The Common Core State Standards in Mathematics

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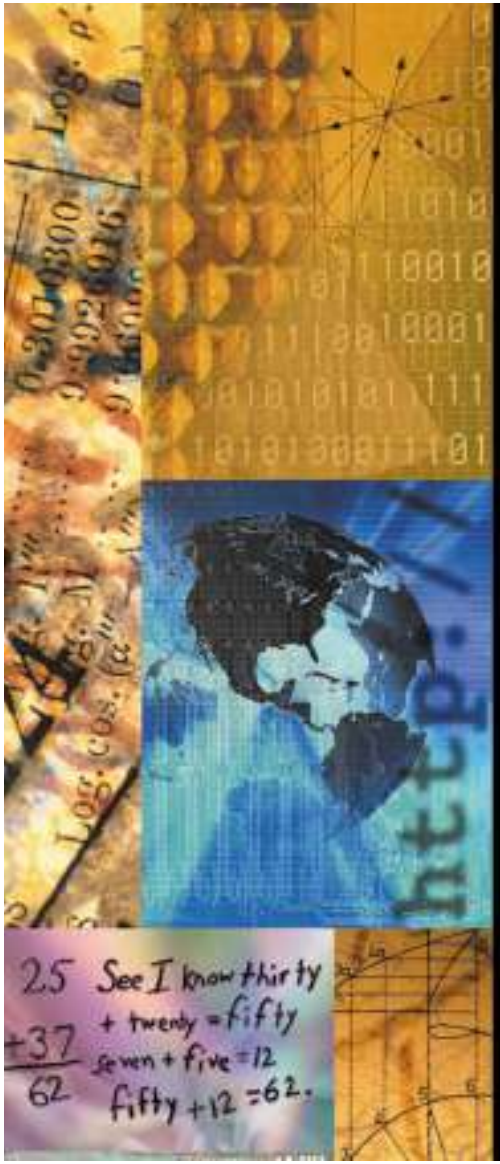
50 US States


Washington
Oregon
Idaho
Montana
Wyoming
Nevada
Utah
Colorado
California
Arizona
New Mexico
Texas
Oklahoma
Kansas
Nebraska
Iowa
Minnesota
Wisconsin
Michigan
Missouri
Kentucky
Tennessee
Alabama
Georgia
Florida
Louisiana
Mississippi
Arkansas
Indiana
Ohio
Maine
Washington D.C.
Seattle
Portland
Salt Lake City
Cheyenne
Denver
Santa Fe
Phoenix
Los Angeles
San Diego
Dallas
Austin
Houston
San Antonio
Topeka
Lincoln
Omaha
Des Moines
Chicago
Detroit
Nashville
Memphis
Atlanta
Jacksonville
Orlando
Miami

16,000 school districts

Figure: Georgia school districts



Principles and Standards for School Mathematics

Grade level where different states introduced addition and subtraction of fractions (CSMC)

Grade	Addition & Subtraction of Fractions
1st grade	2 states
2nd grade	
3rd grade	7 states
4th grade	22 states
5th grade	9 states
6th grade	1 state
7th grade	1 state
8th grade	
Not specified	

Composite of U.S. state standards, 2005

Note that topics are introduced and sustained in a way that produces no visible structure.

TOPIC	GRADE: 1	2	3	4	5	6	7	8
Whole Number Meaning	■	■	■	■	■	□		
Whole Number Operations	■	■	■	■	■	□		
Measurement Units	■	■	■	■	■	■	■	■
Common Fractions	□	■	■	■	■	■	□	□
Equations & Formulas	□	□	■	■	■	■	■	■
Data Representation & Analysis	■	■	■	■	■	■	■	■
2-D Geometry: Basics	■	■	■	■	■	■	■	■
Polygons & Circles	■	■	■	■	■	■	■	■
Perimeter, Area & Volume		□	□	□	■	■	■	■
Rounding & Significant Figures								
Estimating Computations	□	□	■	■	■	■	■	■
Properties of Whole Number Operations	□	□	□	□				
Estimating Quantity & Size			□					
Decimal Fractions			□	■	■	■	□	□
Relationship of Common & Decimal Fractions				□	□	□		
Properties of Common & Decimal Fractions								
Percentages					□	■	■	□
Proportionality Concepts						■	□	
Proportionality Problems						■	■	■
2-D Coordinate Geometry			□	■	□	□	□	■
Geometry: Transformations	■	■	■	■	■	■	■	■
Negative Numbers, Integers & Their Properties						□	■	□
Number Theory					■	□	□	□
Exponents, Roots & Radicals						□	□	■
Exponents & Orders of Magnitude							□	□
Measurement Estimation & Errors	□	□	■	□	■	■	■	□
Constructions w/ Straightedge & Compass								
3-D Geometry	■	■	■	■	■	■	■	■
Congruence & Similarity					□	■	■	□
Rational Numbers & Their Properties						■	■	□
Patterns, Relations & Functions	■	■	■	■	■	■	■	■
Slope & Trigonometry								
Number of topics covered by at least 67% of the states	14	15	18	18	20	25	23	22
Number of additional topics intended by states to complete a typical curriculum at each grade level	8	8	7	8	8	5	6	6

□ – intended by 67% of the states ■ – intended by 83% of the states ■ – intended by 100% of the states

- Mathematics topics intended at each grade by at least two thirds of 21 U.S. States.
- On average a state would have 6–8 more topics per grade level in its complete curriculum.
- From Schmidt, Houang, and Cogan, *American Educator*, 2005.

Composite of high achieving countries

Note that topics are introduced and sustained in a coherent fashion, producing a clear upper-triangular structure.

TOPIC	GRADE: 1	2	3	4	5	6	7	8
Whole Number Meaning	■	■	■	■	■			
Whole Number Operations	■	■	■	■	■			
Measurement Units	□	■	■	■	■	■	■	
Common Fractions			□	■	■	■		
Equations & Formulas			□	■	■	■	■	■
Data Representation & Analysis			□	□	■	■		□
2-D Geometry: Basics			□	■	■	■	■	■
Polygons & Circles				■	■	■	■	■
Perimeter, Area & Volume				■	■	■	■	□
Rounding & Significant Figures				■	■			
Estimating Computations				■	■	■		
Properties of Whole Number Operations				□	■			
Estimating Quantity & Size				□	□			
Decimal Fractions				■	■	■		
Relationship of Common & Decimal Fractions				■	■	■		
Properties of Common & Decimal Fractions					■	■		
Percentages					■	■		
Proportionality Concepts					■	■	■	□
Proportionality Problems					■	■	■	■
2-D Coordinate Geometry					□	□	■	■
Geometry: Transformations						■	■	■
Negative Numbers, Integers & Their Properties						□	■	
Number Theory							■	□
Exponents, Roots & Radicals							■	■
Exponents & Orders of Magnitude							□	□
Measurement Estimation & Errors							□	
Constructions w/ Straightedge & Compass							■	□
3-D Geometry							■	■
Congruence & Similarity								■
Rational Numbers & Their Properties								□
Patterns, Relations & Functions								□
Slope & Trigonometry								□
Number of topics covered by at least 67% of the A+ countries	3	3	7	15	20	17	16	18
Number of additional topics intended by A+ countries to complete a typical curriculum at each grade level	2	6	5	1	1	3	6	3

□ – intended by 67% of the A+ countries ■ – intended by 83% of the A+ countries ■ – intended by 100% of the A+ countries

- A+ countries from 1995 TIMSS
- Topics intended at each grade by at least two thirds of A+ countries.
- On average an A+ country would have 1–6 more topics per grade level in its complete curriculum.

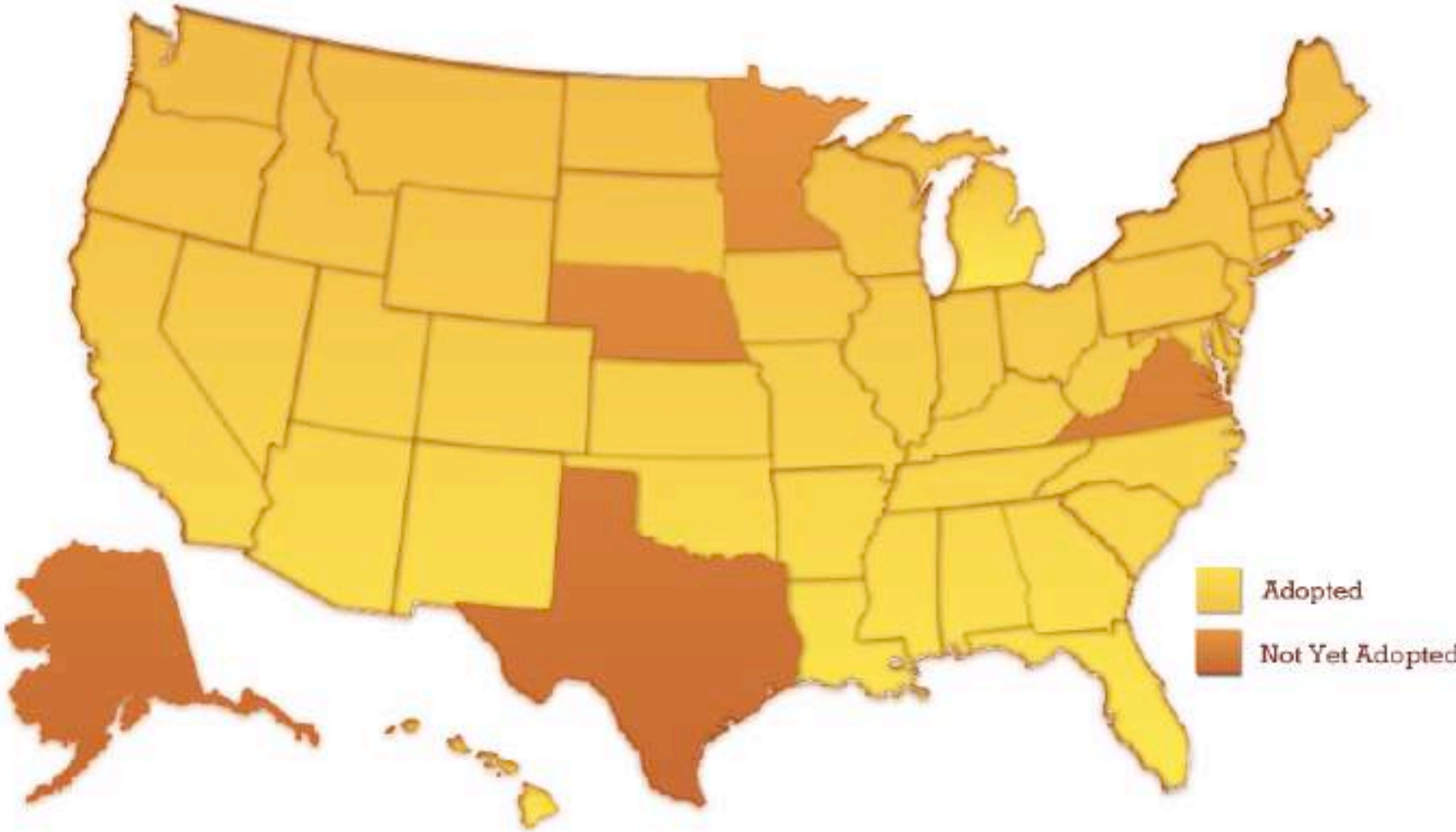
COMMON CORE STATE STANDARDS FOR

Mathematics

Process

- Led by two organizations of states, the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO)—not directed by the federal government.
- 48 states participated in the development of the standards
- 3-person writing team, 59-person work team, 19-person feedback group
- Based on progressions provided by the work team
- Rapid release and review by work team, feedback group, states, national organizations
- Finished Summer 2010, validated by a validation committee, states started making decisions whether or not to adopt the standards

Adoption of the Common Core State Standards, 2012



Sources considered

- Standards of other countries and states
- Mathematics education research
- The structure of the subject
- National reports
- Commissioned research reviews and progressions
- Opinions of many groups and people

What should standards look like?

O Level Mathematics Syllabus — Secondary One

Secondary One	
Topic/Sub-topics	**Content**
Algebraic representation and formulae	Include: • using letters to represent numbers • interpreting notations: * ab as $a \times b$ * $\frac{a}{b}$ as $a \div b$ * a^2 as $a \times a$, a^3 as $a \times a \times a$, a^2b as $a \times a \times b$, . . . * $3y$ as $y - y - y$ or $3 \times y$ * $\frac{3 \pm y}{5}$ as $(3 \pm y) \div 5$ or $\frac{1}{5} \times (3 \pm y)$ • evaluation of algebraic expressions and formulae • translation of simple real-world situations into algebraic expressions • recognising and representing number patterns (including finding an algebraic expression for the nth term)
Algebraic manipulation	Include:

Number and Operations

Standard for Grades 6–8

Instructional programs from prekindergarten through grade 12 should enable all students to—

Expectations

In grades 6–8 all students should–

Understand numbers, ways of representing numbers, relationships among numbers, and number systems

- work flexibly with fractions, decimals, and percents to solve problems;
- compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line;
- develop meaning for percents greater than 100 and less than 1;
- understand and use ratios and proportions to represent quantitative relationships;
- develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation;

A sample state

MATHEMATICS STANDARD ARTICULATED BY GRADE LEVEL

GRADE 6

Strand 1: Number Sense and Operations

Every student should understand and use all concepts and skills from the previous grade levels. The standards are designed so that new learning builds on preceding skills and are needed to learn new skills. Communication, Problem-solving, Reasoning & Proof, Connections, and Representation are the process standards that are embedded throughout the teaching and learning of mathematical strands.

Concept 1: Number Sense Understand and apply numbers, ways of representing numbers, the relationships among numbers and different number systems.
PO 1. Express fractions as ratios, comparing two whole numbers (e.g., ¾ is equivalent to 3:4 and 3 to 4).
PO 2. Compare two proper fractions, improper fractions, or mixed numbers.
PO 3. Order three or more proper fractions, improper fractions, or mixed numbers.
PO 4. Determine the equivalency between and among fractions, decimals, and percents in contextual situations.

What should standards look like?

- NCTM: 14 standards + 7 pages of narrative for Grades 6–8 Number and Operations
- State: 82 standards + 0 pages of narrative for Grade 6 Number and Operations

What should standards look like?

- Flat lists of performance objectives of even grain size, designed to be delivered into the hands of assessment writers without further human intervention?
- Guides to curriculum? A list of things that teachers have to "cover"?
- A statement of what we want our children to understand and be able to do?

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- A statement of what we want our children to understand and be able to do?

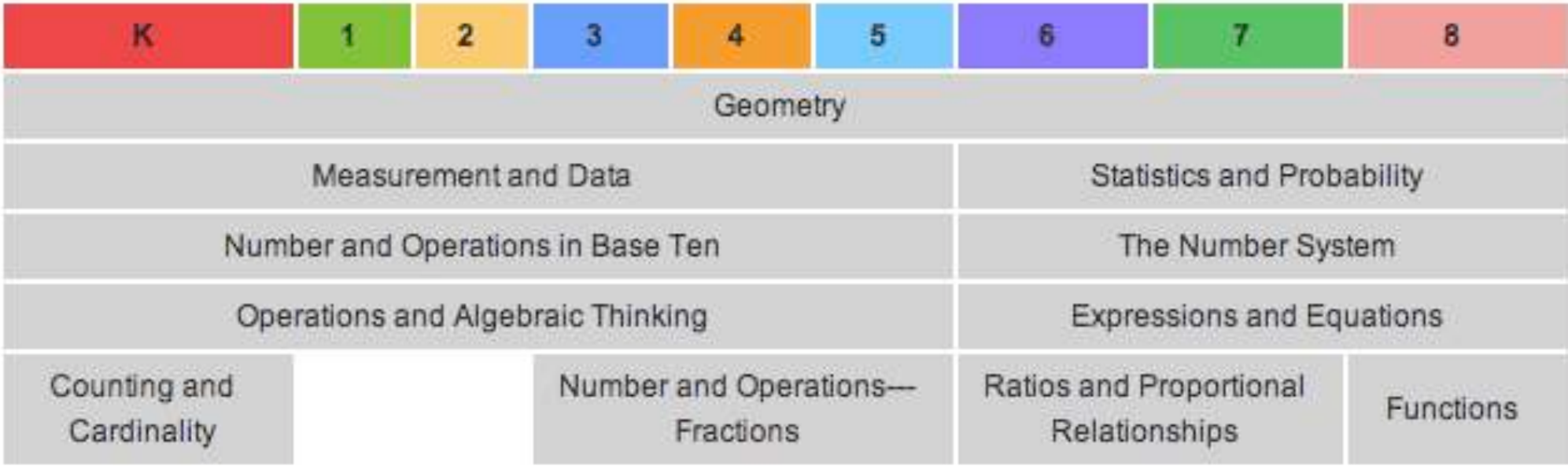
Design principles for the Common Core

Focus attending to fewer topics in greater depth at any given grade level, giving teachers and students time to complete that grade's learning.

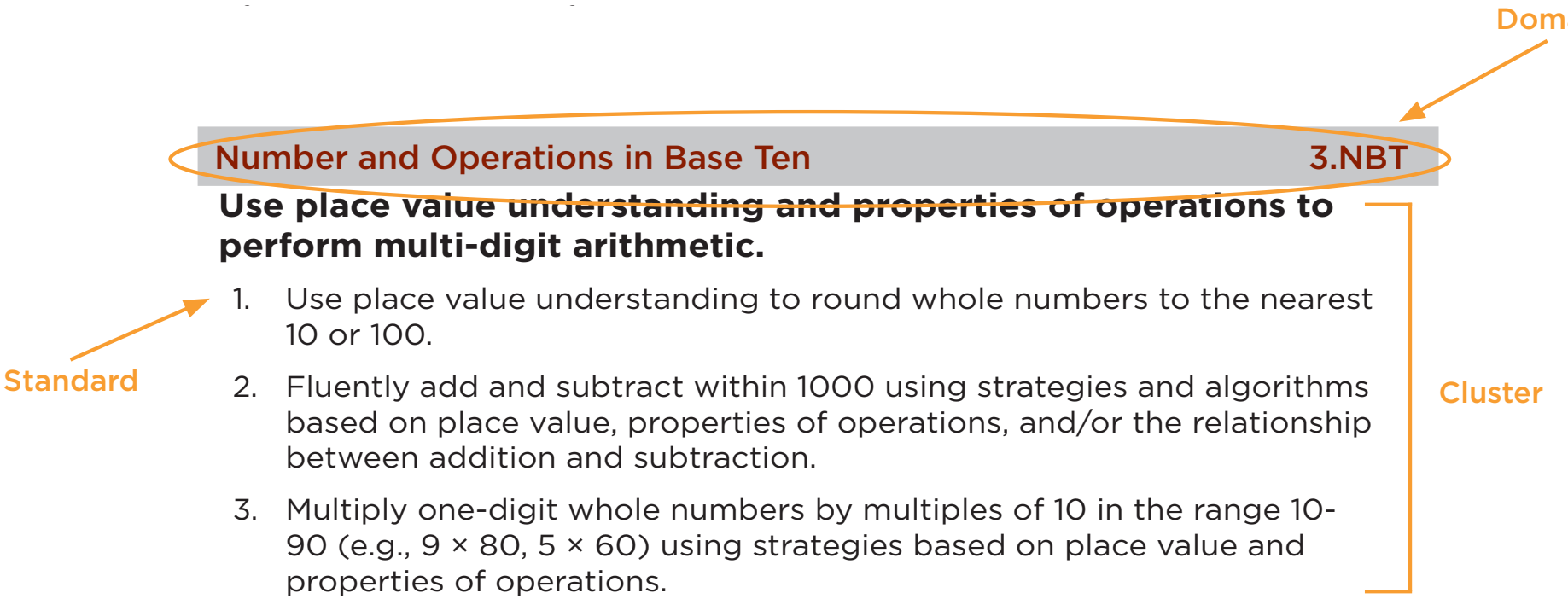
Coherence attending to the structure of mathematics and the natural pathways through that structure, where "natural" means taking into account both the imperatives of logic and the imperatives of cognitive development in designing the sequence of ideas.

Rigor balancing conceptual understanding, procedural fluency, and meaningful applications of mathematics.

Standards for Mathematical Content, K–8



Arrangement of content standards



- *Content standards* define what students should understand and be able to do
- *Clusters* are groups of related standards
- *Domains* are larger groups that progress across grades

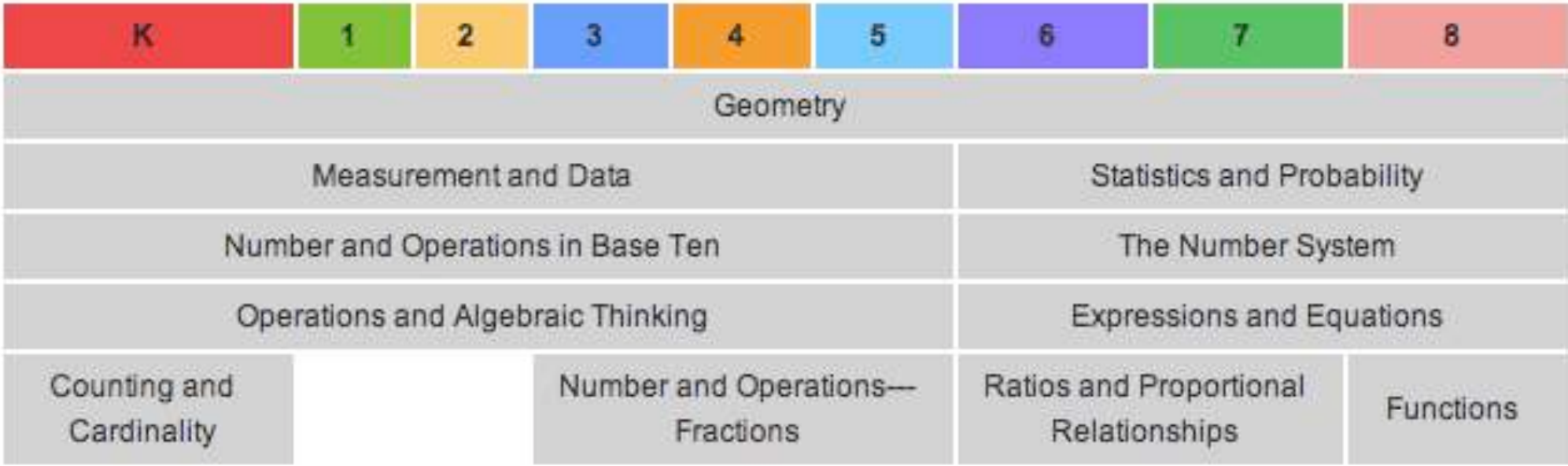
Standards for Mathematical Content, High School

- Number and Quantity
  - ...
- Algebra
  - Seeing Structure in Expressions
  - Arithmetic with Polynomials and Rational Expressions
  - Creating Equations
  - Reasoning with Equations and Inequalities
- Functions
  - Interpreting Functions
  - Building Functions
  - Linear, Quadratic, and Exponential Models
  - Trigonometric Functions
- Modeling
  - ...
- Geometry
  - ...
- Statistics and Probability
  - ...

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Taking focus seriously



Standards not in the number operations domains often support that work:

2.MD.10

Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems.

Taking focus seriously: the case of patterns

- Trickle of pattern standards in elementary school
  - **3.OA.9.** Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations.
  - **4.OA.5.** Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.
  - **5.OA.3.** Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane.
- In middle school, further preparation for functions is provided in the domains Ratios and Proportional Relationships and Expressions and Equations.
- The function concept finally makes its appearance in its own domain in Grade 8, and becomes a major conceptual category in high school.

Focus in Grade 3

Operations and Algebraic Thinking

- **Represent and solve problems involving multiplication and division.**
- **Understand properties of multiplication and the relationship between multiplication and division.**
- **Multiply and divide within 100.**
- **Solve problems involving the four operations, and identify and explain patterns in arithmetic.**

Number and Operations in Base Ten

- **Use place value understanding and properties of operations to perform multi-digit arithmetic.**

Number and Operations — Fractions

- **Develop understanding of fractions as numbers.**

Measurement and Data

- **Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.**
- **Represent and interpret data.**
- **Geometric measurement: understand concepts of area and relate area to multiplication and addition.**
- **Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.**

Geometry

- **Reason with shapes and their attributes.**

Focus in Grade 7

Ratios and Proportional Reasoning

- **Analyze proportional relationships and use them to solve real-world and mathematical problems.**

The Number System

- **Apply and extend previous understandings of operations with fractions to add, subtract, multiply and divide rational numbers.**

Expressions and Equations

- **Use properties of operations to generate equivalent expressions.**
- **Solve real-life and mathematical problems using numerical and algebraic expressions and equations.**

Geometry

- **Draw, construct and describe geometrical figures and describe the relationships between them.**
- **Solve real-life and mathematical problems involving angle measure, area, surface area and volume.**

Statistics and Probability

- **Use random sampling to draw inferences about a population.**
- **Draw informal comparative inferences about two populations.**
- **Investigate chance processes and develop, use, and evaluate probability models.**

Say goodbye to . . .

- The love affair with patterns
- Mindless medians
- Least common denominators
- Factor trees
- Reducing fractions
- “Simplifying”

$$p^{-\frac{1}{4}}$$

Focus in high school

There is a world of difference between a student who can summon a mnemonic device to expand a product such as $(a + b)(x + y)$ and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics . . .

— CCSSM, p. 5

Focus in high school

- Algebra
  - Seeing Structure in Expressions
  - Arithmetic with Polynomials and Rational Expressions
  - Creating Equations
  - Reasoning with Equations and Inequalities
- Functions
  - Intepreting Functions
  - Building Functions
  - Linear, Quadratic, and Exponential Models
  - Trigonometric Functions
- Modeling
  - ...
- Geometry
  - ...
- Statistics and Probability
  - ...

The payoff for focus

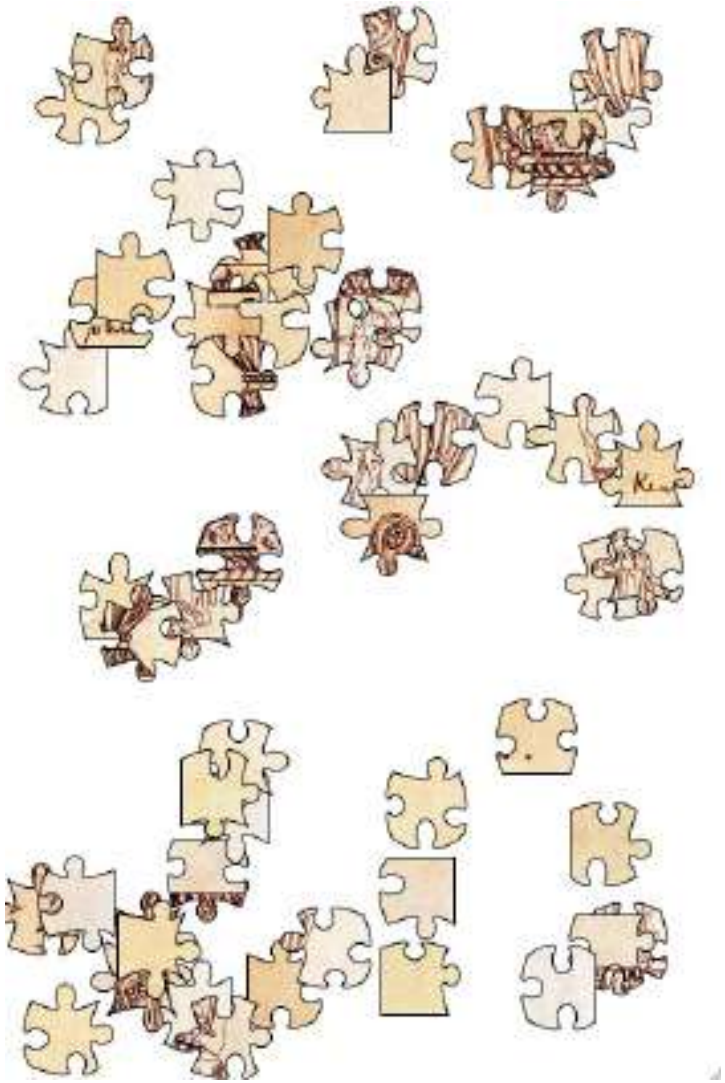
- Taking focus seriously means delaying favored topics until their time, which will be a difficult shift for the educational system in the U.S.
- In high school, the subject matter focus broadens: algebra, functions, geometry, statistics, probability, modeling.
- Focus in high school means not so much a small number of topics as a concentration of skills and practice into a small number of underlying principles.

Coherence

A Grecian Urn

Coherence

A Grecian Urn



Standards for a Grecian Urn

Breaking with the tradition of “equal grain size”

Understand place value (Grade 2)

2.NBT.1. Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

a. 100 can be thought of as a bundle of ten tens—called a “hundred.”

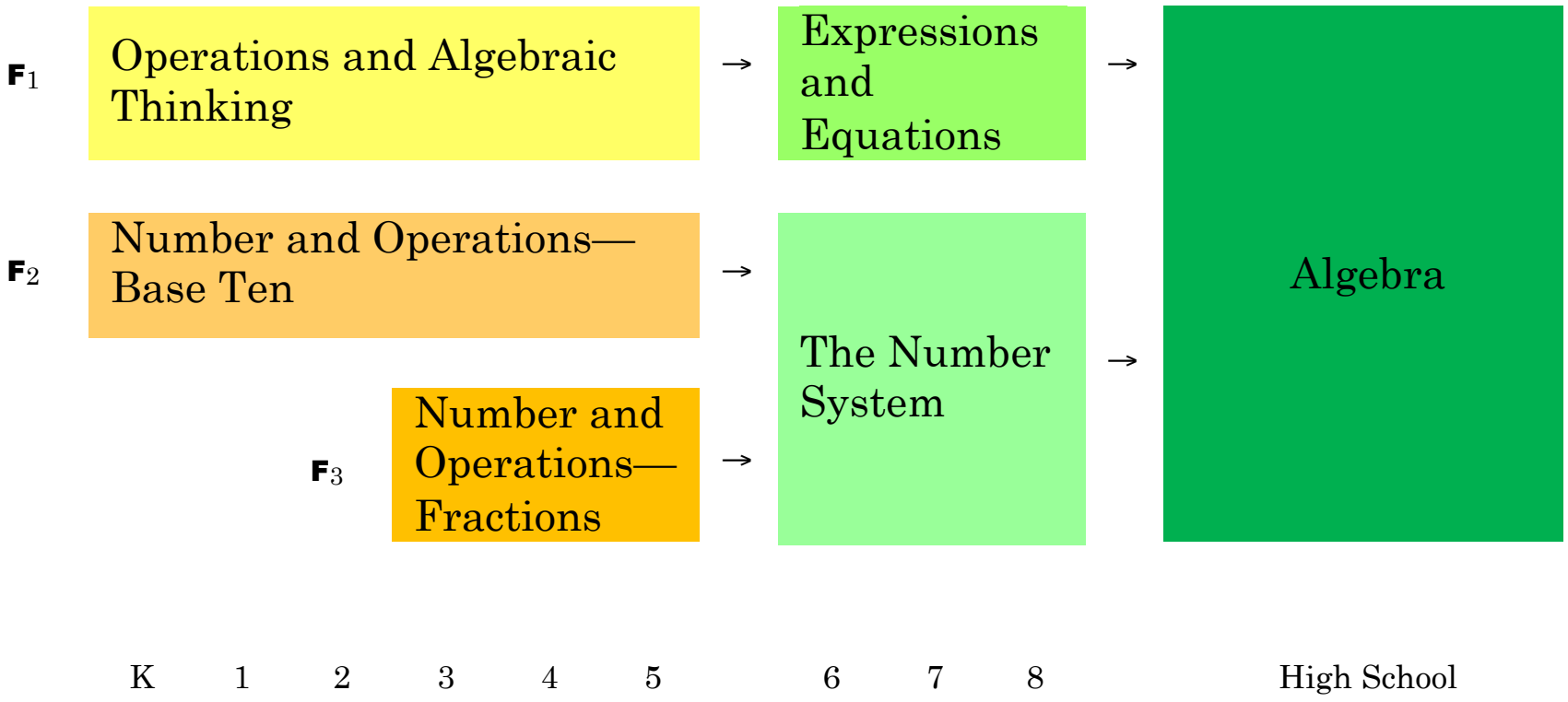
b. The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).

2.NBT.2. Count within 1000; skip-count by 5s, 10s, and 100s.

2.NBT.3 . . .

2.NBT.4 . . .

Coherence flows from focus



Rigor

Understanding **1.OA.7.** Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false.

Fluency **2.OA.2.** Fluently add and subtract within 20 using mental strategies.[1] By end of Grade 2, know from memory all sums of two one-digit numbers.

Applications **2.MD.5.** Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

[1] See standard 1.OA.6 for a list of mental strategies.

“Understand”

- Understand and apply properties of operations and the relationship between addition and subtraction (Grade 2)
- Understand concepts of area and relate area to multiplication and to addition (Grade 3)
- Understand ratio concepts and use ratio reasoning to solve problems (Grade 6)
- Understand congruence and similarity using physical models, transparencies, or geometry software (Grade 8)
- Understand solving equations as a process of reasoning and explain the reasoning (High School)
- Understand and evaluate random processes underlying statistical experiments (High School)

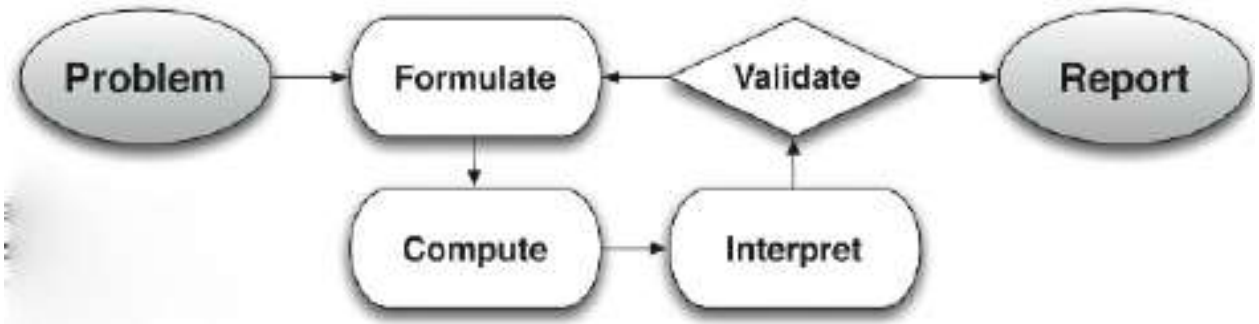
Fluency requirements

- Single digit addition facts in Grade 2
- Single digit multiplication facts in Grade 3
- Addition and subtraction within 1000 in Grade 3
- Addition and subtraction using standard algorithm in Grade 4 (addition and subtraction starts in Grade 1)
- Multiplication using standard algorithm in Grade 5 (multiplication and division starts in Grade 3)
- Division using standard algorithm in Grade 6
- All operations on decimals using standard algorithm in Grade 6
- Solving linear equations with rational coefficients in Grade 7

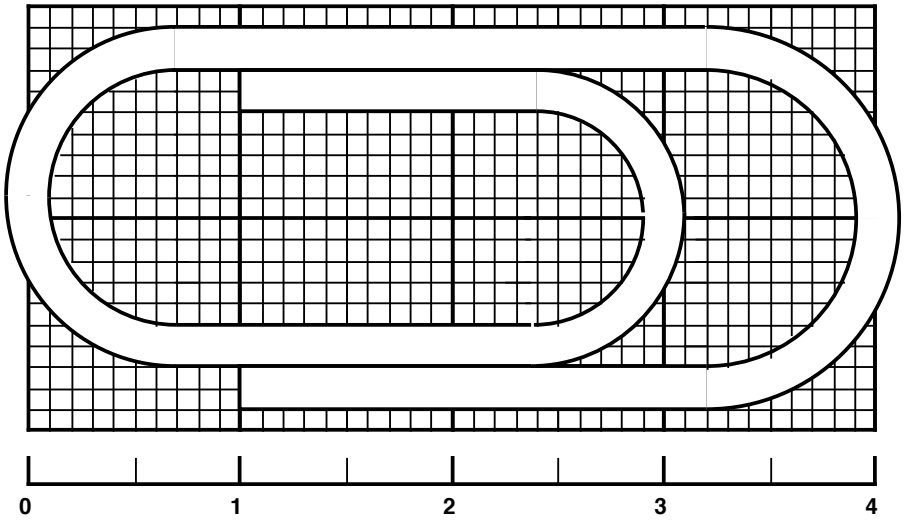
Putting mathematics to work

7.EE.3. Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.

SMP.4. Model with mathematics.



The paper clip below is just over 4 cm long.



How many paper clips like this may be made from a straight piece of wire 10 meters long?

Focus, coherence, and rigor work together

A Grecian Urn

Focus, coherence, and rigor work together

A Grecian Urn

- It's pretty clear what the focus is here; nobody is going to say this is all about grape leaves.

Focus, coherence, and rigor work together

A Grecian Urn

- It's pretty clear what the focus is here; nobody is going to say this is all about grape leaves.
- But the grape leaves are important; details large and small make this a coherent work of art.

Focus, coherence, and rigor work together

A Grecian Urn

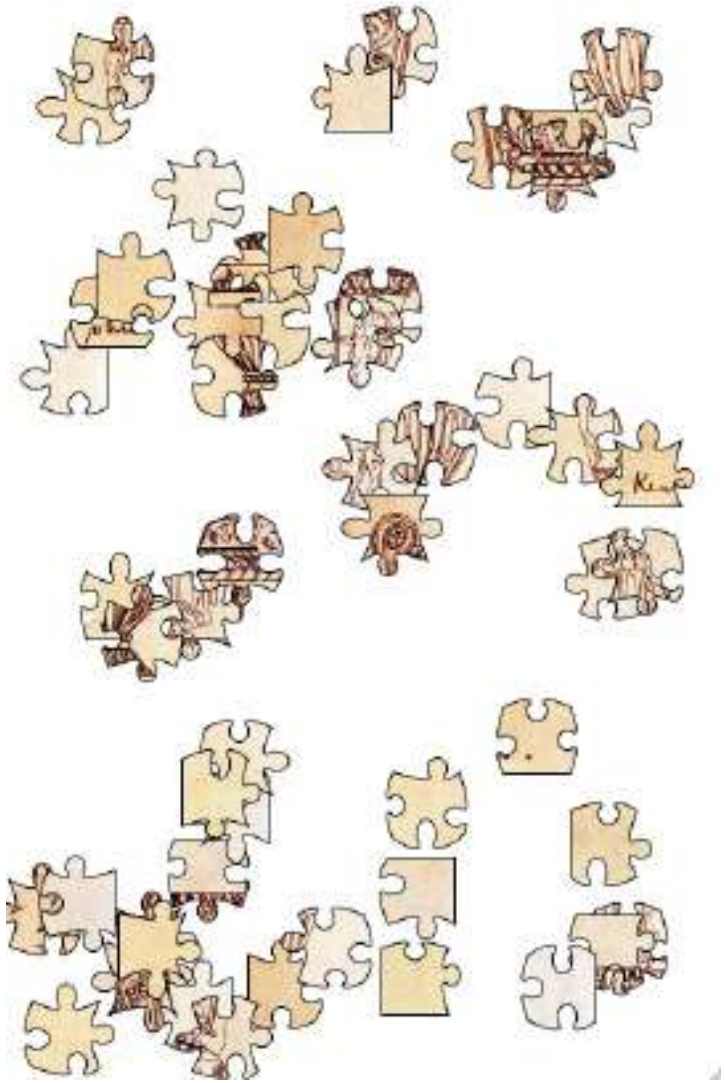
- It's pretty clear what the focus is here; nobody is going to say this is all about grape leaves.
- But the grape leaves are important; details large and small make this a coherent work of art.
- Craftsmanship shows in the balance between between art, technique, and story.

Focus, coherence, and rigor work together

A Grecian Urn

- It's pretty clear what the focus is here; nobody is going to say this is all about grape leaves.
- But the grape leaves are important; details large and small make this a coherent work of art.
- Craftsmanship shows in the balance between between art, technique, and story.
- Rigor is the balance that produces students who can use what they know; who show craft in using mathematics.

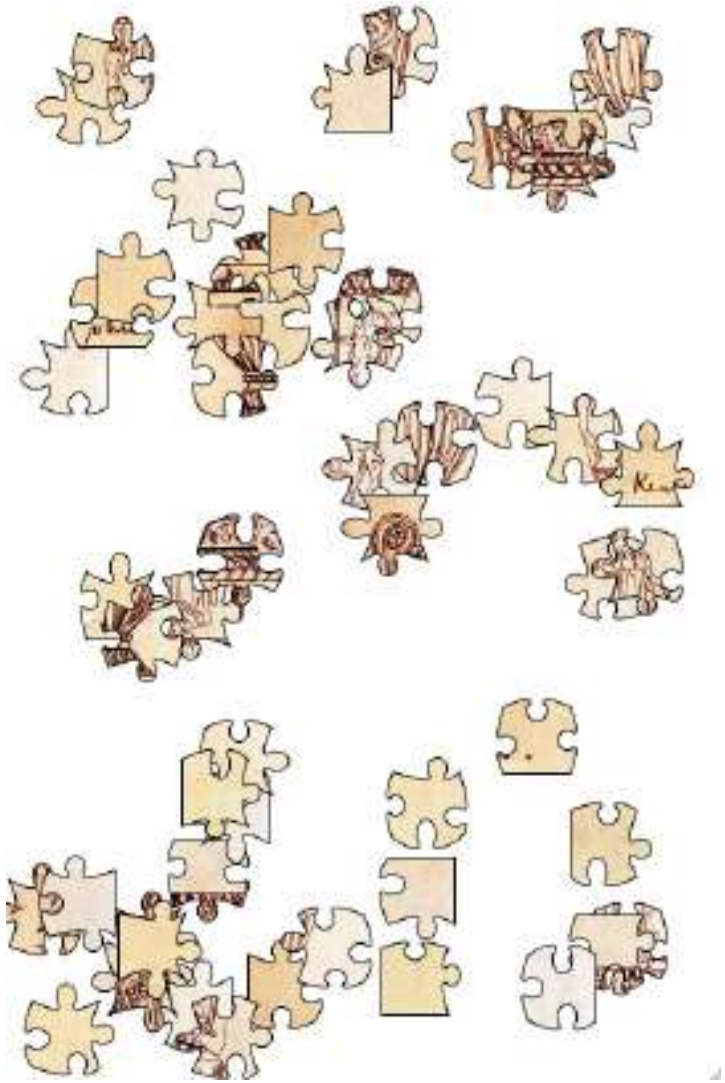
The danger of checklists



Standards for a Grecian Urn

- Standards cannot be read individually, in isolation.

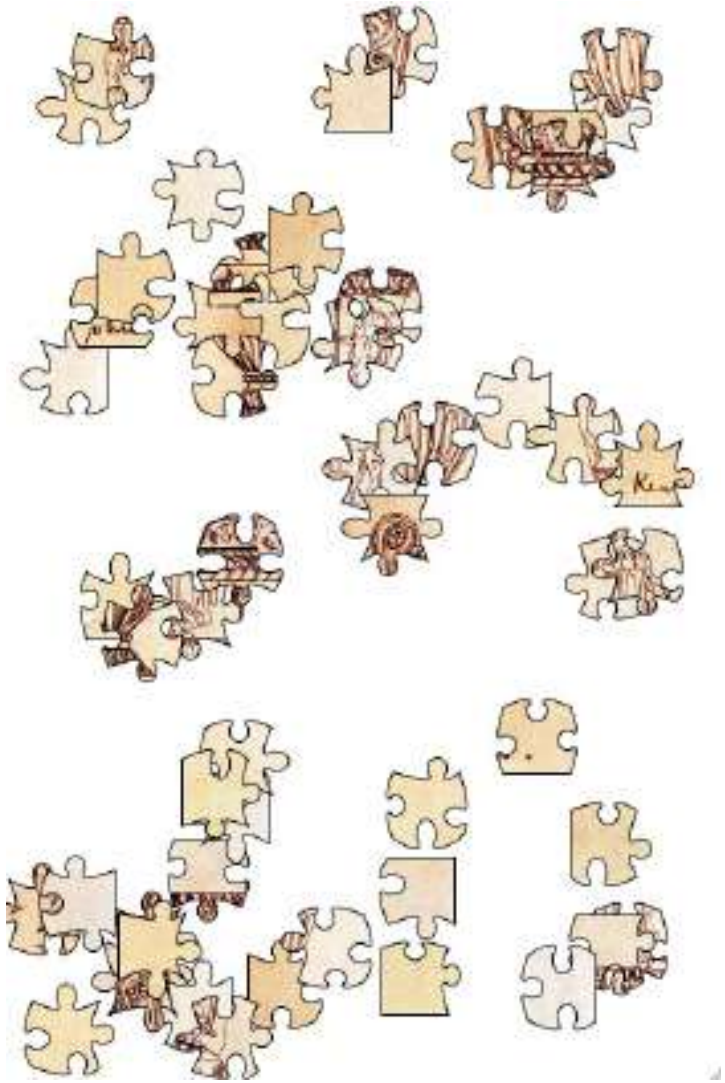
The danger of checklists



Standards for a Grecian Urn

- Standards cannot be read individually, in isolation.
- The cluster headings are part of the standards.

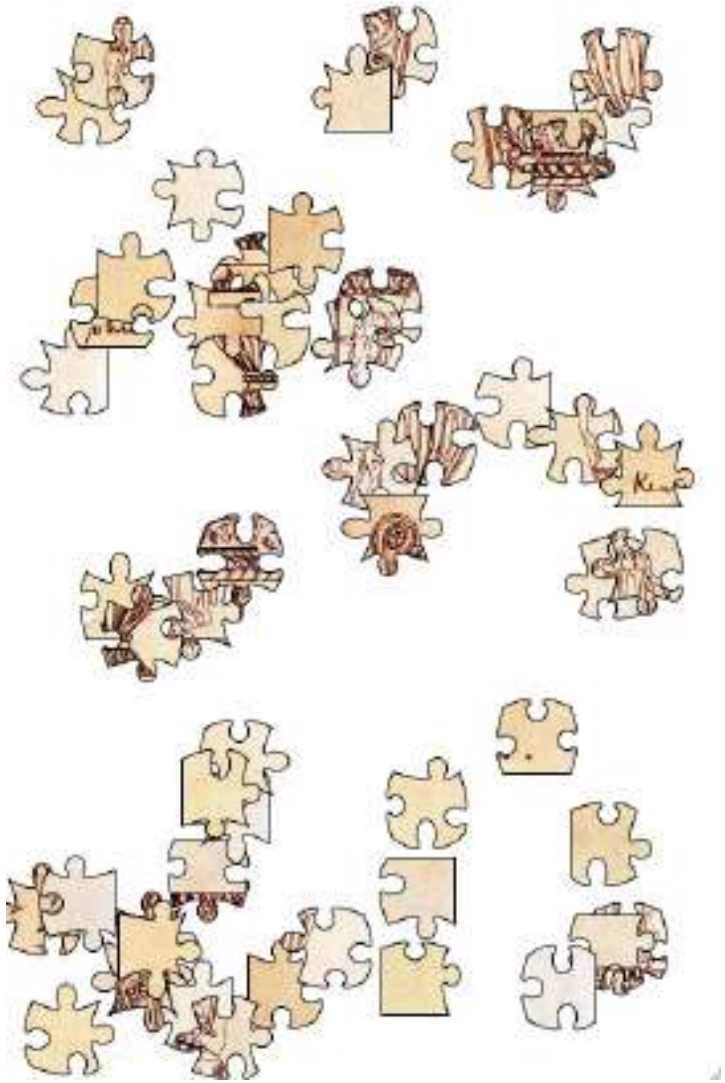
The danger of checklists



Standards for a Grecian Urn

- Standards cannot be read individually, in isolation.
- The cluster headings are part of the standards.
- The domain headings are part of the standards.

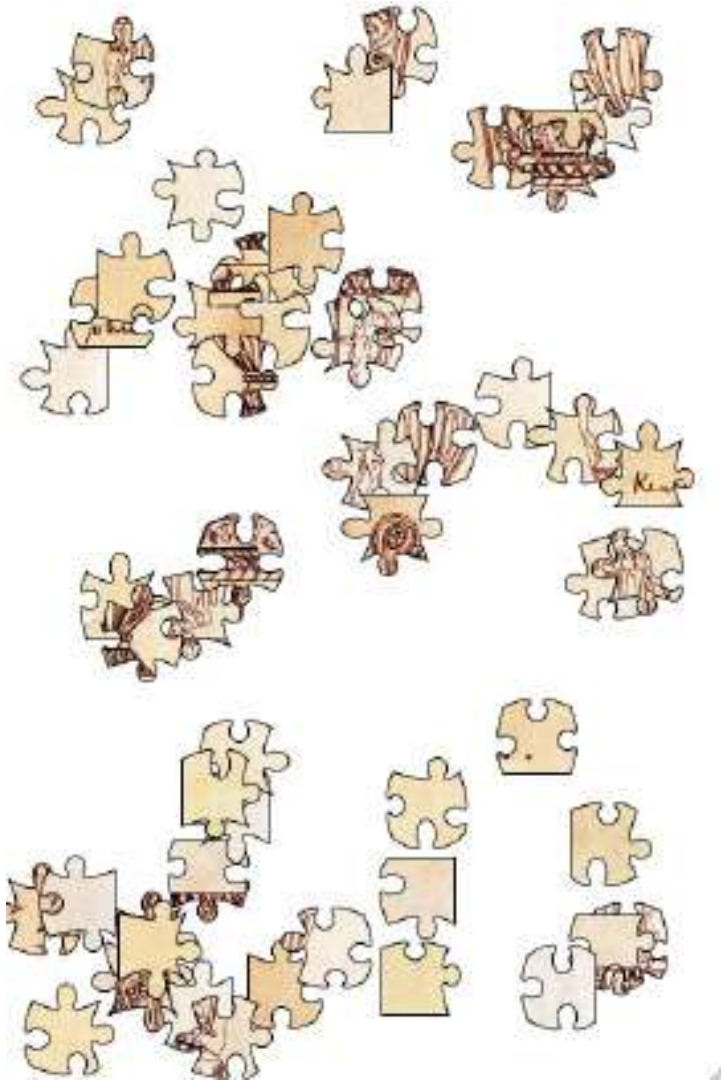
The danger of checklists



Standards for a Grecian Urn

- Standards cannot be read individually, in isolation.
- The cluster headings are part of the standards.
- The domain headings are part of the standards.
- The connections across and within grade levels are part of the standards.

The danger of checklists



Standards for a Grecian Urn

- Standards cannot be read individually, in isolation.
- The cluster headings are part of the standards.
- The domain headings are part of the standards.
- The connections across and within grade levels are part of the standards.
- The structure is the standards.

Tools

- Progressions (http://ime.math.arizona.edu/progressions/)
- Illustrative Mathematics Project (http://illustrativemathematics.org).
- My blog, Tools for the Common Core, http://commoncoretools.wordpress.com
- Institute for Mathematics & Education (http://ime.math.arizona.edu/commoncore/)

Comparison of CCSS with A+ composite

Topic	1	2	3	4	5	6	7	8
Whole Number: Meaning	●	●	●	●	●			
Whole Number: Operations	●	●	●	●	●			
Measurement Units	●	●	●	●	●	●	●	
Common Fractions	●	●	●	●	●	●		
Equations & Formulas				●	●	●	●	●
Data Representation & Analysis	●	●	●	●	●	●	●	●
2-D Geometry: Basics		●	●	●	●	●	●	●
2-D Geometry: Polygons & Circles	●	●	●	●	●	●	●	●
Measurement: Perimeter, Area & Volume		●	●	●	●	●	●	●
Rounding & Significant Figures				●	●			
Estimating Computations				●		●		●
Whole Numbers: Properties of Operations	●	●	●	●	●			
Estimating Quantity & Size								
Decimal Fractions				●	●	●		
Relation of Common & Decimal Fractions			●	●	●	●		
Properties of Common & Decimal Fractions					●	●		
Percentages							●	
Proportionality Concepts						●	●	●
Proportionality Problems						●	●	●
2-D Geometry: Coordinate Geometry					●	●	●	●
Geometry: Transformations				●				●
Negative Numbers, Integers, & Their Properties						●	●	
Number Theory				●				
Exponents, Roots & Radicals						●		
Exponents & Orders of Magnitude								
Measurement: Estimation & Errors								
Constructions Using Straightedge & Compass							●	●
3-D Geometry	●	●			●	●	●	●
Geometry: Congruence & Similarity								●
Rational Numbers & Their Properties						●	●	●
Patterns, Relations & Functions								●
Proportionality: Slope & Trigonometry								●

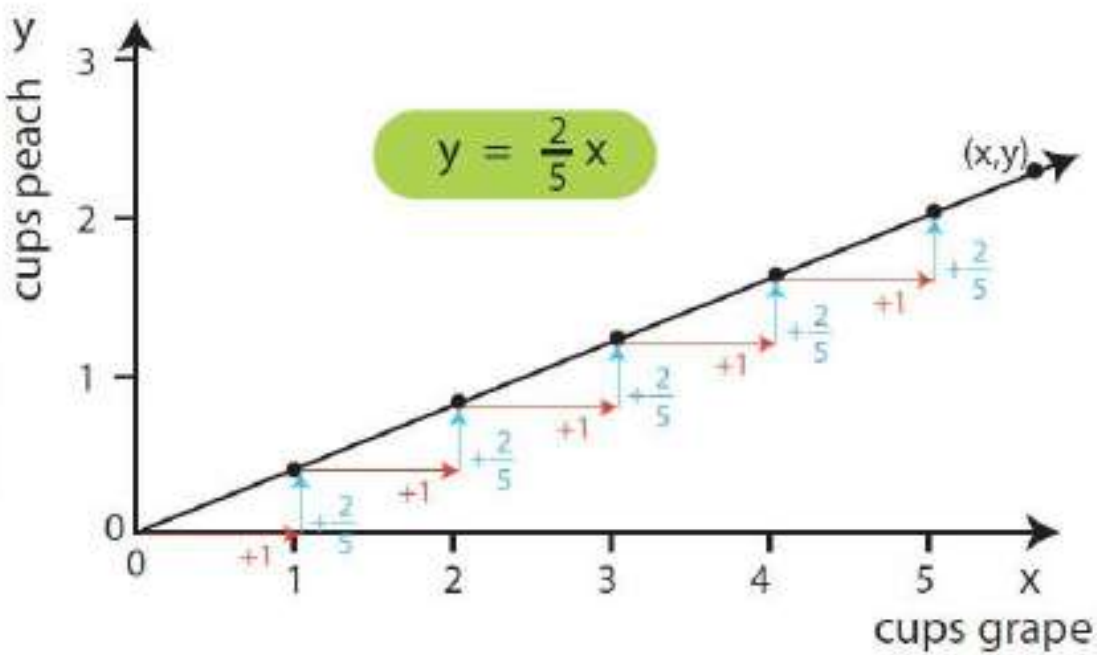
Topic Intended in Common Core Standards ●
Intended in More Than Half of Top Achieving Countries

- The number of extra topics per grade level in CCSS is comparable with A+ countries.

Look for and express regularity in repeated reasoning

For every 5 cups grape juice, mix in 2 cups peach juice.

x cups grape	y cups peach
(0)	(0)
5	2
1	$\frac{2}{5}$
2	$2 \cdot \frac{2}{5}$
3	$3 \cdot \frac{2}{5}$
4	$4 \cdot \frac{2}{5}$
x	$x \cdot \frac{2}{5}$



For each 1 unit you move to the right, move up $\frac{2}{5}$ of a unit.

When you go 2 units to the right, you go up $2 \cdot \frac{2}{5}$ units.
When you go 3 units to the right, you go up $3 \cdot \frac{2}{5}$ units.
When you go 4 units to the right, you go up $4 \cdot \frac{2}{5}$ units.
When you go x units to the right, you go up $x \cdot \frac{2}{5}$ units.

Starting from $(0, 0)$, to get to a point (x, y) on the graph, go x units to the right, so go up $x \cdot \frac{2}{5}$ units.

Therefore $y = x \cdot \frac{2}{5}$

MP7. Look for and make use of structure—High School

Do the equations have a solution? Give a reason for your answer that does not depend on solving the equation.

1. $\dfrac{t+2}{3+t} = 1$
2. $\dfrac{3+t}{3-t} = 1$
3. $\dfrac{t-2}{2-t} = 1$

Shifts in the Standards

A sequence of Grade 3 tasks about placing fractions on the number line, arranged in order of increasing sophistication.

- Locating Fractions Less Than One on the Number Line (http://illustrativemathematics.org/illustrations/168)
- Locating Fractions Greater than One on the Number Line (http://illustrativemathematics.org/illustrations/173)
- Closest to 1/2 (http://illustrativemathematics.org/illustrations/171)
- Find 1 (http://illustrativemathematics.org/illustrations/169)
- Find 2/3 (http://illustrativemathematics.org/illustrations/170)
- Which is Closer to 1? (http://illustrativemathematics.org/illustrations/ 172)