b} = -96 - \cancel{16b}^{\,0} \text{ ft/sec}$$

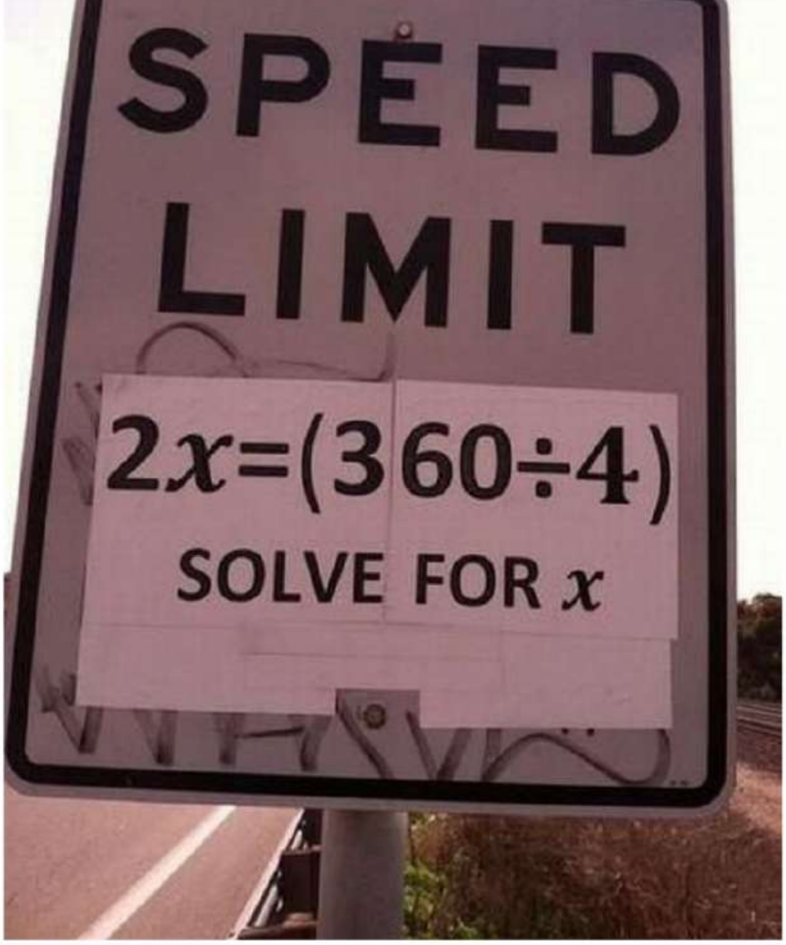
$$\lim_{b \to 0}(-96 - 16b) = -96 \frac{\text{ft}}{\text{sec}}$$

Why is this negative? Because velocity includes direction, and the ball is going down.

Its speed is: $|-96 \text{ ft/sec}| = 96 \text{ ft/sec}$

What are limits of functions?

The limit of a function is the height (y-value) a function gets closer and closer to as you approach a specific x-value. It does not matter if the function actually gets there or not. All that matters is that the function's y-values close in on that value as you approach the given x-value.


SPEED
LIMIT
2x=(360÷4)
SOLVE FOR x

Properties of Limits

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $$\lim_{x \to c} (f(x) + g(x)) = L + M$$

   The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $$\lim_{x \to c} (f(x) - g(x)) = L - M$$

   The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $$\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$$

   The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$

   The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M},\ M \neq 0$$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If r and s are integers, $s \neq 0$, then

   $$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

   provided that $L^{r/s}$ is a real number.

   The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Polynomial and Rational Functions

1. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

2. If $f(x)$ and $g(x)$ are polynomials and c is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad \text{provided that } g(c) \neq 0.$$

In other words, you can find the limit of $f(x)$ as x approaches c by substitution (finding $f(c)$), as long as you don't end up with 0 in the denominator. If the denominator is 0, you'll need to investigate further.

Find:

1. $\lim_{x \to -1}(x^3 + 6x^2 - 3)$

$(-1)^3 + 6(-1)^2 - 3 = -1 + 6 - 3 = 2$

2. $\lim_{x \to 4} \frac{x-7}{x^2}$

$\frac{4-7}{4^2} = -\frac{3}{16}$

3. $\lim_{x \to 5} \frac{x^2 - 4x - 5}{x^2 - 25} = \lim_{x \to 5} \frac{(x-5)(x+1)}{(x-5)(x+5)}$

$= \frac{6}{10} = .6$

$$\frac{0}{1} = 0$$

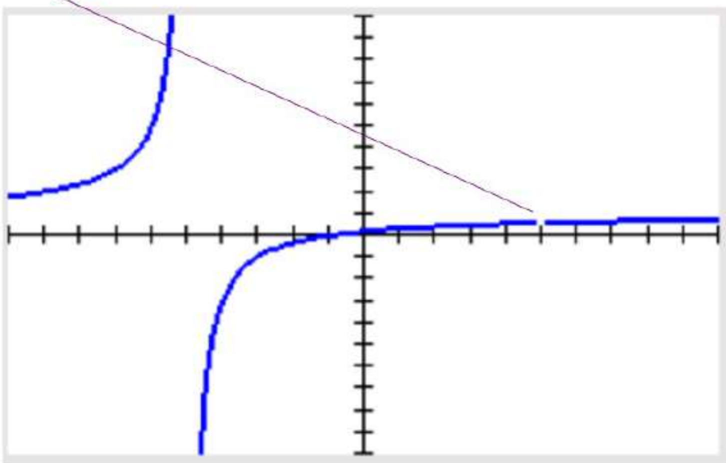
$$\frac{1}{0} = \text{undefined}$$

$$\frac{0}{0} = \text{indeterminate: We need more information.}$$

http://fox8.com/2015/06/30/siri-gets-a-little-snarky-when-asked-whats-zero-divided-by-zero/

"Imagine that you have zero cookies and you split them evenly among zero friends. How many cookies does each person get? See? It doesn't make sense. And Cookie Monster is sad that there are no cookies, and you are sad that you have no friends."

Let's investigate #3 on the calculator. Notice the hole in the graph at (5, .6). If your calculator doesn't show that, then use ZOOM 4:ZoomDecimal and it should show up.

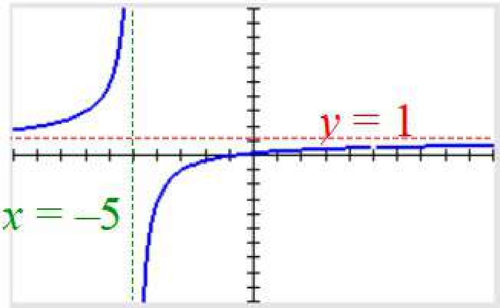


$$\lim_{x \to 5} \frac{x^2 - 4x - 5}{x^2 - 25} = \lim_{x \to 5} \frac{(x-5)(x+1)}{(x-5)(x+5)} = \lim_{x \to 5} \frac{(x+1)}{(x+5)}, x \neq 5$$

Y1=(X²−4X−5)/(X²−25)
Y2=(X+1)/(X+5)

X	Y1	Y2
4.8	.59184	.59184
4.9	.59596	.59596
5	ERROR	.6
5.1	.60396	.60396
5.2	.60784	.60784
5.3	.61165	.61165
5.4	.61538	.61538

Notice that the two expressions are identical, except when $x = 5$. Even though Y1 is not defined at $x = 5$, it does appear to get closer and closer to 0.6 as x approaches 5.



$$f(x) = \frac{x^2 - 4x - 5}{x^2 - 25} = \frac{(x-5)(x+1)}{(x-5)(x+5)} = \frac{(x+1)}{(x+5)}, x \neq 5$$

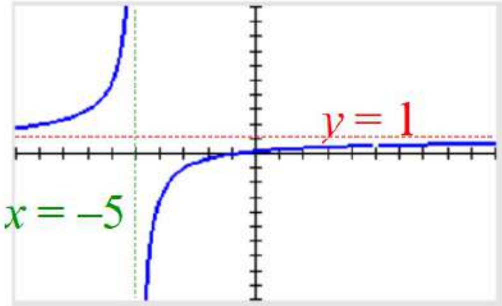
Let's review rational functions and their points of discontinuity.

Common factors in numerator and denominator create holes in the graph.

Vertical asymptotes are created when a factor in the denominator is unmatched by a factor in the numerator after removing common factors (filling in holes). $x + 5 = 0 \Rightarrow x = -5$

Even though it's not necessarily a point of discontinuity, a zero (x-intercept) is created when a factor in the numerator is unmatched by a factor in the denominator after removing common factors (filling in holes).

$$\frac{(x - \text{zero})(x - \text{hole})}{(x - \text{v.a.})(x - \text{hole})}$$



$$f(x) = \frac{x^2 - 4x - 5}{x^2 - 25} = \frac{(x-5)(x+1)}{(x-5)(x+5)} = \frac{(x+1)}{(x+5)}, x \neq 5$$

Horizontal asymptotes can be found by comparing the degree of the numerator (n) to the degree of the denominator (d).

$n < d \Rightarrow y = 0$

$n = d \Rightarrow y = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$

$n > d \Rightarrow$ none

In this function, $n = d$, so $y = 1/1 = 1$ is a horizontal asymptote.

One- and Two-Sided Limit Notation

right-hand: $\lim_{x \to c^+} f(x)$ *The limit of f as x approaches c from the right.*

$x \to c^+$

left-hand: $\lim_{x \to c^-} f(x)$ *The limit of f as x approaches c from the left.*

$x \to c^-$

We sometimes call $\lim_{x \to c} f(x)$ the two-sided limit of *f* at *c* to distinguish it from the one-sided right-hand and left-hand limits of *f* at *c*.

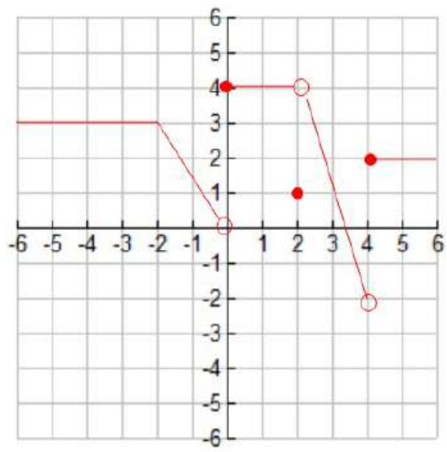
THEOREM 3 One-sided and Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the right-hand and left-hand limits at c exist and are equal. In symbols,

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

In other words, the one-sided limits have to match for the two-sided limit to exist.

Consider the following piece-wise function. Provide the given limits:



1. $\lim_{x \to -2} f(x) =$ 3

2. $\lim_{x \to 0^-} f(x) =$ 0

3. $\lim_{x \to 0^+} f(x) =$ 4

4. $\lim_{x \to 0} f(x) =$ dne

5. $\lim_{x \to 2^-} f(x) =$ 4

6. $\lim_{x \to 2^+} f(x) =$ 4

7. $\lim_{x \to 2} f(x) =$ 4

8. $\lim_{x \to 4^-} f(x) =$ -2

9. $\lim_{x \to 4^+} f(x) =$ 2

10. $\lim_{x \to 4} f(x) =$ dne

Find the requested limits for the following function.

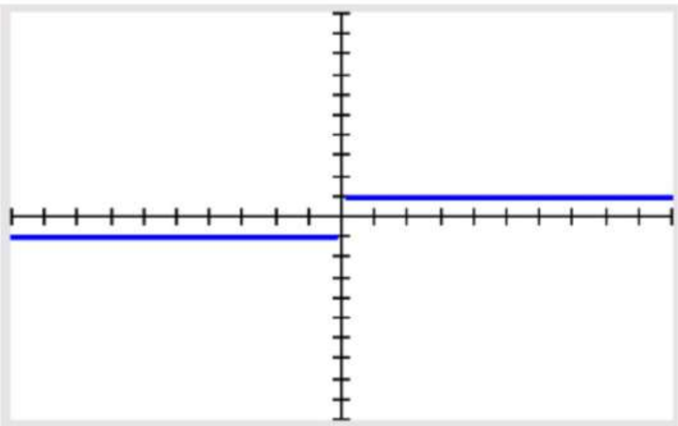
$$f(x) = \frac{|x|}{x}$$

$a)\lim_{x\to 3} f(x) =$ 1

$b)\lim_{x\to 0^-} f(x) =$ -1

$c)\lim_{x\to 0^+} f(x) =$ 1

$d)\lim_{x\to 0} f(x) =$ dne



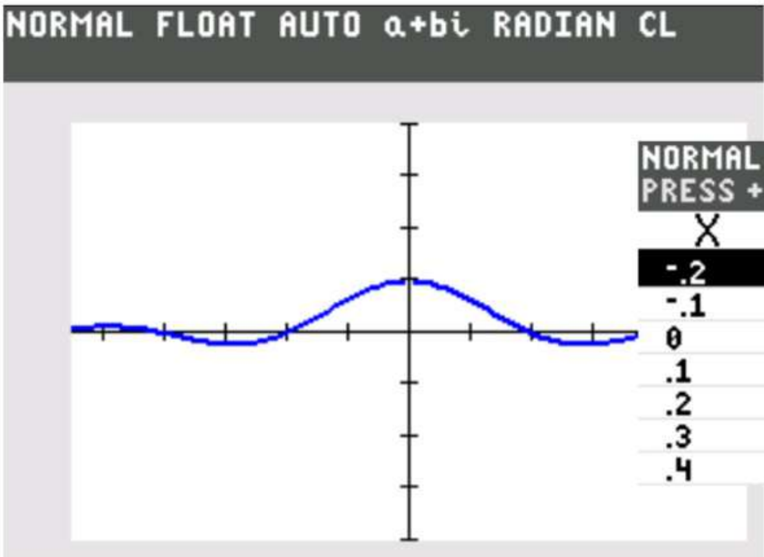
X	Y1
-.2	-1
-.1	-1
0	ERROR
.1	1
.2	1

Find the requested limits for the following function.

$$f(x)=\frac{\sin x}{x}$$

$a)\lim_{x\to 3} f(x) =$ $\frac{\sin 3}{3} \approx .047$ $\quad b)\lim_{x\to 0^-} f(x) =$ 1

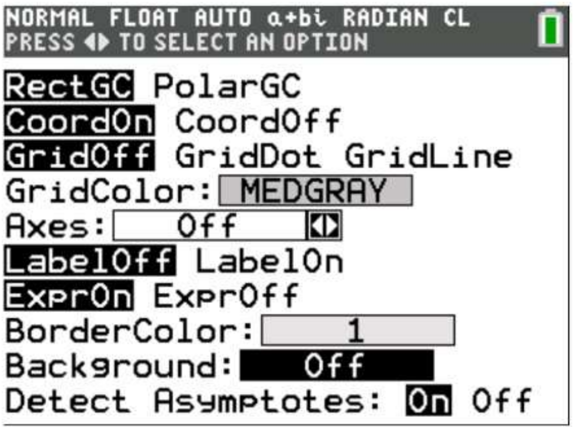
$c)\lim_{x\to 0^+} f(x) =$ 1 $\quad d)\lim_{x\to 0} f(x) =$ 1

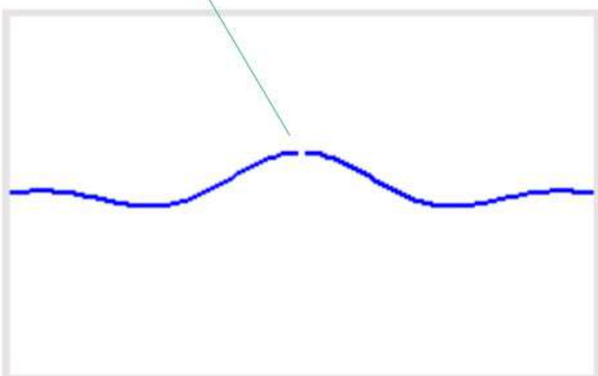


NORMAL FLOAT AUTO a+bi RADIAN CL
PRESS + FOR △Tbl

X	Y_1			
-.2	.99335			
-.1	.99833			
0	ERROR			
.1	.99833			
.2	.99335			
.3	.98507			
.4	.97355			

Note: If you go to FORMAT (2ND ZOOM) on the calculator you can turn the axes off and see the hole at $x = 0$.





Find the requested limits for the function.

$$f(x) = \begin{cases} 3x, x < -2 \\ x+4, -2 \le x \le 3 \\ 6, x > 3 \end{cases}$$

1. $\lim_{x \to -2^-} f(x)$ — $3(-2) = -6$

2. $\lim_{x \to -2^+} f(x)$ — $-2+4=2$

3. $\lim_{x \to -2} f(x)$ — dne

4. $\lim_{x \to 3^-} f(x)$ — $3+4=7$

5. $\lim_{x \to 3^+} f(x)$ — 6

6. $\lim_{x \to 3} f(x)$ — dne

7. $\lim_{x \to 0} f(x)$ — $0+4=4$

8. $\lim_{x \to -\infty} f(x)$ — $-\infty$

9. $\lim_{x \to \infty} f(x)$ — 6

The Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

Note: This theorem is sometimes called the Squeeze Theorem.

Sometimes we have trouble finding a limit directly, but we can find it indirectly using the Sandwich Theorem.

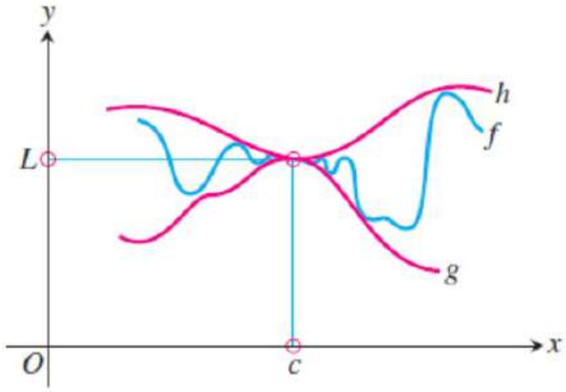
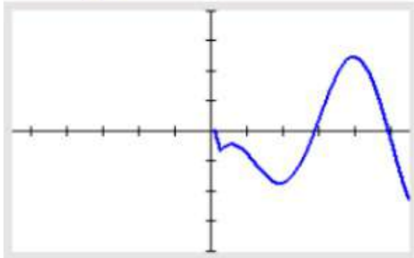


Figure 2.7 Sandwiching f between g and h forces the limiting value of f to be between the limiting values of g and h.

Example: Find the limit graphically. Use the Sandwich Theorem to confirm your answer.

$$\lim_{x \to 0^+} \sqrt{x} \cos\left(x + \frac{1}{x}\right)$$



$-1 \le \quad \le 1$

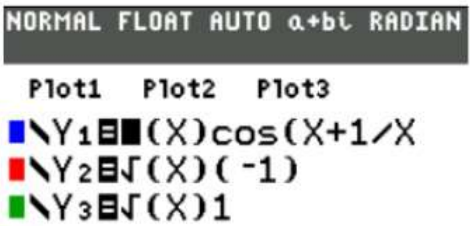
It looks like the function approaches 0 from the right. Using the Sandwich (or Squeeze) Theorem, we know that $-1 \le \cos(x + 1/x) \le 1$. So

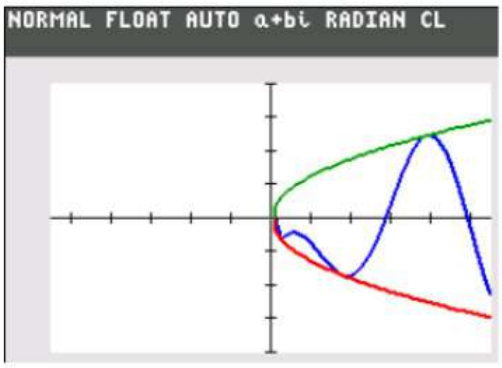
$$\lim_{x \to 0^+} \sqrt{x} \cdot (-1) \le \lim_{x \to 0^+} \sqrt{x} \cos\left(x + \frac{1}{x}\right) \le \lim_{x \to 0^+} \sqrt{x} \cdot 1 \ .$$

Therefore, $0 \le \lim_{x \to 0^+} \sqrt{x} \cos\left(x + \frac{1}{x}\right) \le 0$.

So $\lim_{x \to 0^+} \sqrt{x} \cos\left(x + \frac{1}{x}\right) = 0$.

Graphically,





Find the requested limits for the function.

$$f(x)=\begin{cases}3x, x<-2\\ x+4, -2\le x\le 3\\ 6, x>3\end{cases}$$

1. $\lim_{x\to -2^-} f(x)$ 2. $\lim_{x\to -2^+} f(x)$ 3. $\lim_{x\to -2} f(x)$

4. $\lim_{x\to 3^-} f(x)$ 5. $\lim_{x\to 3^+} f(x)$ 6. $\lim_{x\to 3} f(x)$

7. $\lim_{x\to -\infty} f(x)$ 8. $\lim_{x\to \infty} f(x)$