CCGPS Frameworks Student Edition

Mathematics

CCGPS Analytic Geometry
Unit 6: Modeling Geometry

Dr. John D. Barge, State School Superintendent
"Making Education Work for All Georgians"

Unit 6
Modeling Geometry

Table of Contents

OVERVIEW

In this unit students will:

- use Algebra to model Geometric ideas.
- spend time developing equations from geometric definition of circles and parabolas.
- address equations in standard and general forms.
- graph by hand and by using graphing technology.
- address parabolas as a type of conic section and make connections as to how parabolas were addressed in previous course work.
- develop the idea of algebraic proof in conjunction with writing formal geometric proofs.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under "Evidence of Learning" be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics. Students will not only have the opportunity to use different forms of circles and parabolas, but also to derive the formulas for themselves. This should lead to a deeper understanding of the conceptual ideas of circles and parabolas. Throughout the process, students will be developing formulas and algebraic proofs.

KEY STANDARDS

Expressing Geometric Properties with Equations

Translate between the geometric description and the equation for a conic section

MCC9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

MCC9-12.G.GPE.2 Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically

MCC9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).*

MCC9-12.A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line y = –3x and the circle* $x^2 + y^3 = 3$.

RELATED STANDARDS

Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. **Make sense of problems and persevere in solving them.** High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.** High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making

mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

8. **Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

ENDURING UNDERSTANDINGS

- Write and interpret the equation of a circle
- Derive the formula for a circle using the Pythagorean Theorem
- Recognize, write, and interpret equations of parabolas
- Prove properties involving parabolas
- Prove properties involving circles
- Apply algebraic formulas and ideas to geometric figures and definitions
- The intersection of a line and a quadratic figure is the point where the two equations are equal.

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- number sense
- computation with whole numbers and decimals, including application of order of operations
- addition and subtraction of common fractions with like denominators
- applications of the Pythagorean Theorem
- usage of the distance formula, including distance between a point and a line.
- Finding a midpoint
- graphing on a coordinate plane
- completing the square

- operations with radicals
- methods of proof

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The website below is interactive and includes a math glossary.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website. Links to external sites are particularly useful.

- **Center of a Circle**: The point inside the circle that is the same distance from all of the points on the circle.

- **Circle:** The set of all points in a plane that are the same distance, called the radius, from a given point, called the center.

- **Diameter**: The distance across a circle through its center. The line segment that includes the center and whose endpoints lie on the circle.

- **Directrix of a Parabola:** every point on a parabola is equidistant from a fixed point (focus) and a fixed line (**directrix**).

- **Focus of a Parabola:** every point on a parabola is equidistant from a fixed point (**focus**) and a fixed line (directrix).

- **General Form of a Circle:** $Ax^2+By^2+Cx+Dy+E=0$

- **Parabola**: the set of all points in a plane equidistant from a fixed line, the directrix, and a fixed point, the focus, in the plane

- **Pythagorean Theorem**: A theorem that states that in a right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs.

- **Radius:** The distance from the center of a circle to any point on the circle. Also, the line segment that has the center of the circle as one endpoint and a point on the circle as the other endpoint.

- **Standard Form of a Circle:** $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center and r is the radius.

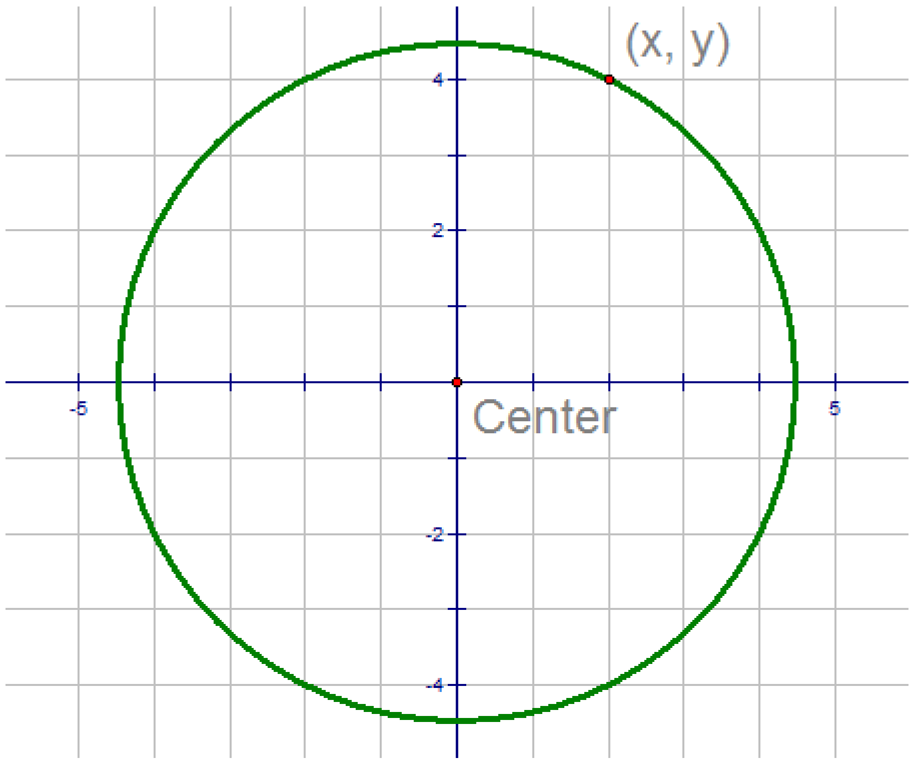
DERIVING THE GENERAL EQUATION OF A CIRCLE

Standard Addressed

MCC9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Part 1: Finding the Radius

Consider the circle below. Notice the center is at the origin and a point is on the circle (x, y).



Answer the following questions or perform the requested constructions.

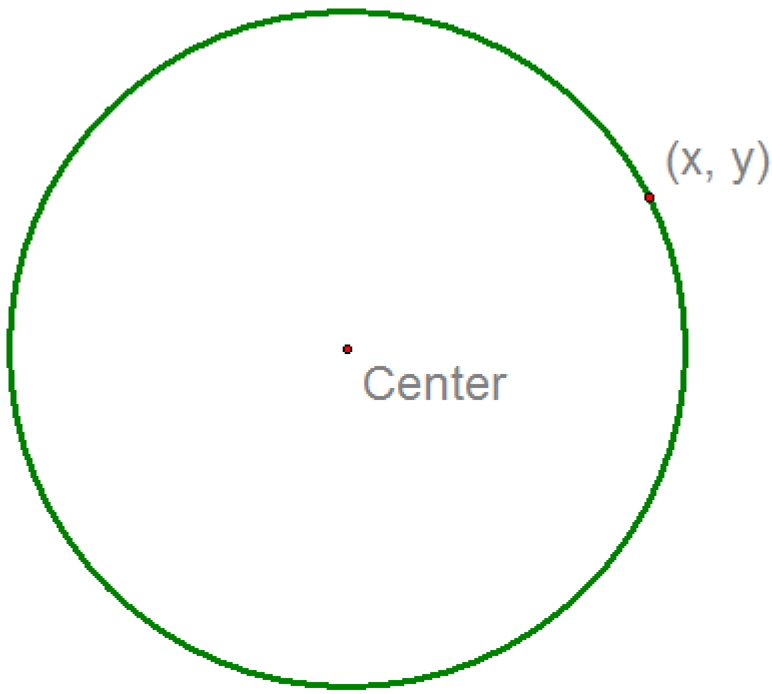
1. Construct a line segment from the center to the point (x, y) on the circle and label it "r". What is this line segment called?

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point (x, y)?

3. What is the measure of r? Explain your method for calculating it.

Part 2: Circles Centered at the Origin.

Consider the circle below. The center is located at the origin.



Answer the following questions or perform the requested constructions.

1. Construct a radius from the center to the point (x, y). Label it "r".

2. Construct a right triangle with r as the hypotenuse. What are the coordinates of the point where the legs meet?

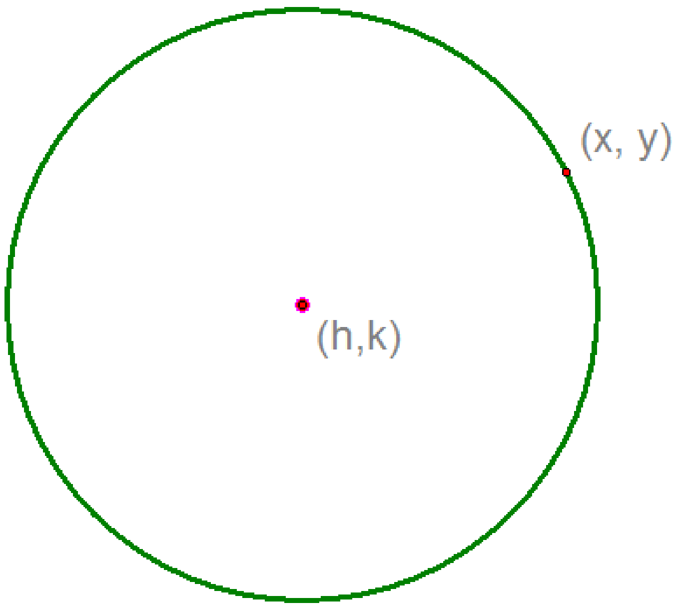
3. Write an expression for the distance from the center to the point from #2. Label the triangle accordingly.

4. Write an expression for the distance from (x, y) to the point from #2. Label the triangle accordingly.

5. Now use your method from part one to write an expression for r^2

Part 3: Circles centered anywhere!

In the previous section, you found that $x^2 + y^2 = r^2$. This is the general equation for a circle centered at the origin. However, circles are not always centered at the origin. Use the following circle and directions to find the general equation for a circle centered anywhere.



Answer the following questions and perform the requested constructions.

1. Construct a radius between (h, k) and (x, y). Then create a right triangle with the radius as the hypotenuse. Find the coordinates for the point where the legs meet.

2. Write an expression for the distance between (x, y) and the point from #1. Label the triangle.

3. Write an expression for the distance between (h, k) and the point from #1. Label the triangle.

4. Now write an expression for r^2.

CONVERTING STANDARD FORM TO GENERAL FORM

Standard Addressed

MCC9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

In Task 1, you used the Pythagorean Theorem to derive: $(x-h)^2+(y-k)^2=r^2$, which is known as the **Standard Form of a Circle**.

By expanding the binomial terms this equation can be written as

$x^2-2hx+h^2+y^2-2ky+k^2=r^2$ or

$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$

Then, by using variables for coefficients, and realizing that h^2, k^2 and r^2 are all real numbers and can be added, we derive the **General Form equation of a Circle**:

$$Ax^2+By^2+Cx+Dy+E=0$$

******Note: In order to be a circle, A and B must be equal.*

Occasionally, it becomes necessary to convert the equation of a circle from Standard to General Form. Take the circle with a center at (3, 4) and a radius of 6, for example.

The Standard Equation would be: $(x-3)^2+(y-4)^2=6^2$

By expanding the binomial terms, we would then have: $x^2-6x+9+y^2-8y+16=36$.
Grouping the monomials according to degree would yield:
$x^2+y^2-6x-8y+9+16-36=0$

Through arithmetic, the General Form equation would be: $x^2+y^2-6x-8y-11=0$

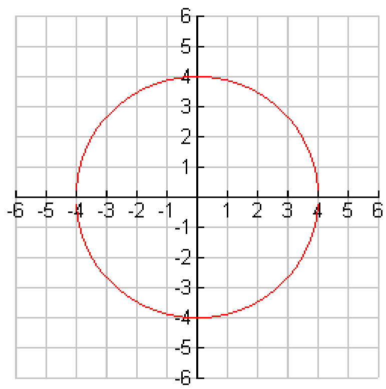
Write the General form equations for the following circles:

1. A circle with center *(1, -6)* and radius *4*
2. A circle with center *(6, 8)* and radius *10*
3. A circle with center *(0, 3)* and radius $2\sqrt{3}$
4. A circle with center *(-0.5, 5.5)* and radius *8.4*
5. A circle with center *(a, b)* and radius *c*

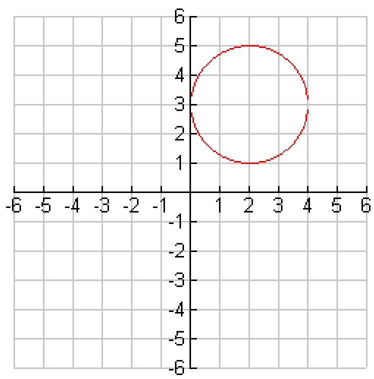
COMPLETING THE SQUARE IN A CIRCLE?

1. Write equations for the following circle graphs in both standard form and general form.

a.



b.



2. Take a moment to compare your General form and Standard form equations. Which form would be easier to graph? Why do you think so?

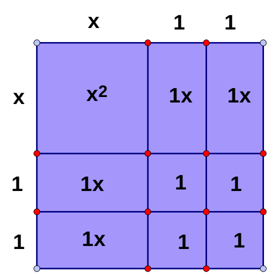
In Task 2, you converted the Standard form equation to a General form equation. Today you will convert from General form to Standard form.

To change from general form to standard form, it is necessary to "complete the square" for x and y. **Completing the square** is an algebraic tool used to change equations of circles given in general form, $Ax^2 + By^2 + Cx + Dy + E = 0$, to standard form, $(x-h)^2 + (y-k)^2 = r^2$. Standard form is the form used to graph circles.

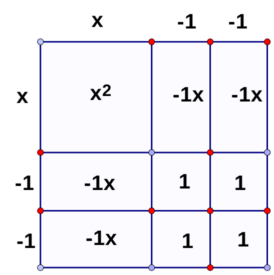
Perfect squares are numbers or expressions which have exactly two identical factors.

$(2)(2) = 4$ $(-5)(-5) = 25$ $(3x)(3x) = 9x^2$ $(x + 2)(x + 2) = x^2 + 4x + 4$

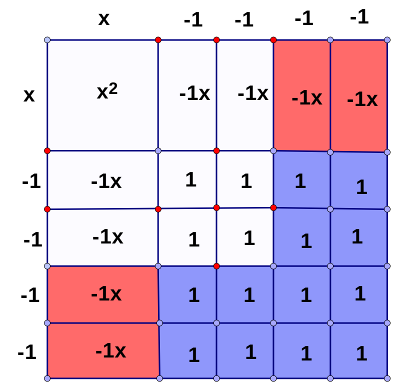
Consider the following geometric area models of three perfect squares. The area is given as both factors and as a quadratic expression.



area = (x + 2)(x + 2)
= x2 + 4x + 4



area = (x - 2)(x - 2)
= x2 - 4x + 4



area = (x - 4)(x - 4)
= x2 - 8x + 16

3. Find the products of the following expressions.

a. $(x + 1)^2 = (x + 1)(x + 1) =$
b. $(x - 3)^2 = (x - 3)(x - 3) =$
c. $(x - 5)^2 = (x - 5)(x - 5) =$
d. $(x + 7)^2 = (x + 7)(x + 7) =$
e. $(x + n)^2 = (x + n)(x + n) =$

4. Each of the products in #3 is a perfect square. Use the results of #3 to complete each of the squares and show their factored forms. Include Geometric diagrams to illustrate the perfect squares.

a. $x^2 + 20x +$ ____ $= (x +$ ____$)^2$

b. $x^2 - 12x +$ ____ $= (x -$ ____$)^2$

c. $x^2 + 18x +$ ____ $= (x +$ ____$)^2$

d. $x^2 - 7x +$ ____ $= (x -$ ____$)^2$

e. $x^2 + 2nx +$ ____ $= (x +$ ____$)^2$

In order to graph a circle given in general form, it is necessary to change to standard form. In order to rewrite $x^2 + y^2 + 2x - 4y - 11 = 0$ in standard form to facilitate graphing, it is necessary to complete the square for both x and y.

$\mathbf{x^2 + y^2 + 2x - 4y - 11 = 0}$	
$(x^2 + 2x \quad) + (y^2 - 4y \quad) = 11$	group the x and y terms
$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 11 + 1 + 4$	complete the square on x and y balance the equation by adding 1 and 4 to both sides of the equation
$\mathbf{(x + 1)^2 + (y - 2)^2 = 16}$	factor

circle with center at (-1, 2) and radius 4

To change $x^2 + y^2 + 2x - 4y - 11 = 0$ to standard form, it is necessary to remove a factor of 2 before completing the square for both x and y.

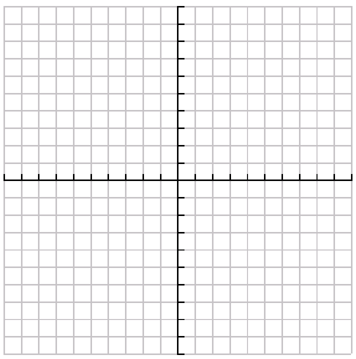
$\mathbf{2x^2 + 2y^2 - 4x + 6y - 4 = 0}$	remove a factor of 2
$(x^2 - 2x \quad) + (y^2 + 3y \quad) = 2$	group the x and y terms
$(x^2 - 2x + 1) + (y^2 + 3y + {}^9/_4) = 2 + 1 + {}^9/_4$	complete the square on x and y balance the equation by adding 1 and ${}^9/_4$ to both sides of the equation
$(x - 1)^2 + (y + {}^3/_2)^2 = {}^{21}/_4$	factor

$\mathbf{(x - 1)^2 + (y + 1.5)^2 = 5.25}$

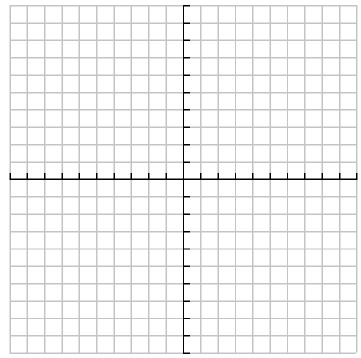
circle with center at (1, -1.5) and radius 5.25

5. Change the following equations to standard form. Graph the circles; identify the centers and the radii.

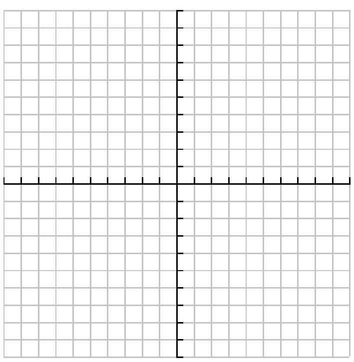
a. $x^2 + y^2 + 2x + 4y - 20 = 0$



b. $x^2 + y^2 - 4y = 0$



c. $x^2 + y^2 - 6x - 10y = 2$



GRAPHING CIRCLES ON A GRAPHING CALCULATOR

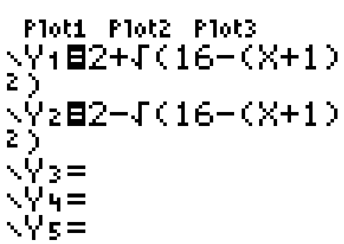
Standard Addressed

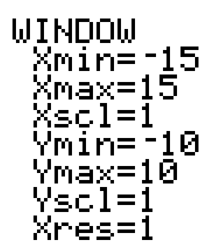
MCC9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

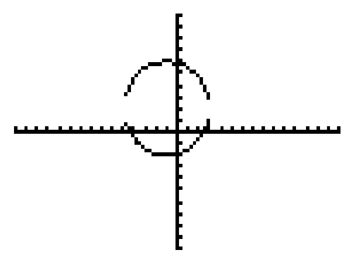
To graph the circle $x^2 + y^2 + 2x - 4y - 11 = 0$ using a TI83/TI84 it is necessary to solve for y after changing the equation to standard form.

$$(x + 1)^2 + (y - 2)^2 = 16$$
$$(y - 2)^2 = 16 - (x + 1)^2$$
$$\sqrt{(y-2)^2} = \pm\sqrt{16-(x+1)^2}$$
$$y - 2 = \pm\sqrt{16-(x+1)^2}$$
$$y = 2 \pm \sqrt{16-(x+1)^2}$$

Enter this result as two functions $y1 = 2 + \sqrt{(16-(x+1)^2)}$ and $y2 = 2 - \sqrt{(16-(x+1)^2)}$. In order to minimize the distortion caused by the rectangular screen of the graphing calculator, use a window with a x to y ratio of 3 to 2. Otherwise circles appear as ellipses.







1. Write the equations as you would enter them in a graphing calculator and list an appropriate graphing window to show the entire circle graph.

   a. $x^2 + y^2 + 2x + 4y - 20 = 0$

   b. $x^2 + y^2 - 4y = 0$

   c. $x^2 + y^2 - 6x - 10y = 2$

RADIO STATION LISTENING AREAS

Standard Addressed

MCC9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

1. Radio signals emitted from a transmitter form a pattern of concentric circles. Write equations for three concentric circles.

2. Randy listens to radio station WYAY from Atlanta. Randy's home is located 24 miles east and 32 miles south of the radio station's transmitter. His house is located on the edge of WYAY's maximum broadcast range.

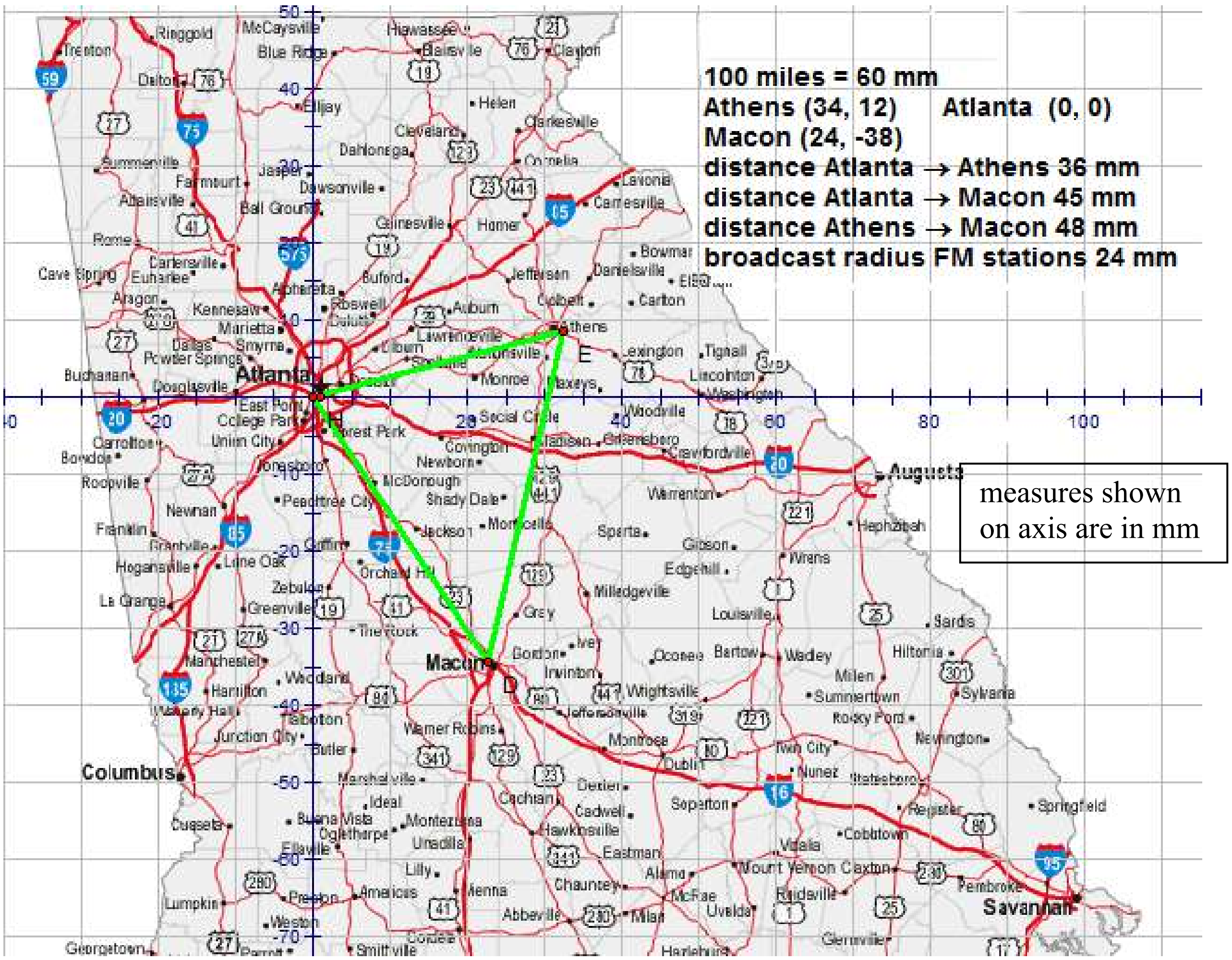
   a. When a radio signal reaches Randy's house, how far has it traveled? Sketch WYAY's listening area of the partial map of Georgia given. On the map let Atlanta's WYAY have coordinates (0, 0) and use the scale as 100 miles = 60 mm.

   b. Find an equation which represents the station's maximum listening range.

   c. Determine four additional locations on the edge of WYAY's listening area, give coordinates correct to tenths.

3. Randy likes to listen to country music. Several of his friends have suggested that in addition to WYAY, he try station WXAG in Athens and WDEN in Macon. WYAY, WXAG, and WDEN are FM stations which normally have an average broadcast range of 40 miles. Use the map included with the indicated measures to answer the following questions.

   a. Given the location of Randy's home, can he expect to pick up radio signals from WXAG and WDEN? Explain how you know.


100 miles = 60 mm
Athens (34, 12)
Atlanta (0, 0)
Macon (24, -38)
distance Atlanta → Athens 36 mm
distance Atlanta → Macon 45 mm
distance Athens → Macon 48 mm
broadcast radius FM stations 24 mm
measures shown on axis are in mm
Atlanta
Athens
Macon
Augusta
Columbus
Savannah
E
D

DERIVING THE GENERAL EQUATION OF A PARABOLA

Standard Addressed

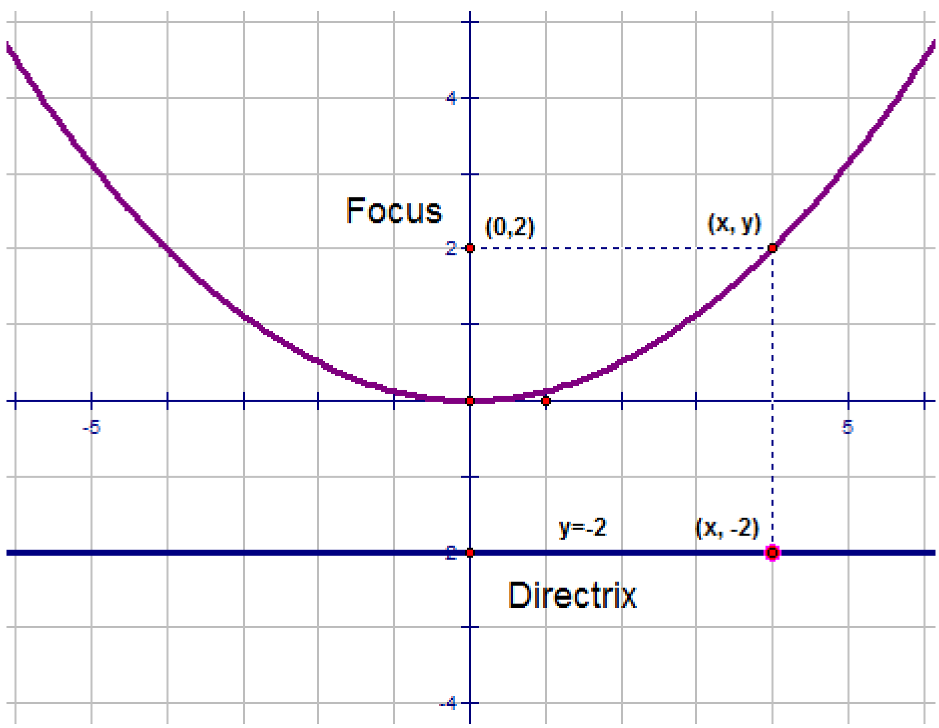
MCC9-12.G.GPE.2 Derive the equation of a parabola given a focus and directrix.

Parabolas were studied in the previous unit as quadratic functions where the equations were based on the position of the vertex and additional points were found using values of x on either side of the axis of symmetry. Equations were in the vertex form, $y = a(x-h)^2 + k$, or in general quadratic form $y = ax^2 + bx + c$. We will use these forms and expand our study by including the geometric definition of the parabola.

Definition: A parabola is the set of all points that are the same distance from a fixed point, the focus, and a fixed line, the directix.

Part 1: Finding a specific equation.

Consider the parabola below. Notice the vertex at the origin and a point on the parabola (x, y).



Follow the steps to find the equation of the parabola.

1. Write equations for the distance from the focus to *(x, y)* and the distance from the directrix to *(x, y)*.

2. Because the definition of a parabola says that the distances you wrote in #1 are the same, write an equation stating this fact.

3. Now square both sides to eliminate the radicals. Square the binomials and write your answer below.

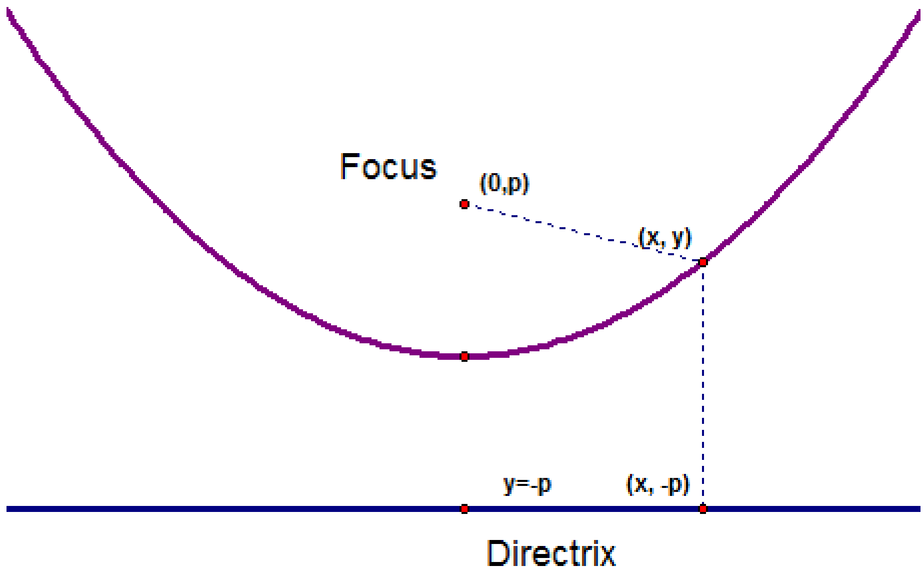
4. Collect like terms and solve the equation for y.

5. Write your equation in terms of y below.

Part 2: Writing the general equation for a Parabola.

Consider the graph of the parabola below. It's vertex is at the origin, the focus is at *(0,p)*, and the directrix is the line *y=-p*.

Follow the steps to find the equation of the parabola.



1. Write equations for the distance from the focus to *(x, y)* and the distance from the directrix to *(x, y)*.

2. Because the definition of a parabola says that the distances you wrote in #1 are the same, write an equation stating this fact.

3. Now square both sides to eliminate the radicals. Square the binomials and write your answer below.

4. Collect like terms and solve the equation for *y*.

5. Write your equation in terms of *y* below.

6. What does the *"p"* represent in your equation?

Consider that the vertex of a parabola is not always at the origin. It could be translated to any point on the graph. Use what you know about transformations and your formula above to write the following equations.

7. Vertex at *(2, 3)*; *p=4*.

8. Vertex at *(-4, 8); p=-3*.

9. Vertex at *(5, -9); p=0.5*.

10. Vertex at *(h, k); p=p*.

PARABOLAS IN OTHER DIRECTIONS

Standard Addressed

MCC9-12.G.GPE.2 Derive the equation of a parabola given a focus and directrix.

When we studied quadratic functions in the last unit, we used only parabolas that opened up or down. When we use a geometric definition of parabolas, we also can write equations for parabolas that open right or left.

Consider the parabola $y = \frac{1}{4}x^2$. When we compare it to our general form equation, we see that:

$$\frac{1}{4p} = \frac{1}{4}$$
$$4p = 4$$
$$p = 1$$

Because *p* is positive, we know that the parabola is going to open in a positive direction. If *p* had been negative, it would have opened in a negative direction.

We also know from our work with quadratics, that the x^2 term represents a parabola that opens in a vertical direction.

The general form for a horizontal parabola is $x - h = \frac{1}{4p}(y-k)^2$. When we compare the parabola $x - 2 = -2(y+1)^2$, we see that:

$$\frac{1}{4p} = -2$$
$$-8p = 1$$
$$p = -\frac{1}{8}$$

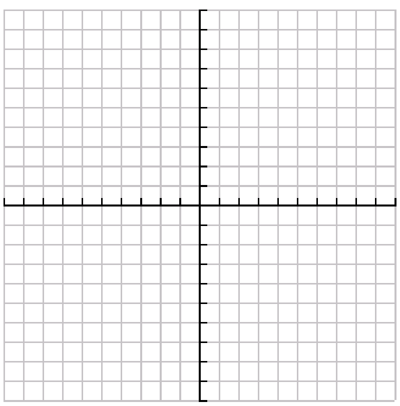
Because *p* is negative, we know that the parabola opens in a negative direction. Also notice that the vertex is *(2, -1)*.

Complete the following chart:

Equation	Vertex	P=	Opens	Squared Variable?	Horizontal or Vertical?
$x+4=16(y-3)^2$	(-4, 3)	$\frac{1}{64}$	(Positive) Right	Y	Horizontal
$y=\frac{1}{4}(x+6)^2$					
$x+1=-6(y+10)^2$					
$y=\frac{3}{4}(x-9)^2$					

1. In each of the following problems, list the coordinates of the vertex, the coordinates of the focus, the equation of the directrix, and graph the parabola.

a. $(y-3)^2=-12(x+2)$

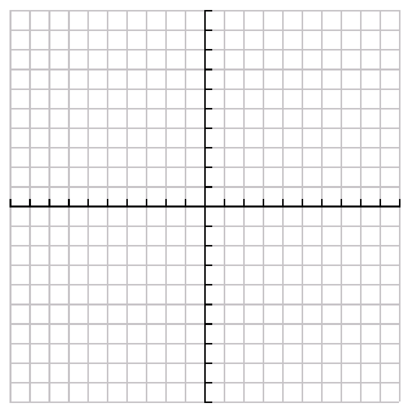


Vertex: ________________

Focus: ________________

Directrix: ______________

b. $y=-\frac{1}{8}(x-1)^2$

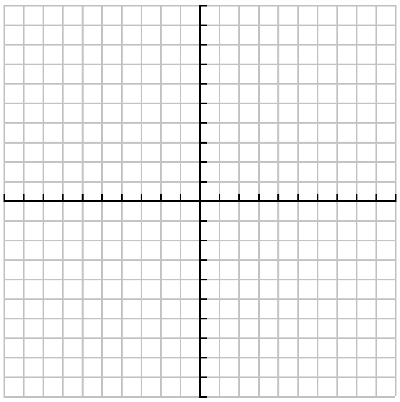


Vertex: __________________

Focus: ___________________

Directrix: _________________

c. $x = (y-4)^2$

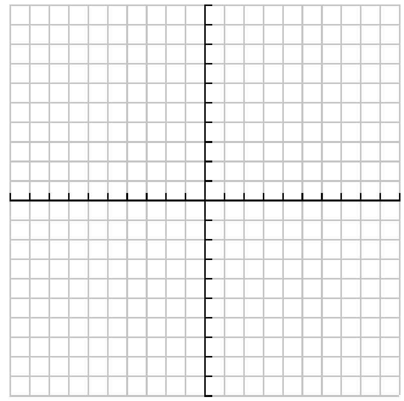


Vertex: _______________

Focus: _______________

Directrix: _____________

d. $(x+1)^2 = 2(y+3)$

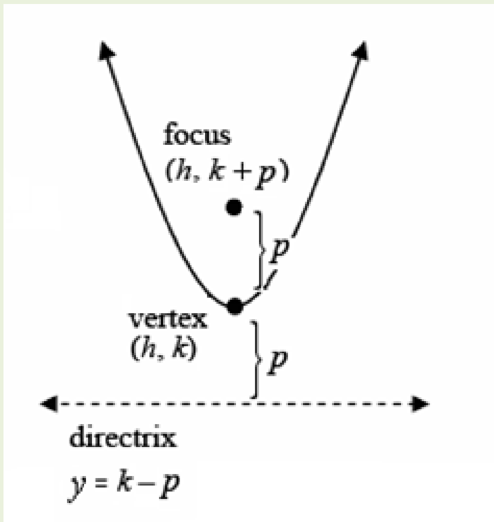


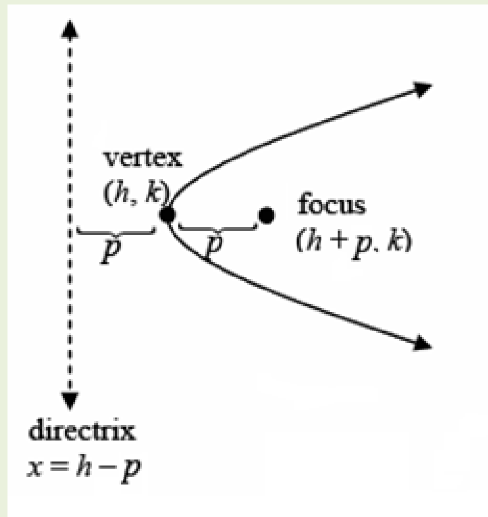
Vertex: _________________

Focus: __________________

Directrix: ________________

Summary of Parabola Information





$$Ax^2 + By^2 + Cx + Dy + E = 0$$
where A = 0 or B = 0 but not both = 0

vertex at (h, k)

Horizontal Directrix

$$y - k = \frac{1}{4p}(x - h)^2$$

$p > 0$ opens up
$p < 0$ opens down
focus $(h, k + p)$
directrix $y = k - p$
axis of symmetry $x = h$

Vertical Directrix

$$x - h = \frac{1}{4p}(y - k)^2$$

$p > 0$ opens right
$p < 0$ opens left
focus $(h + p, k)$
directrix $x = h - p$
axis of symmetry $y = k$

WRITING THE EQUATIONS OF PARABOLAS

Standard Addressed

MCC9-12.G.GPE.2 Derive the equation of a parabola given a focus and directrix.

In this task, you will be writing the equations of a parabola given certain information about it. First, here is something to think about:

1. What relationship does the vertex have with the focus and directrix?

Example: Write the equation of the parabola with focus (3, 5), and directrix $y = -1$.

Solution: First off, we know that this is a vertical parabola because the directrix is a horizontal line. This means we will be writing our equation in the form of $y-k=\frac{1}{4p}(x-h)^2$. Now we can find the vertex. It is the midpoint between the focus and the point directly below it on the directrix. In this case, it is *(3,- 1)*. So the midpoint is:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{3+3}{2},\frac{5+(-1)}{2}\right)=(3,2)$$

The final step is to find the value for p. This is simply the distance from the focus to the vertex. From *(3, 5)* to *(3, 2)* the distance is 3and we keep it positive because the parabola is opening up.

Now we write the equation.

$$y-2=\frac{1}{4(3)}(x-3)^2$$

$$y-2=\frac{1}{12}(x-3)^2$$

Remember, p is the distance from the vertex to the focus. It also determines which direction the parabola will open.

For the following information, write the equation for the parabola in standard form.

2. Focus: (2, -5) ; Directrix: $x = -2$

3. Vertex: (5, -3); Focus: (0, -3)

4. Focus: (1.5, -4); Vertex: (1, -4)

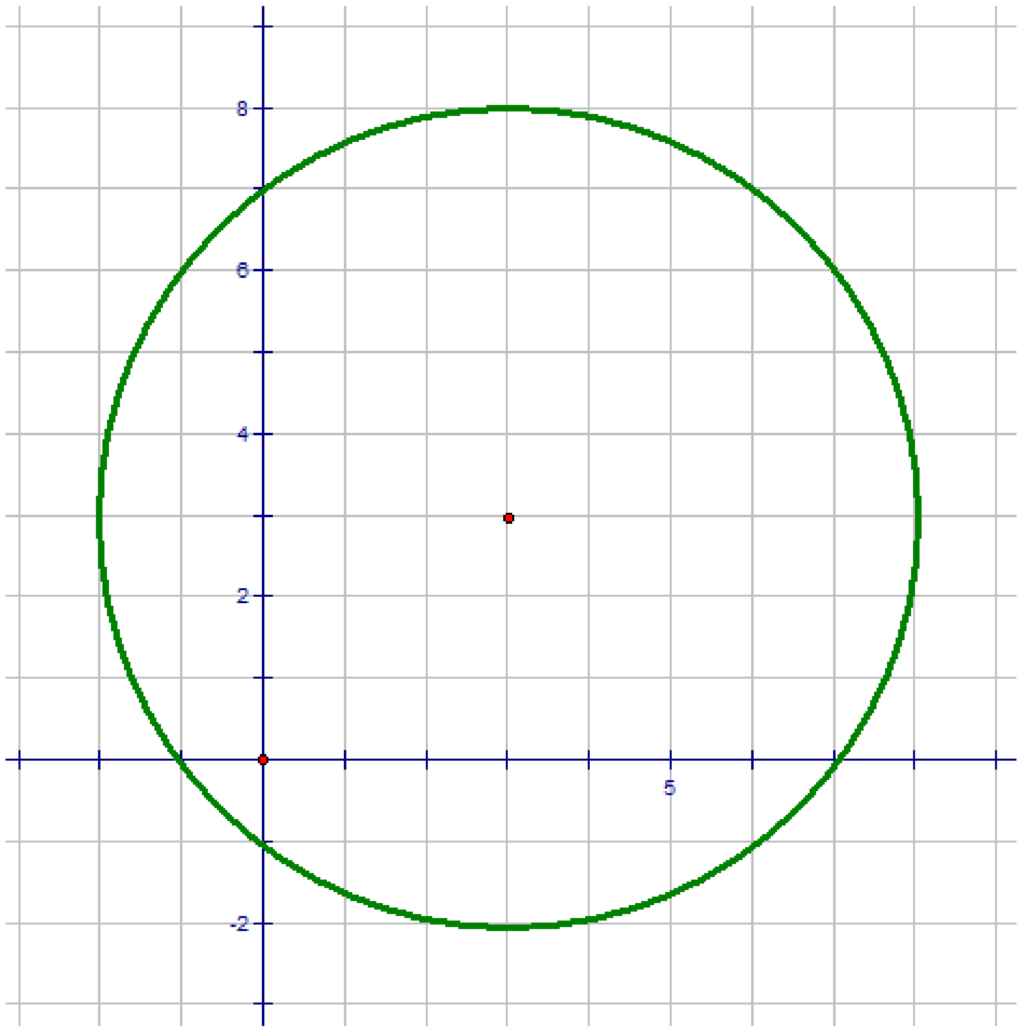
5. Focus: (2, -4.5); Directrix: $y = -5.5$

THE INTERSECTION OF A LINE AND A QUADRATIC

Standard Addressed

MCC9-12.A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = -3x$ *and the circle* $x^2 + y^3 = 3$.

Using this circle, graph the given lines, estimate the intersection points and then check your estimates by solving algebraically.



1. $y = -2$
2. $x = 3$
3. $y = x + 5$
4. $2x + y = 7$
5. $y = \frac{3}{4}x - 3$
6. $2x + 3y = 18$

ALGEBRAIC PROOF

Standard Addressed

MCC9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).*

Already in Analytic Geometry you have been exposed to and written proofs about geometric theorems and properties. It is now time to mix in some algebraic proofs. For this unit, we will restrict our algebraic proofs to problems involving the coordinate plane.

First, we should examine our "toolbox" to see what math concepts we have at our disposal for these types of proofs:

Distance Formula: Useful for determining distances between two points.

Slope Formula: Useful for determining if lines are parallel or perpendicular.

Substitution: Useful for determining if points satisfy given equations.

1. Proof #1: Do these points form a rectangle? Q (0, 0); R (3, 0); S (3, -4); T (0, -4)

a. What information do we need to show in order to prove this is a rectangle?

b. Show algebraically that the opposite sides are congruent.

c. Show algebraically that the figure has 2 right angles

d. Is it necessary to show both of the above to prove it is a rectangle? Justify your answer.

e. Now use the facts in a-d to write a paragraph proof stating if the figure is a rectangle.

2. Proof #2: Prove or disprove that the point $\left(1, \sqrt{3}\right)$ lies on the circle centered at the origin and passing through the point (0, 2).

a. What do we need to show in order to prove or disprove this statement?

b. Write an equation for the circle described in the problem.

c. Substitute the point in for the equation and comment on the results. Did you prove the statement or disprove it?

d. Now use the facts in a-c to write a paragraph proof stating if the figure is a rectangle.

Now you are ready to try some on your own. Use the questions above as a guide and write algebraic proofs for the following.

3. Do the points A(1, 2) , B(4, 5) and C(5, -2) form a right triangle?

4. Prove or disprove that the points A(-1, 3), B(-3, 1) and C(1, -1) make up an equilateral triangle.

5. Prove or disprove that the point A(10, 3) lies on a circle centered at C(5, -2) and passing through the point B(6, 5).

6. Show that the points A(2, -1), B(4, 1), C(2, 3) and D(0, 1) are the corners of a square that is inscribed in the circle centered at O(2, 1), and passing through E$\left(3, 1+\sqrt{3}\right)$.