AP CALCULUS BC
SUMMER PACKET

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Help Session: Friday, July 28, 3:30 – 5:30
Due Date: Wednesday, August 2
Prerequisites Quiz: Friday, August 4

Work on this packet over the summer. If needed, hire a tutor or find a friend or family member that can help you.

This packet will be graded for completion and accuracy. This is your first graded assignment of the fall semester in AP Calculus BC. You must SHOW YOUR WORK to receive credit. You must complete this packet and pass the Prerequisites Quiz to keep your position in AP Calculus BC. If you fail to complete this packet or fail the Prerequisite Quiz, I will recommend that you be removed from the AP Calculus BC class. The answers to all of these problems are NOT attached. Discuss your work with other students and come to the first day of class with a complete problem set and ready to check your answers with me.

<u>**Submission Instructions:**</u> **Work all problems on your own paper. Organize your work and answers so that they can be easily graded.**

"We are what we repeatedly do. Excellence, therefore, is not an act, it is a habit. Who are you? What do you want to be? Look at the habits of your heart, your mind, and your behavior. You! Me! We are what we repeatedly do. Where our attention goes, we go!" - Unknown

"Success is a lousy teacher. It seduces smart people into thinking they can't lose." – Bill Gates

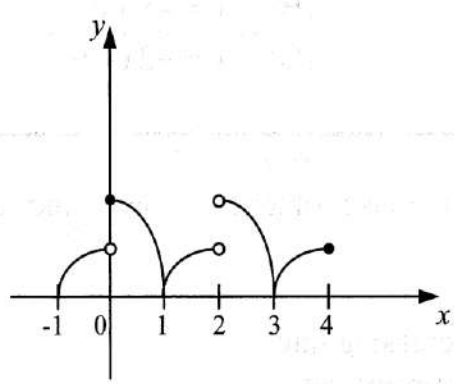
AP Calculus AB Review & Challenge Problems

Solve the following problems 1 – 32 without the use of a calculator. Show your work, organized by problem number on a separate sheet of paper.

1. The $\lim_{x \to 4} \frac{4-x}{\sqrt{x}-2} =$

(A) $-\infty$ (B) -4 (C) 0 (D) 4 (E) ∞

2.



The function shown above is defined on the closed interval $-1 \le x \le 4$ for

(A) all x
(B) all x except $x = 0$
(C) all x except $x = 1$
(D) all x except $x = 2$
(E) all x except $x = 0$ and $x = 2$

3.

x	-8	-6	-4	-2	0	2	4
$f(x)$	0	5	0	-2	-4	-6	-4
$f'(x)$	4	0	-4	-2	-1	0	1
$f''(x)$	-2	-6	-2	0	1	4	3

For some key values of x, the values of $f(x)$, $f'(x)$ and $f''(x)$ are given in the table above. The equation of the tangent to the curve $y = f(x)$ at the point of inflection shown in the table is:

(A) $y = 4x$ (B) $y = 4x + 8$
(C) $y = -6x + 24$ (D) $y = -2x - 6$ (E) $y = -x + 3$

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4. Using the table in the previous problem, at $x = -8$ the function $y = f(x)$ is

(A) at a relative minimum
(B) increasing at an increasing rate
(C) increasing at a decreasing rate
(D) decreasing at an increasing rate
(E) decreasing at a decreasing rate

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5. The graph of the function $f(x) = \dfrac{|x^3|}{x^3 - 8}$ has the following asymptotes:

(A) one vertical and one horizontal
(B) two vertical and one horizontal
(C) one vertical and two horizontal
(D) three vertical and one horizontal
(E) two vertical and two horizontal

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6. Find the limit as $a \to 0$ of the y-intercept of the normal to the parabola $y = x^2$ at the point (a, a^2).

7.▪ Let $f(x) = \begin{cases} bx^2 + 6x, \text{ if } x \le 2 \\ ax^3, \text{ if } x > 2 \end{cases}$. Find the values of a and b such that $f(x)$ is differentiable at $x = 2$.

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8. Determine the constants a, b, c, and d so that the graph of $y = ax^3 + bx^2 + cx + d$ has a relative maximum at (2, 4) and a point of inflection at the origin.

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9. The function $f(x)$ is continuous on the closed interval $[-3,5]$ and differentiable on the open interval $(-3,5)$. If $f'(x) > 0$ over the interval and if $f(-3) = -4$ and $f(5) = 12$, then $f(-1)$ cannot equal

(A) -6 (B) -1 (C) 4 (D) 5 (E) 10

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10. Let $f(x) = \begin{cases} \dfrac{x^3+8}{x+2}, & \text{if } x \neq -2 \\ 4, & \text{if } x = -2 \end{cases}$. Which of the following four statements are true?

I. $f(x)$ is defined at $x = -2$
II. $f(x)$ is continuous at $x = -2$
III. $\lim_{x \to -2} f(x)$ exists
IV. $f(x)$ is differentiable at $x = -2$

(A) I only (B) I and II (C) I and III
(D) II and III (E) I, II, III and IV

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11. For which of the following functions $f(x)$ do $f(0)$ and $f'(0)$ exist, but not $f''(0)$?

(A) $f(x) = x^{-\frac{1}{2}}$ (B) $f(x) = x^{\frac{1}{3}}$ (C) $f(x) = x^{\frac{4}{3}}$
(D) $f(x) = x^{\frac{7}{2}}$ (E) $f(x) = x^{\frac{11}{5}}$

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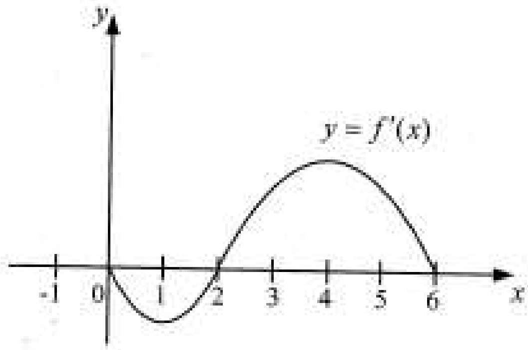
12. For a right circular cylinder with radius r and height h, volume $V = \pi r^2 h$ and surface area $S = 2\pi r^2 + 2\pi rh$. If the radius is a function of time and the height of the cylinder is equal to the diameter, then $\dfrac{dV}{dt} =$

(A) $r\dfrac{dS}{dt}$ (B) $2r\dfrac{dS}{dt}$ (C) $\dfrac{r}{2}\dfrac{dS}{dt}$ (D) $\dfrac{1}{r}\dfrac{dS}{dt}$ (E) $\dfrac{1}{r^2}\dfrac{dS}{dt}$

13.



The graph of $f'(x)$ is shown above. For which of the following values of x is $f(x)$ concave down?

(A) $x = \frac{1}{2}$ (B) $x = \frac{3}{2}$ (C) $x = 2$ (D) $x = \frac{5}{2}$ (E) $x = 3$

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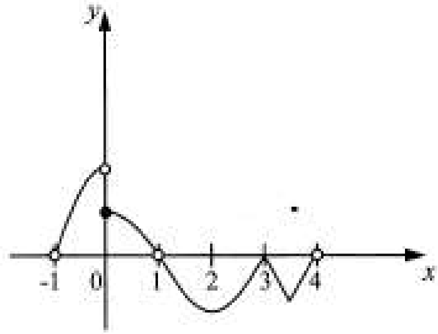
14. Find the coordinates of the point where the line tangent to the parabola $y^2 = 8x$ at (2, 4) intersects the axis of symmetry of the parabola.

15.■ The acceleration of a particle is given by $a(t) = 36t^2 - 12$ and $s(t)$ is the position function. If $s(-1) = -2$ and $s(2) = 37$, find the velocity of the particle at $t = 1$.

16.■ Chord $\overline{AB}$ of the parabola $y = x^2$ moves up from the vertex at $\frac{3}{4}$ units/sec, always remaining parallel to the x-axis. Triangle ABC is formed by $\overline{AB}$ and the two tangents to the parabola at A and at B. The triangle increases in area as $\overline{AB}$ moves up. Find the rate at which the area is increasing in units^2/sec at the instant when $\overline{AB}$ is 4 units above the vertex.

17.



The function, shown above, is differentiable at

(A) $x=1$ (B) $x=2$ (C) $x=3$ (D) $x=2$ and $x=3$
(E) $x=0$, $x=3$, and $x=4$

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18. If $p>0$ and $q>0$, $\int_{p}^{p+q} \frac{dx}{x^2} =$

(A) q (B) $\frac{p+q}{p}$ (C) $\frac{p}{p+q}$ (D) $\frac{q-p}{pq}$ (E) $\frac{q}{p(p+q)}$

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19. If $h(x)=2f^2(x)-3g^2(x)$, $f'(x)=g(x)$, and $g'(x)=-f(x)$, then $h'(x)=$

(A) $-10f(x)g(x)$ (B) $-2f(x)g(x)$
(C) $2f(x)g(x)$ (D) $10f(x)g(x)$
(E) $2f'(x)g'(x)$

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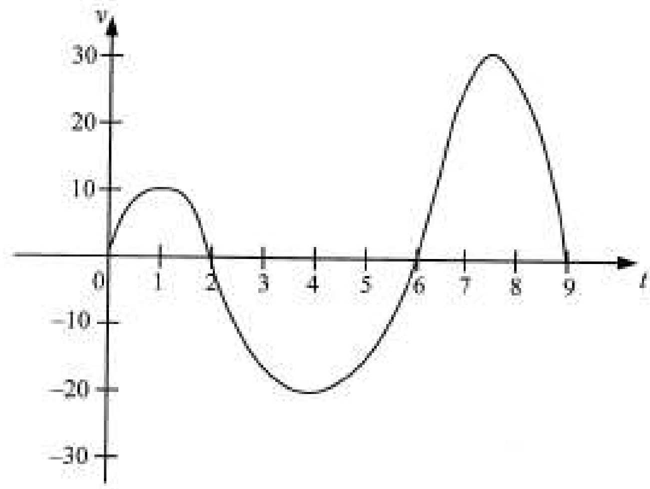
20. If $x>0$ and $n>1$, then the area bounded by the curves $y=x^n$ and $y=\sqrt[n]{x}$ is

(A) 1 (B) $n-1$ (C) $n+1$ (D) $\frac{n-1}{n+1}$ (E) $\frac{n+1}{n-1}$

21.



The graph of the velocity of a particle traveling along the x-axis for $0 \le t \le 9$ is shown above. At $t = 0$, $x = 0$. Approximately how many units to the right of the origin is the position of the particle at $t = 9$?

(A) 2 (B) 8 (C) 15 (D) 27 (E) 31

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22. Given the curve $y^2 = 150 - 10x$, $0 \le x \le 15$, find the coordinates of the point on the curve in the first quadrant that is nearest the origin.

23.■ The curve $y = \dfrac{ax+b}{(x-4)(x-1)}$ has a relative maximum at the point $(2, -1)$. Find a and b.

24.■ The average value of the function $f(x) = ax^2 - 2ax + a$ over the interval [1, 4] equals 13. Find the value of a.

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25. If the function $f(x)$ is such that $f(1)=4$, $f(2)=4$, and $f''(x)$ exists and is positive on the closed interval [0, 4], then we must have

(A) $f'(1.5)=0$ (B) $f'(1.5)>0$
(C) $f'(3)>0$ (D) $f'(3)<0$
(E) none of the above

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26. Find the range of $y=\dfrac{e}{x^2-\pi^2}$ in terms of e and π.

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27. If $f(x)=e^x$, $g(x)=\sin x$, and $h(x)=f\left(g(x)\right)$, then $h'\left(\dfrac{\pi}{2}\right)=$

(A) -1 (B) 0 (C) $e^{-\frac{\pi}{2}}$ (D) 1 (E) $e^{\frac{\pi}{2}}$

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28. If $f'(x)=6x^2$ and $f(2)=1$, then $\int_0^2 f(x)\,dx=$

(A) -22 (B) -16 (C) 2 (D) 8 (E) 18

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29. The expression $e^1+e^2+e^3$ is a

(A) left-hand Riemann sum with 3 subintervals for $\int_0^4 e^x\,dx$

(B) left-hand Riemann sum with 3 subintervals for $\int_0^3 e^x\,dx$

(C) right-hand Riemann sum with 3 subintervals for $\int_0^3 e^x\,dx$

(D) right-hand Riemann sum with 3 subintervals for $\int_1^4 e^x\,dx$

(E) midpoint Riemann sum with 4 subintervals for $\int_0^4 e^x\,dx$

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30. The rate of growth of an investment, which is compounded continuously, is proportional to the current balance with the constant of proportionality being the rate of interest. How much money (to the nearest dollar) should a 20-year-old person invest at $9\frac{1}{2}\%$ annual interest compounded continuously to have half a million dollars at age 70?

(A) \$433 (B) \$1,336 (C) \$4,326 (D) \$8,756 (E) \$23,781

31.▪ The area in the first quadrant bounded above by $y = \sin x$, below by the x-axis, and to the right by $x = \frac{\pi}{2}$ is divided into two equal parts by the line $x = a$. Find a.

32.▪ The base of a solid is the region in the first quadrant bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Each cross-section perpendicular to the x-axis is an isosceles right triangle with a leg as the base. Find the volume of the solid.

Disregard the jump in problem numbers. The last problem in this set is numbered 32. The first problem in the next set is number 51.

AP Calculus BC Prerequisite Skills from PreCalculus

Problems 51 – 60 review the prerequisite skills from your PreCalculus course that are required for AP Calculus BC. The prerequisite topics are outlined below and in some cases brief reminders are provided before problems.

Except where indicated, you should be able to solve these problems without the use of a calculator.

I) Parametric Equations & Curves
   a. Sketch parametrically defined curves by plotting points.
   b. Determine the direction of trace for a parametric curve.
   c. Determine where parametrically defined curves intersect and collide.

II) Polar Equations & Curves
   a. Sketch polar curves by plotting points.
   b. Convert polar coordinates into Cartesian coordinates and vice versa.
   c. Convert polar equations into Cartesian form and vice versa.

III) Vectors
   a. Sketch vectors given (*i*) direction and magnitude and (*ii*) horizontal and vertical components
   b. Combine vectors using addition and subtraction geometrically and algebraically.
   c. Perform scalar multiplication geometrically and algebraically.
   d. Determine the magnitude (length) and direction of a vector geometrically and algebraically.
   e. Determine the direction and magnitude of a vector given its horizontal and vertical components.
   f. Determine the horizontal and vertical components of a vector given its direction and magnitude.

IV) Sequences & Series
   a. Write the formula for the general term of a sequence or series.
   b. Use a generating function to find the terms of a sequence or a series.
   c. Identify geometric series and compute their sums.

I. <u>Parametric Equations and Curves</u>

(51) Sketch each parametric curve (without using your calculator) by plotting points on the indicated *t*-interval. Use arrows to indicate the direction in which the curve is traced.

(a) $x(t)=2$
$y(t)=-t^2$
$-2\le t\le 2$

(b) $(x,y)=(4\cos t, 2\sin t)$
$0\le t\le \pi$

(c) $x=2+t^2$
$y=2+t$
$-3\le t\le 3$

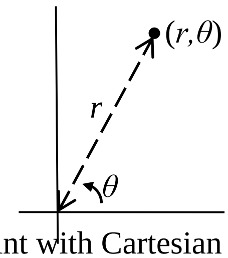
(52) Use algebra to determine where each pair of parametric curves (*i*) intersect and (*ii*) collide. Note: parametric curves $(x_1(t),y_1(t))$ and $(x_2(t),y_2(t))$ intersect when $x_1(t_1)=x_2(t_2)$ <u>and</u> $y_1(t_1)=y_2(t_2)$ for some t_1 and t_2. The intersection is a collision only when $t_1=t_2$.

(a) $(x_1,y_1)=(t,4-t)$
$(x_2,y_2)=(t^2-t,t)$

(b) $(x_1,y_1)=(\cos t,\sin t)$
$(x_2,y_2)=(t,t)$

II. <u>Polar Coordinates, Equations, and Curves</u>

- A **polar equation** of the form $r = f(\theta)$ describes a **polar curve**. The polar equation determines a radial distance from the origin, r, for an angle θ, measured relative the positive x-axis as illustrated below with counter-clockwise angles defined as positive.
- Polar coordinates are usually recorded in the form (r,θ). Angles are always reported in radians.



- Polar coordinates are not unique. The point with Cartesian coordinates $(x,y)=(1,1)$ can be described using polar coordinates in many ways: $(r,\theta)=(\sqrt{2},\pi/4)=(-\sqrt{2},5\pi/4)=(\sqrt{2},-7\pi/4)$
- You can convert a polar equations or coordinates into a Cartesian equation or coordinates and vice versa using the formulas $r^2 = x^2 + y^2$, $\tan\theta = y/x$, $x = r\cos\theta$, and $y = r\sin\theta$.

(53) Sketch the polar curve by plotting points for the given θ-interval.

(a) $r = 3\sin\theta$, $0 \le \theta \le 2\pi$ (b) $r = 10/\theta$, $0 < \theta \le 10$ (c) $r = \cos(2\theta)$, $0 \le \theta \le 2\pi$

(54) (a) Convert from Cartesian (x,y) into polar (r,θ) coordinates: (i) $(\sqrt{12},6)$ and (ii) $(-2,0)$

(b) Convert from polar (r,θ) into Cartesian (x,y) coordinates: (i) $(5,\pi/2)$ and (ii) $(-1,\pi/4)$

(55) (a) Convert each polar equation into Cartesian form: (i) $r = 2$ and (ii) $r = \tan\theta$.

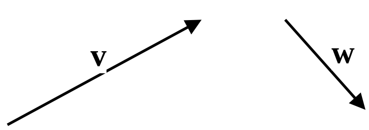
(b) Convert each polar equation into a set of parametric equations with parameter θ:

(i) $r = 2\cos\theta$ and (ii) $r = \ln\theta$.

III. <u>Vectors</u>

- $|\mathbf{v}|$ denotes the magnitude (length) of a vector $\mathbf{v}$.
- All direction angles are given relative to the positive x-direction with counter-clockwise angles defined as positive.
- The component representation $\mathbf{v} = \langle a,b\rangle$ describes a vector $\mathbf{v}$ with horizontal component a and vertical component b.
- If $\mathbf{v} = \langle a,b\rangle$, then the magnitude (length) of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{a^2+b^2}$ and the direction of $\mathbf{v}$ is $\theta = \tan^{-1}(b/a)$.

(56) Given vectors $\mathbf{v}$ and $\mathbf{w}$ shown below, (i) make a sketch of each of the following: (a) $2\mathbf{v}$, (b) $-\mathbf{v}$, (c), $\mathbf{v}+\mathbf{w}$, and (d) $\mathbf{v}-\mathbf{w}$. (ii) Use a ruler and a protractor to approximate the magnitude, direction, vertical component, and horizontal component of each resulting vector.



(57) Given vectors $\mathbf{v} = \langle 2,-1\rangle$ and $\mathbf{w} = \langle 5,3\rangle$, ($i$) find the component representation of each of the following: (a) $2\mathbf{v}$, (b) $-\mathbf{v}$, (c), $\mathbf{v}+\mathbf{w}$, and (d) $\mathbf{v}-\mathbf{w}$. (ii) <u>Without using a ruler or protractor</u>, find the exact magnitude and direction of each of the resulting vectors. You may use your calculator as needed.

IV. <u>Sequences and Series</u>

- A **sequence** is an ordered list of numbers that may or may not follow a pattern.
- The notation $\{a_n\}_{n=1}^{k} = \{a_1, a_2, a_3, \ldots a_{k-1}, a_k\}$ is used to describe a sequence with k terms whose n^{th} term is given by the formula a_n, called the **general term of the sequence**. Sometimes, the notation is simplified to $\{a_n\}$ or just a_n if the number of terms is not important.
- Examples of sequences:

  (*i*) $\{2n\}_{n=1}^{3} = \{2,4,6\}$ (*ii*) $\{n^2\}_{n=0}^{4} = \{0,1,4,9,16\}$ (*iii*) $\{\mathbf{1/2}\}_{\boldsymbol{n=1}}^{\infty} = \{\mathbf{1/2}\,,\mathbf{1/2}\,,\mathbf{1/2}\,,\ldots\}$

- Sequences can have either a finite (as in *i* and *ii* above) or an infinite number of terms (as in *iii* above).
- A **series** is the sum of the terms of a sequence.
- The notation $\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_{k-1} + a_k$ is used to describe a series with k terms whose n^{th} term is given by the formula a_n, called the **general term of the series**. Sometimes, the notation is simplified to $\sum a_n$ if the number of terms is not important. The symbol Σ is the Greek uppercase letter sigma, or S, which in this case stands for sum.
- Examples of series:

  (*i*) $\sum_{n=1}^{3} -n = -1 + (-2) + (-3) = -6$

  (*ii*) $\sum_{n=1}^{5} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$

  (*iii*) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$

  (*iv*) $\sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + \cdots + 2^n + \cdots$

- Examples (*iii*) & (*iv*) above are **infinite series** with infinitely many terms. Example (*iii*) "converges" to a sum of 1, but example (*iv*) "diverges," meaning that the sum increases without bound as additional terms are added.
- A **geometric series** is of the form $\sum_{n=0}^{k} a(r)^n = a + ar + ar^2 + \cdots + ar^{k-1} + ar^k$. Geometric series are recognized by the **common ratio**, r, which is multiplied by each term to produce the next term.
- The **sum of a finite geometric series** is found using the formula $\sum_{n=0}^{k} a(r)^n = a\frac{1-r^{k+1}}{1-r}$ for $r \neq \pm 1$.

  If $r = \pm 1$, then $\sum_{n=0}^{k} a(r)^n = \sum_{n=0}^{k} a = \overbrace{a + a + a + \cdots + a}^{k+1 \text{ terms}} = a(k+1)$.

- The **sum of an infinite geometric series** is found using the formula $\sum_{n=0}^{\infty} a(r)^n = \frac{a}{1-r}$ for $-1 < r < 1$. If $r \leq -1$ or $r \geq 1$, the infinite series "diverges."

(58) Write out the terms of the sequence or series. If the sequence or series is infinite, write the first five terms.

(a) $\left\{(-1)^n / n^2\right\}_{n=1}^{5}$

(b) $\left\{\sin(n\pi/2)\right\}_{n=0}^{\infty}$

(c) $\sum_{n=0}^{4} \frac{3^n}{n!}$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$

(59) (i) Write the next three terms of the sequence or series and (ii) find a formula for the general term of each sequence or series:

(a) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$

(b) $\{1, -1, 1, -1, \ldots\}$

(c) $\frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2} + \cdots$

(d) $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \cdots$

(60) (i) Classify each series as geometric or non-geometric. (ii) If the series is geometric, find the common ratio r. (iii) Find the sum of the series or explain why there is no sum.

(a) $\sum_{n=0}^{100} \frac{1}{4^n}$

(b) $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

$\frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \frac{4}{3} + \cdots$

(c) $3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$