



Georgia Standards of Excellence Frameworks

Mathematics

GSE Algebra II/Advanced Algebra

Unit 2: Operations with Polynomials



Richard Woods, Georgia's School Superintendent
"Educating Georgia's Future"

Unit 2
Operations with Polynomials

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***Revised standard indicated in bold red font.**

OVERVIEW

In this unit students will:

- understand the definition of a polynomial
- interpret the structure and parts of a polynomial expression including terms, factors, and coefficients
- simplify polynomial expressions by performing operations, applying the distributive property, and combining like terms
- use the structure of polynomials to identify ways to rewrite them and write polynomials in equivalent forms to solve problems
- perform arithmetic operations on polynomials and understand how closure applies under addition, subtraction, and multiplication
- divide one polynomial by another using long division
- use Pascal's Triangle to determine coefficients of binomial expansion
- use polynomial identities to solve problems
- use complex numbers in polynomial identities and equations
- find inverses of simple functions

This unit develops the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students will find inverse functions and verify by composition that one function is the inverse of another function.

STANDARDS ADDRESSED IN THIS UNIT

Mathematical standards are interwoven and should be addressed throughout the year in as many different units and activities as possible in order to emphasize the natural connections that exist among mathematical topics.

Perform arithmetic operations on polynomials

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Use polynomial identities to solve problems

MGSE9-12.A.APR.5 Know and apply that the Binomial Theorem gives the expansion of $(x + y)^n$ in powers of x and y for a positive integer n , where x and y are any numbers, with coefficients determined using Pascal's Triangle.

Rewrite rational expressions

MGSE9-12.A.APR.6 Rewrite simple rational expressions in different forms using inspection, long division, or a computer algebra system; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$.

Build a function that models a relationship between two quantities

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1b Combine standard function types using arithmetic operations in contextual situations (Adding, subtracting, and multiplying functions of different types).

MGSE9-12.F.BF.1c Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.*

Build new functions from existing functions

MGSE9-12.F.BF.4 Find inverse functions.

MGSE9-12.F.BF.4a Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2(x^3)$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.

MGSE9-12.F.BF.4b Verify by composition that one function is the inverse of another.

MGSE9-12.F.BF.4c Read values of an inverse function from a graph or a table, given that the function has an inverse.

RELATED STANDARDS

Interpret the structure of expressions

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Use polynomial identities to solve problems

MGSE9-12.A.APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*

Use complex numbers in polynomial identities and equations.

MGSE9-12.N.CN.8 Extend polynomial identities to include factoring with complex numbers. For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.

STANDARDS FOR MATHEMATICAL PRACTICE

Refer to the Comprehensive Course Overview for more detailed information about the Standards for Mathematical Practice.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

ENDURING UNDERSTANDINGS

- Viewing an expression as a result of operations on simpler expressions can sometimes clarify its underlying structure.
- Factoring and other forms of writing polynomials should be explored.
- Determine the inverse to a simple function and how it relates to the original function.

ESSENTIAL QUESTIONS

- How can we write a polynomial in standard form?
- How can we write a polynomial in factored form?
- How do we add, subtract, multiply, and divide polynomials?
- In which operations does closure apply?
- How can we apply Pascal's Triangle to expand $(x + y)^n$?
- How can you find the inverse of a simple function?

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- Combining like terms and simplifying expressions
- Long division
- The distributive property
- The zero property
- Properties of exponents
- Simplifying radicals with positive and negative radicands
- Factoring quadratic expressions

SELECT TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real

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life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The websites below are interactive and include a math glossary suitable for high school children. **Note – At the high school level, different sources use different definitions. Please preview any website for alignment to the definitions given in the frameworks**

<http://www.amathsdictionaryforkids.com/>

This web site has activities to help students more fully understand and retain new vocabulary.

<http://intermath.coe.uga.edu/dictionary/homepg.asp>

Definitions and activities for these and other terms can be found on the Intermath website.

- **Coefficient:** a number multiplied by a variable.
- **Degree:** the greatest exponent of its variable
- **End Behavior:** the value of $f(x)$ as x approaches positive and negative infinity
- **Pascal's Triangle:** an arrangement of the values of ${}_n C_r$ in a triangular pattern where each row corresponds to a value of n
- **Polynomial:** a mathematical expression involving a sum of nonnegative integer powers in one or more variables multiplied by coefficients. A polynomial in one variable with constant coefficients can be written in $a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ form.
- **Remainder Theorem:** states that the remainder of a polynomial $f(x)$ divided by a linear divisor $(x - c)$ is equal to $f(c)$.
- **Roots:** solutions to polynomial equations.
- **Synthetic Division:** Synthetic division is a shortcut method for dividing a polynomial by a linear factor of the form $(x - a)$. It can be used in place of the standard long division algorithm.
- **Zero:** If $f(x)$ is a polynomial function, then the values of x for which $f(x) = 0$ are called the **zeros** of the function. Graphically, these are the x intercepts.

EVIDENCE OF LEARNING

By the conclusion of this unit, students should be able to demonstrate the following competencies:

- perform operations on polynomials (addition, subtraction multiplication, long division, and synthetic division)
- identify and which operations are closed under polynomials and explain why
- write polynomials in standard and factored forms
- perform binomial expansion by applying Pascal’s Triangle
- find the inverse of simple functions and verify inverses with the original function

FORMATIVE ASSESSMENT LESSONS (FAL)

Formative Assessment Lessons are intended to support teachers in formative assessment. They reveal and develop students’ understanding of key mathematical ideas and applications. These lessons enable teachers and students to monitor in more detail their progress towards the targets of the standards. They assess students’ understanding of important concepts and problem solving performance, and help teachers and their students to work effectively together to move each student’s mathematical reasoning forward.

More information on Formative Assessment Lessons may be found in the Comprehensive Course Guide.

SPOTLIGHT TASKS

A Spotlight Task has been added to each GSE mathematics unit in the Georgia resources for middle and high school. The Spotlight Tasks serve as exemplars for the use of the Standards for Mathematical Practice, appropriate unit-level Georgia Standards of Excellence, and research-based pedagogical strategies for instruction and engagement. Each task includes teacher commentary and support for classroom implementation. Some of the Spotlight Tasks are revisions of existing Georgia tasks and some are newly created. Additionally, some of the Spotlight Tasks are 3-Act Tasks based on 3-Act Problems from Dan Meyer and Problem-Based Learning from Robert Kaplinsky.

3-ACT TASKS

A Three-Act Task is a whole group mathematics task consisting of 3 distinct parts: an engaging and perplexing Act One, an information and solution seeking Act Two, and a solution discussion and solution revealing Act Three.

More information along with guidelines for 3-Act Tasks may be found in the Comprehensive Course Guide.

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TASKS

The following tasks represent the level of depth, rigor, and complexity expected of all Algebra II students. These tasks, or tasks of similar depth and rigor, should be used to demonstrate evidence of learning. It is important that all elements of a task be addressed throughout the learning process so that students understand what is expected of them. While some tasks are identified as a performance task, they may also be used for teaching and learning (learning/scaffolding task).

Task Name	Task Type <i>Grouping Strategy</i>	Content Addressed	SMPs Addressed
Classifying Polynomials	Introductory Learning Task	Classifying polynomials based on characteristics of expressions	1, 2, 3, 5
We've Got to Operate	Scaffolding/Learning Task <i>Individual/Partner Task</i>	Define and operate with polynomials	1, 6
A Sum of Functions	Learning Task <i>Individual/Partner</i>	Adding and Subtracting Functions	1, 4, 7
Building By Composition	Learning Task <i>Individual/Whole Class</i>	Composition of Functions	1, 4, 7
Nesting Functions	Learning Task <i>Individual/Partner</i>	Composition of Inverse Functions	1, 4, 7
Changes in Latitude	Learning Task <i>Individual</i>	The Role of the Inverse Function	1, 4, 7
Cardboard Box	Learning Task <i>Partner/Small Group Task</i>	Rate of change of quadratic functions Writing expressions for quadratic functions	1-4, 7, 8
What's Your Identity	Learning Task <i>Partner/Small Group Task</i>	Develop and apply polynomial identities	1, 2, 3, 5, 6, 7, 8

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Rewriting Rational Expressions	Learning Task <i>Individual/Partner</i>	Divide polynomials using long division as well as synthetic division	1, 2, 3
Finding Inverses Task	Learning Task <i>Small Group</i>	Finding inverses of simple functions	1, 2, 3, 6

Classifying Polynomials

Mathematical Goals

- Understand the definition of a polynomial
- Classify polynomials by degree and number of terms

Georgia Standards of Excellence

Interpret the structure of expressions

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Look for and make use of structure.
6. Look for and express regularity in repeated reasoning.

Introduction

In this task, we are going to explore the definition and classification of polynomial functions. We will identify different parts of these expressions and explain their meaning within the context of problems

Materials

- Pencil
- Handout

Classifying Polynomials

Previously, you have learned about linear functions, which are first degree polynomial functions that can be written in the form $f(x) = a_1x^1 + a_0$ where a_1 is the slope of the line and a_0 is the y-intercept (Recall: $y = mx + b$, here m is replaced by a_1 and b is replaced by a_0 .) Also, you have learned about quadratic functions, which are 2nd degree polynomial functions and can be expressed as $f(x) = a_2x^2 + a_1x^1 + a_0$.

These are just two examples of polynomial functions; there are countless others. A polynomial is a mathematical expression involving a sum of nonnegative integer powers in one or more variables multiplied by coefficients. A polynomial in one variable with constant coefficients can be written in $a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$ form where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers.

1. What are whole numbers? *The set of numbers {0, 1, 2, 3...}*
2. What are real numbers? *The set of all rational and irrational numbers*
3. Decide whether each function below is a polynomial. If it is, explain how you know and write the function in standard form. If it is not, explain why.

a. $f(x) = 2x^3 + 5x^2 + 4x + 8$
yes, it is already in standard form

c. $f(x) = 5 - x + 7x^3 - x^2$

yes, $f(x) = 7x^3 + 5x^2 - x + 5$

e. $f(x) = 2\sqrt{x}$
no, the exponent is not a whole number

b. $f(x) = 2x^2 + x^{-1}$

no because the exponent is not a whole number

d. $f(x) = \frac{2}{3}x^2 - x^4 + 5 + 8x$

yes, $f(x) = -x^4 + 2/3x^2 + 8x + 5$

g. $f(x) = \frac{1}{3x^2} + \frac{6}{x} - 2$

no because the exponent is not a whole number

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4. Polynomials can be classified by the number terms as well as by the degree of the polynomial. The degree of the polynomial is the same as the term with the highest degree. Complete the following chart. Make up your own expressions for the last three rows.

Polynomial	Number of Terms	Classification	Degree	Classification
$f(x) = 2$	<i>One</i>	monomial	<i>0th</i>	constant
$f(x) = 3x - 1$	<i>Two</i>	binomial	<i>1st</i>	linear
$f(x) = x^2 - 2x + 1$	<i>Three</i>	trinomial	<i>2nd</i>	quadratic
$f(x) = 8x^3 + 125$	<i>Two</i>	binomial	<i>3rd</i>	cubic
$f(x) = x^4 + 10x^2 + 16$	<i>Three</i>	trinomial	<i>4th</i>	quartic
$f(x) = -x^5$	<i>one</i>	monomial	<i>5th</i>	quintic
<i>Answers will vary</i>				
<i>Answers will vary</i>				
<i>Answers will vary</i>				

Classifying Polynomials

Previously, you have learned about linear functions, which are first degree polynomial functions that can be written in the form $f(x) = a_1x^1 + a_0$ where a_1 is the slope of the line and a_0 is the y-intercept (Recall: $y = mx + b$, here m is replaced by a_1 and b is replaced by a_0 .) Also, you have learned about quadratic functions, which are 2nd degree polynomial functions and can be expressed as $f(x) = a_2x^2 + a_1x^1 + a_0$.

These are just two examples of polynomial functions; there are countless others. A polynomial is a mathematical expression involving a sum of nonnegative integer powers in one or more variables multiplied by coefficients. A polynomial in one variable with constant coefficients can be written in $a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$ form where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers.

1. What are whole numbers?

2. What are real numbers?

3. Decide whether each function below is a polynomial. If it is, write the function in standard form. If it is not, explain why.

a. $f(x) = 2x^3 + 5x^2 + 4x + 8$

b. $f(x) = 2x^2 + x^{-1}$

c. $f(x) = 5 - x + 7x^3 - x^2$

d. $f(x) = \frac{2}{3}x^2 - x^4 + 5 + 8x$

e. $f(x) = 2\sqrt{x}$

g. $f(x) = \frac{1}{3x^2} + \frac{6}{x} - 2$

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4. Polynomials can be classified by the number terms as well as by the degree of the polynomial. The degree of the polynomial is the same as the term with the highest degree. Complete the following chart. Make up your own expressions for the last three rows.

Polynomial	Number of Terms	Classification	Degree	Classification
$f(x) = 2$		monomial		constant
$f(x) = 3x - 1$		binomial		linear
$f(x) = x^2 - 2x + 1$		trinomial		quadratic
$f(x) = 8x^3 + 125$		binomial		cubic
$f(x) = x^4 + 10x^2 + 16$		trinomial		quartic
$f(x) = -x^5$		monomial		quintic

We've Got to Operate

Mathematical Goals

- Add, subtract, and multiply polynomials and understand how closure applies

Georgia Standards of Excellence

Interpret the structure of expressions.

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Perform arithmetic operations on polynomials.

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Standards for Mathematical Practice

7. Make sense of problems and persevere in solving them.
8. Reason abstractly and quantitatively.
9. Construct viable arguments and critique the reasoning of others.
10. Model with mathematics.
11. Look for and make use of structure.
12. Look for and express regularity in repeated reasoning.

Introduction

In this task, we will perform operations on polynomials (addition, subtraction, multiplication) and simplify these expressions by combining like terms and using the distributive property. Finally, we will learn how closure applies to these operations on polynomials.

Materials

- Pencil
- Handout

We've Got to Operate

Previously, you learned how to use manipulatives to add and subtract like terms of polynomial expressions. Now, in this task, you will continue to use strategies that you previously developed to simplify polynomial expressions. To simplify expressions and solve problems, you learned that we sometimes need to perform operations with polynomials. We will further explore addition and subtraction in this task.

Answer the following questions and justify your reasoning for each solution.

Teachers should have completed a lesson using algebra tiles, algeblocks or virtual manipulative prior to presenting this task. Students should see and comprehend the concept conceptually first. Jumping directly to the abstract idea would not be beneficial for most students.

1. Bob owns a small music store. He keeps inventory on his xylophones by using x^2 to represent his professional grade xylophones, x to represent xylophones he sells for recreational use, and constants to represent the number of xylophone instruction manuals he keeps in stock. If the polynomial $5x^2 + 2x + 4$ represents what he has on display in his shop and the polynomial $3x^2 + 6x + 1$ represents what he has stocked in the back of his shop, what is the polynomial expression that represents the entire inventory he currently has in stock?

$$(5x^2 + 2x + 4) + (3x^2 + 6x + 1) = 8x^2 + 8x + 5$$

2. Suppose a band director makes an order for 6 professional grade xylophones, 13 recreational xylophones and 5 instruction manuals. What polynomial expression would represent Bob's inventory **after** he processes this order? Explain the meaning of each term.

$(8x^2 + 8x + 5) - (6x^2 + 13x + 5) = 2x^2 - 5x$ which means *he has two professional grade xylophones left in stock, has to order 5 recreational xylophones or short-change his customer, and he has no xylophone manuals left in stock.*

3. Find the sum or difference of the following using a strategy you acquired in the previous lesson:

$$\begin{array}{r} \text{a. } 5x^2 + 2x - 8 \\ + 3x^2 - 7x - 1 \\ \hline 8x^2 - 5x - 9 \end{array}$$

$$\begin{array}{r} \text{b. } 2x^2 - 2x + 7 \\ (-) x^2 + 2x + 1 \\ \hline x^2 - 4x + 6 \end{array}$$

$$\begin{array}{r} \text{c. } (7x - 5) + (2x + 8) \\ 9x + 3 \end{array}$$

$$\begin{array}{r} \text{d. } (2a^2 - 5a + 1) + (a^2 + 3a) \\ 3a^2 - 2a + 1 \end{array}$$

$$\text{e. } (-2x^2 - 5x + 9) - (-3x^2 + 2x + 4)$$

$$\text{f. } (5x^2 + 2xy - 7y^2) - (3x^2 - 5xy + 2y^2)$$

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$$x^2 - 7x + 5$$

$$2x^2 + 8xy - 9y^2$$

4. You have multiplied polynomials previously, but may not have been aware of it. When you utilized the distributive property, you were just multiplying a polynomial by a monomial. In multiplication of polynomials, the central idea is the distributive property.

a. An important connection between arithmetic with integers and arithmetic with polynomials can be seen by considering whole numbers in base ten to be polynomials in the base $b = 10$. Compare the product 213×47 with the product $(2b^2 + 1b + 3)(4b + 7)$:

$2b^2 + 1b + 3$	$200 + 10 + 3$	213
$\times 4b + 7$	$\underline{40 + 7}$	$\times \underline{47}$
$14b^2 + 7b + 21$	$1400 + 70 + 21$	1491
$\underline{8b^3 + 4b^2 + 12b + 0}$	$\underline{8000 + 400 + 120 + 0}$	$\underline{8520}$
$8b^3 + 18b^2 + 19b + 21$	$8000 + 1800 + 190 + 21$	10011

b. Now compare the product 135×24 with the product $(1b^2 + 3b + 5)(2b + 4)$. Show your work!

$2b^3 + 10b^2 + 22b + 20$	$100 + 30 + 5$	135
	$\underline{20 + 4}$	$\times \underline{24}$
	$400 + 120 + 20$	3240
	$\underline{2000 + 600 + 100 + 0}$	
	$2000 + 1000 + 220 + 20$	

5. Find the following products. Be sure to simplify results.

<p>a. $3x(2x^2 + 8x + 9)$ $6x^3 + 24x^2 + 27x$</p> <p>c. $(2x + 7)(2x - 5)$ $4x^2 + 4x - 35$</p> <p>e. $(x - 3)(2x^2 + 3x - 1)$ $2x^3 - 3x^2 - 10x + 3$</p> <p>g. $(4x - 7y)(4x + 7y)$ $16x^2 - 49y^2$</p> <p>i. $(x - 1)^3$ $x^3 - 3x^2 + 3x - 1$</p>	<p>b. $-2x^2(5x^2 - x - 4)$ $-10x^4 + 2x^3 + 8x^2$</p> <p>d. $(4x - 7)(3x - 2)$ $12x^2 - 29x + 14$</p> <p>f. $(6x + 4)(x^2 - 3x + 2)$ $6x^3 - 14x^2 + 8$</p> <p>h. $(3x - 4)^2$ $9x^2 - 24x + 16$</p> <p>j. $(x - 1)^4$ $x^4 - 4x^3 + 6x^2 - 4x - 1$</p>
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6. A set has the **closure property** under a particular **operation** if the result of the operation is always an element in the set. If a set has the **closure property** under a particular **operation**, then we say that the set is “**closed** under the **operation**.”

It is much easier to understand a property by looking at examples than it is by simply talking about it in an abstract way, so let's move on to looking at examples so that you can see exactly what we are talking about when we say that a set has the **closure property**.

a. The set of integers is **closed** under the **operation** of addition because the sum of any two integers is always another integer and is therefore in the set of integers. Write a few examples to illustrate this concept:

Answers will vary

b. The set of integers is not **closed** under the **operation** of division because when you divide one integer by another, you don't always get another integer as the answer. Write an example to illustrate this concept: *Answers will vary*

c. Go back and look at all of your answers to problem number 5, in which you added and subtracted polynomials. Do you think that polynomial addition and subtraction is closed? Why or why not?

Yes because every sum and difference is a polynomial

d. Now, go back and look at all of your answers to problems 6 and 7, in which you multiplied polynomials. Do you think that polynomial multiplication is closed? Why or why not?

Yes because every product is a polynomial

We've Got to Operate

Previously, you learned how to use manipulatives to add and subtract like terms of polynomial expressions. Now, in this task, you will continue to use strategies that you previously developed to simplify polynomial expressions. To simplify expressions and solve problems, you learned that we sometimes need to perform operations with polynomials. We will further explore addition and subtraction in this task.

Answer the following questions and justify your reasoning for each solution.

1. Bob owns a small music store. He keeps inventory on his xylophones by using x^2 to represent his professional grade xylophones, x to represent xylophones he sells for recreational use, and constants to represent the number of xylophone instruction manuals he keeps in stock. If the polynomial $5x^2 + 2x + 4$ represents what he has on display in his shop and the polynomial $3x^2 + 6x + 1$ represents what he has stocked in the back of his shop, what is the polynomial expression that represents the entire inventory he currently has in stock?

2. Suppose a band director makes an order for 6 professional grade xylophones, 13 recreational xylophones and 5 instruction manuals. What polynomial expression would represent Bob's inventory **after** he processes this order? Explain the meaning of each term.

3. Find the sum or difference of the following using a strategy you acquired in the previous lesson:

$$\begin{array}{r} \text{a. } 5x^2 + 2x - 8 \\ + 3x^2 - 7x - 1 \\ \hline \end{array}$$

$$\begin{array}{r} \text{b. } 2x^2 - 2x + 7 \\ (-) x^2 + 2x + 1 \\ \hline \end{array}$$

$$\text{c. } (7x - 5) + (2x + 8)$$

$$\text{d. } (2a^2 - 5a + 1) + (a^2 + 3a)$$

$$\text{e. } (-2x^2 - 5x + 9) - (-3x^2 + 2x + 4)$$

$$\text{f. } (5x^2 + 2xy - 7y^2) - (3x^2 - 5xy + 2y^2)$$

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4. You have multiplied polynomials previously, but may not have been aware of it. When you utilized the distributive property, you were just multiplying a polynomial by a monomial. In multiplication of polynomials, the central idea is the distributive property.

a. An important connection between arithmetic with integers and arithmetic with polynomials can be seen by considering whole numbers in base ten to be polynomials in the base $b = 10$. Compare the product 213×47 with the product $(2b^2 + 1b + 3)(4b + 7)$:

$2b^2 + 1b + 3$	$200 + 10 + 3$	213
$\times 4b + 7$	$\quad 40 + 7$	$\times 47$
$14b^2 + 7b + 21$	$1400 + 70 + 21$	1491
$8b^3 + 4b^2 + 12b + 0$	$8000 + 400 + 120 + 0$	8520
$8b^3 + 18b^2 + 19b + 21$	$8000 + 1800 + 190 + 21$	10011

b. Now compare the product 135×24 with the product $(1b^2 + 3b + 5)(2b + 4)$. Show your work!

5. Find the following products. Be sure to simplify results.

a. $3x(2x^2 + 8x + 9)$

b. $-2x^2(5x^2 - x - 4)$

c. $(2x + 7)(2x - 5)$

d. $(4x - 7)(3x - 2)$

e. $(x - 3)(2x^2 + 3x - 1)$

f. $(6x + 4)(x^2 - 3x + 2)$

g. $(4x - 7y)(4x + 7y)$

h. $(3x - 4)^2$

i. $(x - 1)^3$

j. $(x - 1)^4$

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6. A set has the **closure property** under a particular **operation** if the result of the operation is always an element in the set. If a set has the **closure property** under a particular **operation**, then we say that the set is “**closed** under the **operation**.”

It is much easier to understand a property by looking at examples than it is by simply talking about it in an abstract way, so let's move on to looking at examples so that you can see exactly what we are talking about when we say that a set has the **closure property**.

- a. The set of integers is **closed** under the **operation** of addition because the sum of any two integers is always another integer and is therefore in the set of integers. Write a few examples to illustrate this concept:

- b. The set of integers is not **closed** under the **operation** of division because when you divide one integer by another, you don't always get another integer as the answer. Write an example to illustrate this concept:

- c. Go back and look at all of your answers to problem number 5, in which you added and subtracted polynomials. Do you think that polynomial addition and subtraction is closed? Why or why not?

- d. Now, go back and look at all of your answers to problems 6 and 7, in which you multiplied polynomials. Do you think that polynomial multiplication is closed? Why or why not?

A Sum of Functions

Source: *Illustrative Mathematics*

<https://www.illustrativemathematics.org/content-standards/HSF/BF/A/1/tasks/230>

Mathematical Goals

- To study the result of adding or subtracting two functions.

Essential Questions

- What is the result of adding two functions from different function families?

TASK COMMENTS

This task leads students through adding two functions (a rational plus a linear) using the graphs of the functions alone. By using graphs, the idea of adding together two function outputs that correspond to the same input is emphasized. Students are also encouraged to subtract the two functions and study the result.

The task, *A Sum of Functions* is a Performance task that can be found at the website:

<https://www.illustrativemathematics.org/content-standards/HSF/BF/A/1/tasks/230>

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.F.BF.1b Combine standard function types using arithmetic operations in contextual situations (Adding, subtracting, and multiplying functions of different types).

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

1. **Make sense of problems and persevere in solving them**
2. **Model with mathematics**
7. **Look for and make use of structure.**

Grouping

- Individual or Partner

Time Needed

- 20-30 minutes

Building by Composition

Source: Illustrative Mathematics

<https://www.illustrativemathematics.org/content-standards/HSF/BF/B/3/tasks/744>

Mathematical Goals

- To understand how composition of functions works and how it can be used to create new functions.

Essential Questions

- How can composition of functions be used to create new functions?

TASK COMMENTS

Students and teachers may find this task to be a good follow-up to an introductory lesson on function composition. The task takes a unique approach to demonstrating the flexibility and possibility that composing functions allows the creative mathematician. The Illustrative Mathematics site indicates that “this task is intended for instruction...”

The task, *Building an Explicit Quadratic Function by Composition* is a Performance task that can be found at the website:

<https://www.illustrativemathematics.org/content-standards/HSF/BF/B/3/tasks/744>

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.F.BF.1c Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.*

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

- 1. Make sense of problems and persevere in solving them**
- 4. Model with mathematics**
- 7. Look for and make use of structure.**

Grouping

- Partner or Whole-Class

Time Needed

- 20-40 minutes

Nesting Functions

Source: *Illustrative Mathematics*

<https://www.illustrativemathematics.org/content-standards/HSF/BF/B/4/tasks/615>

Mathematical Goals

- Use common logarithms and base-10 exponentials to show that the composition of two inverse functions is the identity function.

Essential Questions

- Why does composition of inverse functions produce the identity function?

TASK COMMENTS

This short discovery task leads students through some basic composition of logarithms and exponentials. Though the course of these compositions, students discover that the composition of two inverse functions is the identity function. Attention is also given to the different domains that are produced by commuting the order of composition of logs and exponentials.

The task, *Exponentials and Logarithms II* is a task that can be found at the website:

<https://www.illustrativemathematics.org/content-standards/HSF/BF/B/4/tasks/615>

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.F.BF.4b Verify by composition that one function is the inverse of another.

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

1. **Make sense of problems and persevere in solving them**
7. **Look for and make use of structure.**

Grouping

- Individual or Partner
- Guided Practice

Time Needed

- 20-40 minutes

Changes in Latitude

Source: Illustrative Mathematics

<https://www.illustrativemathematics.org/content-standards/HSF/BF/B/4/tasks/1114>

Mathematical Goals

- Students will use a table of values to create the inverse of a function and be able to interpret the meaning of that inverse in the context of the problem.

Essential Questions

- What uses do inverse functions have in modeling real-world phenomena?

TASK COMMENTS

This task leads students through an introduction to inverse functions from a table of values and challenge the student to come up with an explanation for the purpose of the inverse function they discover. Some interesting extension questions are also offered.

The task, *Latitude* is a task that can be found at the website:

<https://www.illustrativemathematics.org/content-standards/HSF/BF/B/4/tasks/1114>

GEORGIA STANDARDS OF EXCELLENCE

MGSE9-12.F.BF.4c Read values of an inverse function from a graph or a table, given that the function has an inverse.

STANDARDS FOR MATHEMATICAL PRACTICE

This task uses all of the practices with emphasis on:

- 1. Make sense of problems and persevere in solving them**
- 4. Model with mathematics**
- 7. Look for and make use of structure.**

Grouping

- Individual or Partner

Time Needed

- 30-40 minutes

Cardboard Box (Spotlight Task)

Georgia Standards of Excellence

Perform arithmetic operations on polynomials.

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

STANDARDS FOR MATHEMATICAL PRACTICE

- 1. Make sense of problems and persevere in solving them.**
- 2. Reason abstractly and quantitatively.**
- 3. Construct viable arguments and critique the reasoning of others.**
- 4. Model with mathematics.**
- 5. Use appropriate tools strategically.**
- 6. Attend to precision.**
- 7. Look for and make use of structure.**
- 8. Look for and express regularity in repeated reasoning.**

ESSENTIAL QUESTIONS

- What information do you need to make sense of this problem?
- How can you use estimation strategies to find out important information about the picture provided?

MATERIALS REQUIRED

- Access to video
- Student Recording Sheet
- Pencil

TIME NEEDED

- 1 day

TEACHER NOTES

In this task, students will watch the video, generate questions that they would like to answer, make reasonable estimates, and then justify their estimates mathematically. This is a student-centered task that is designed to engage learners at the highest level in learning the mathematics content. During Act 1, students will be asked to discuss what they wonder or are curious about after watching the quick video. These questions should be recorded on a class chart or on the board. Students will then use mathematics, collaboration, and prior knowledge to answer their own questions. Students will be given additional information needed to solve the problem based

on need. When they realize they don't have a piece of information they need to help address the problem and ask for it, it will be given to them.

Task Description

ACT 1:

Watch the video:

<http://real.doe.k12.ga.us/vod/gso/math/videos/Cardboard-box.wmv>

Ask students what they want to know.

The students may say the following:

- How much space can be taken up inside the box?
- What is the volume of the box?
- What are the dimensions of the box?
- How tall is the box?
- How wide is the box?
- How deep is the box?
- What are the measurements of the cut-out portion?

Give students adequate “think time” between the two acts to discuss what they want to know. Focus in on one of the questions generated by the students, i.e. What is the volume of the box?, and ask students to use the information from the video in the first act to figure it out.

Circulate throughout the classroom and ask probing questions, as needed.

ACT 2:

Reveal the following information as requested:



Ask students how they would use this information to solve the problem to find the specific dimensions and the volume of the box.

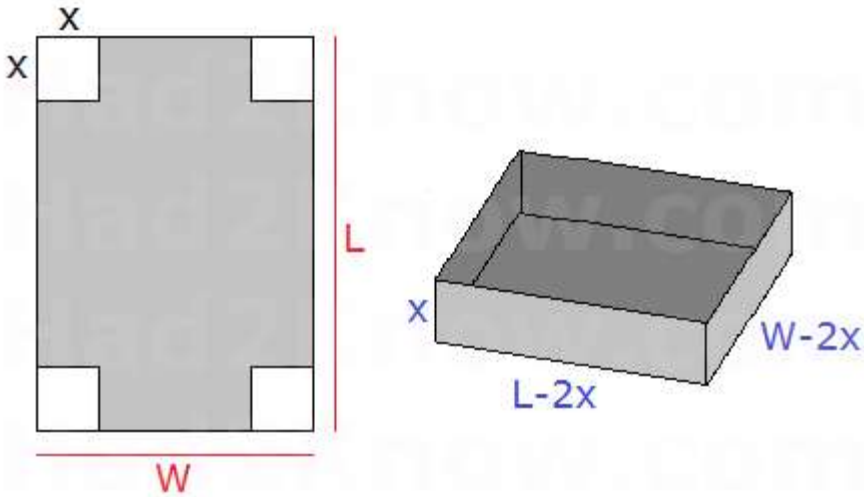
Give students time to work in groups to figure it out.

Circulate throughout the classroom and ask probing questions, as needed.

ACT 3

Students will compare and share solution strategies.

- Reveal the answer. Discuss the theoretical math versus the practical outcome.
- How appropriate was your initial estimate?
- Share student solution paths. Start with most common strategy.
- Revisit any initial student questions that weren't answered.



Intervention:

Ask specific, probing questions, such as:

What do you need to know about the original problem to help you find your solution?

What information is given?

How do you determine the volume of a rectangular prism?

How do you determine the measurement of each side using the given information?

Formative Assessment Check

A toy manufacturer has created a new card game. Each game is packaged in an open-top cardboard box, which is then wrapped with clear plastic. The box for the game is made from a 20-cm by 30-cm piece of cardboard. Four equal squares are cut from the corners, one from each corner of the cardboard piece. Then, the sides are folded and the edges that touch are glued. What must be the dimensions of each square so that the resulting box has maximum volume?

Student Recording Sheet

Task Title: _____

Name: _____

ACT 1

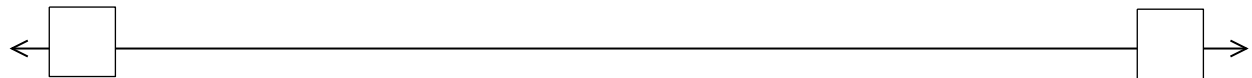
What did/do you notice?

What questions come to your mind?

Main Question: _____

Estimate the result of the main question? Explain?

Place an estimate that is too high and too low on the number line



Low estimate

Place an "x" where your estimate belongs

High estimate

ACT 2

What information would you like to know or do you need to solve the question posed by the class?

Record the given information you have from Act 1 and any new information provided in Act 2.

If possible, give a better estimate using this information: _____

Act 2 (continued)

Use this area for your work, tables, calculations, sketches, and final solution.

ACT 3

What was the result?

Which Standards for Mathematical Practice did you use?	
<input type="checkbox"/> Make sense of problems & persevere in solving them	<input type="checkbox"/> Use appropriate tools strategically.
<input type="checkbox"/> Reason abstractly & quantitatively	<input type="checkbox"/> Attend to precision.
<input type="checkbox"/> Construct viable arguments & critique the reasoning of others.	<input type="checkbox"/> Look for and make use of structure.
<input type="checkbox"/> Model with mathematics.	<input type="checkbox"/> Look for and express regularity in repeated reasoning.

What's Your Identity?

Mathematical Goals

- Illustrate how polynomial identities are used to determine numerical relationships
- Prove polynomial identities by showing steps and providing reasons
- Understand that polynomial identities include, but are not limited to, the product of the sum and difference of two terms, the difference of squares, the sum or difference of cubes, the square of a binomial, etc.
- Extend the polynomial identities to complex numbers. Notice: this is a (+) standard
- For small values of n , use Pascal's Triangle to determine the coefficients and terms in binomial expansion. Notice: this is a (+) standard
- Use the Binomial Theorem to find the n^{th} term in the expansion of a binomial to a positive integer power. Notice: this is a (+) standard

Georgia Standards of Excellence

Interpret the structure of expressions.

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Perform arithmetic operations on polynomials.

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

Use polynomial identities to solve problems.

MGSE9-12.A.APR.4 Prove polynomial identities and use them to describe numerical relationships. For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.

MGSE9-12.A.APR.5 Know and apply that the Binomial Theorem gives the expansion of $(x + y)^n$ in powers of x and y for a positive integer n , where x and y are any numbers, with coefficients determined using Pascal's Triangle.

Use complex numbers in polynomial identities and equations.

MGSE9-12.N.CN.8 Extend polynomial identities to include factoring with complex numbers. For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Introduction

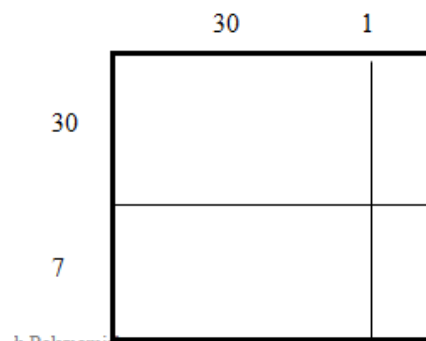
Equivalent algebraic expressions, also called algebraic identities, give us a way to express results with numbers that always work a certain way. In this task you will explore several “number tricks” that work because of basic algebra rules. You will extend these observations to algebraic expressions in order to prove polynomial identities. Finally, you will learn and apply methods to expand binomials. It is recommended that you do this task with a partner.

Materials

- Pencil
- Handout
- Calculator

1. First, you will explore an alternate way to multiply two digit numbers that have the same digit in the ten's place.

- a. For example, $(31)(37)$ can be thought of as
 $(30 + 1)(30 + 7)$.



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Using this model, we find the product equals $900 + 210 + 30 + 7 = 1147$. Verify this solution using a calculator. Are we correct? *Yes*

Rewrite these similarly and use area models to calculate each of the following products:

b. $(52)(57)$
 $2500 + 100 + 350 + 14 = 2964$

c. $(16)(13)$
 $100 + 60 + 30 + 18 = 208$

d. $(48)(42)$
 $1600 + 320 + 80 + 16 = 2016$

e. $(72)(75)$
 $4900 + 140 + 350 + 10 = 5400$

2. All of the previous products involved addition, how do you think it would be different if they also included subtraction? What if the products involved both addition and subtraction?

a. $(27)(37)$ can be thought of as $(30 - 3)(30 + 7)$.

	30	-3
30	900	-90
7	210	-21

Using this model, we find the product equals $900 + 210 + -90 + -21 = 999$. Verify this solution using a calculator. Are we correct? *Yes*

Use both addition and subtraction to rewrite these similarly and use the area models to calculate each of the following products:

b. $(46)(57)$
 $2500 - 200 + 350 - 28 = 2622$

c. $(16)(25)$
 $400 - 80 + 100 - 20 = 400$

d. $(38)(42)$
 $1600 - 80 + 80 - 4 = 1596$

e. $(62)(75)$
 $4900 - 560 + 350 - 40 = 4650$

3. Look at problem 2d above; is there anything special about the binomials that you wrote and the answer that you got? *(Answers may vary) Both factors were two away from 40 since $40 - 2 = 38$ and $40 + 2 = 42$. The rectangles with 80 and -80 canceled.*

a. With a partner compose three other multiplication questions that use the same idea. Explain your thinking. What must always be true for this special situation to work?

Answers may vary, but both factors should be equally away from the tens number

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Now calculate each of the following using what you have learned about these special binomials.

b. $(101)(99)$

$10,000 + 100 - 100 - 1 = 9999$

**Students may cancel these middle two terms and should begin seeing the “shortcut”*

c. $(22)(18)$

$400 + 40 - 40 - 4 = 396$

d. $(45)(35)$

$1600 + 200 - 200 - 25 = 1575$

e. $(2.2)(1.8)$

$4 + 0.4 - 0.4 - 0.04 = 3.96$

4. In Question 3, you computed several products of the form $(x + y)(x - y)$ verifying that the product is always of the form $x^2 - y^2$.

a. If we choose values for x and y so that $x = y$ what will the product be? *zero*

b. Is there any other way to choose numbers to substitute for x and y so that the product $(x + y)(x - y)$ will equal 0? *Yes, if x and y have opposite signs (additive inverses)*

c. In general, if the product of two numbers is zero, what must be true about one of them? *at least one must be zero*

d. These products are called are called conjugates. Give two examples of other conjugates. *any binomials in $a + b$ and $a - b$ form*

e. $(x + y)(x - y) = x^2 - y^2$ is called a polynomial **identity** because this statement of equality is true for all values of the variables.

f. Polynomials in the form of $a^2 - b^2$ are called the **difference of two squares**. Factor the following using the identity you wrote in problem 4e:

$x^2 - 25 = (x + 5)(x - 5)$

$x^2 - 121 = (x + 11)(x - 11)$

$x^4 - 49 = (x + 7)(x - 7)$

$4x^4 - 81 = (2x + 9)(2x - 9)$

5. Previously, you’ve probably been told you couldn’t factor the **sum of two squares**. These are polynomials that come in the form $a^2 + b^2$. Well you can factor these; just not with real numbers.

a. Recall $\sqrt{-1} = i$. What happens when you square both sides? *You get $i^2 = -1$*

b. Now multiply $(x + 5i)(x - 5i) = x^2 + 25i - 25i - 25i^2 = x^2 - 25i^2 = x^2 + 25$

Describe what you see. *I see that the result is a sum of two squares.*

c. I claim that you can factor the sum of two squares just like the difference of two squares, just with i 's after the constant terms. Do you agree? Why or why not? *Yes*

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d. This leads us to another polynomial identity for the sum of two squares.

$$a^2 + b^2 = (a + bi)(a - bi)$$

e. Factor the following using the identity you wrote in problem 5d:

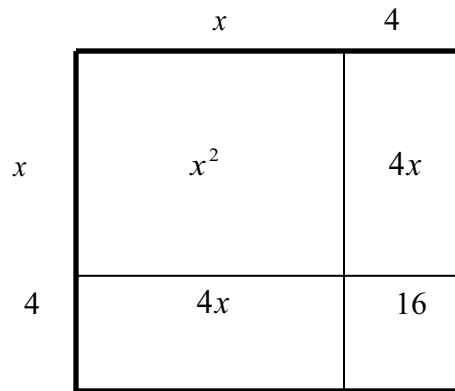
$$x^2 + 25 \quad (x + 5i)(x - 5bi)$$

$$x^2 + 121 \quad (x + 11i)(x - 11i)$$

$$x^2 + 49 \quad (x + 7i)(x - 7i)$$

$$4x^2 + 81 \quad (x + 9i)(x - 9i)$$

6. Now, let's consider another special case to see what happens when the numbers are the same. Start by considering the square below created by adding 4 to the length of each side of a square with side length x .



a. What is the area of the square with side length $= x$? x^2

b. What is the area of the rectangle with length $= x$ and width $= 4$? $4x$

c. What is the area of the rectangle with length $= 4$ and width $= x$? $4x$

d. What is the area of the square with side length $= 4$? 16

e. What is the total area of the square in the model above? $x^2 + 4x + 4x + 16 = x^2 + 8x + 16$

f. Draw a figure to illustrate the area of a square with side length $(x + y)$ assuming that x and y are positive numbers. Use your figure to explain the identity for a **perfect square trinomial**: $(x + y)^2 = x^2 + 2xy + y^2$ *Answers will vary*

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7. This identity gives a rule for squaring a sum. For example, 103^2 can be written as $(100 + 3)(100 + 3)$. Use this method to calculate each of the following by making convenient choices for x and y .

a. $302^2 = (300 + 2)(300 + 2) = 90,000 + 2(600) + 4 = 91,204$

b. $54^2 = (50 + 4)(50 + 4) = 2500 + 2(200) + 16 = 2,916$

c. $65^2 = (60 + 5)(60 + 5) = 3600 + 2(300) + 25 = 4,225$

d. $2.1^2 = (2 + 0.1)(2 + 0.1) = 4 + 2(0.2) + .01 = 4.41$

8. Determine the following identity: $(x - y)^2 = x^2 - 2xy + y^2$ Explain or show how you came up with your answer. *Answers will vary*

9. We will now extend the idea of identities to cubes.

a. What is the volume of a cube with side length 4? $4^3 = 64$

b. What is the volume of a cube with side length x ? x^3

c. Now we'll determine the volume of a cube with side length $x + 4$.

First, use the rule for squaring a sum to find the area of the base of the cube:

$$x^2 + 8x + 16$$

Now use the distributive property to multiply the area of the base by the height, $x + 4$, and simplify your answer: $x^3 + 12x^2 + 48x + 64$

d. Repeat part 8c for a cube with side length $x + y$. Write your result as a rule for the cube of a sum.

First, use the rule for squaring a sum to find the area of the base of the cube:

$$x^2 + 2xy + y^2$$

Now use the distributive property to multiply the area of the base by the height, $x + y$, and simplify your answer: $x^3 + 3x^2y + 3xy^2 + y^3$

e. So the identity for a **binomial cubed** is $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

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f. Determine the following identity: $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

Explain or show how you came up with your answer. *Answers will vary*

10. Determine whether the cube of a binomial is equivalent to the sum of two cubes by exploring the following expressions:

a. Simplify $(x + 2)^3$ $x^3 + 6x^2 + 12x + 8$

b. Simplify $x^3 + 2^3$ $x^3 + 8$

c. Is your answer to part a equivalent to your answer in part b? *no*

d. Simplify $(x + 2)(x^2 - 2x + 4)$ $x^3 + 8$

e. Is your answer to part b equivalent to your answer in part d? *yes*

f. Your answers to parts b and d should be equivalent. They illustrate two more commonly used polynomial identities:

$$\text{The Sum of Two Cubes: } a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$\text{The Difference of Two Cubes: } a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

g. Simplify the following and describe your results in words:

$$(x - 3)(x^2 + 3x + 9) \quad x^3 - 27$$

$$(2x + 5)(4x^2 - 10x + 25) \quad 8x^3 + 125$$

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11. Complete the table of polynomial identities to summarize your findings:

Description	Identity
Difference of Two Squares	$(a + b)(a - b) = a^2 - b^2$
Sum of Two Squares	$(a + bi)(a - bi) = a^2 + b^2$
Perfect Square Trinomial	$(a + b)^2 = a^2 + 2ab + b^2$
Perfect Square Trinomial	$(a - b)^2 = a^2 - 2ab + b^2$
Binomial Cubed	$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
Binomial Cubed	$(a - b)^3 = a^3 - 3a^2b + 3ab^2 + b^3$
Sum of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Two Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

12. Finally, let's look further into how we could raise a binomial to any power of interest. One way would be to use the binomial as a factor and multiply it by itself n times. However, this process could take a long time to complete. Fortunately, there is a quicker way. We will now explore and apply the binomial theorem, using the numbers in Pascal's triangle, to expand a binomial in $(a + b)^n$ form to the n^{th} power.

Binomial Expansion	Pascal's Triangle	n^{th} row
$(a + b)^0$	1	$n = 0$
$(a + b)^1$	1 1	$n = 1$
$(a + b)^2$	1 2 1	$n = 2$
$(a + b)^3$	1 3 3 1	$n = 3$
$(a + b)^4$	1 4 6 4 1	$n = 4$

a. Use the fourth row of Pascal's triangle to find the numbers in the fifth row:

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

Use the fifth row of Pascal's triangle to find the numbers in the sixth row:

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

Use the sixth row of Pascal's triangle to find the numbers in the seventh row:

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

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b. The binomial coefficients from the third row of Pascal's Triangle are 1, 3, 3, 1, so the expansion of $(x + 2)^3 = (1)(x^3)(2^0) + (3)(x^2)(2^1) + (3)(x^1)(2^2) + (1)(x^0)(2^3)$. Describe the pattern you see, then simplify the result: $x^3 + 6x^2 + 12x + 8$

c. Use Pascal's triangle in order to expand the following:

$$(x + 5)^3 = x^3 + 15x^2 + 75x + 125$$

$$(x + 1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$$

$$(x + 3)^5 = x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243$$

d. To expand binomials representing differences, rather than sums, the binomial coefficients will remain the same but the signs will alternate beginning with positive, then negative, then positive, and so on. Simplify the following and compare the result to problem 12b.

$$(x - 2)^3 = (1)(x^3)(2^0) - (3)(x^2)(2^1) + (3)(x^1)(2^2) - (1)(x^0)(2^3) = x^3 - 6x^2 + 12x - 8$$

e. Use Pascal's triangle in order to expand the following:

$$(x - 5)^3 = x^3 - 15x^2 + 75x - 125$$

$$(x - 2)^4 = x^4 - 8x^3 + 24x^2 - 32x + 1$$

$$(x - 10)^5 = x^5 - 50x^4 + 1,000x^3 - 10,000x^2 + 50,000x - 100,000$$

What's Your Identity?

Introduction

Equivalent algebraic expressions, also called algebraic identities, give us a way to express results with numbers that always work a certain way. In this task you will explore several “number tricks” that work because of basic algebra rules. You will extend these observations to algebraic expressions in order to prove polynomial identities. Finally, you will learn and apply methods to expand binomials. It is recommended that you do this task with a partner.

Materials

- Pencil
- Handout
- Calculator

1. First, you will explore an alternate way to multiply two digit numbers that have the same digit in the ten's place.

- a. For example, $(31)(37)$ can be thought of as $(30 + 1)(30 + 7)$.

	30	1
30	900	30
7	210	7

Using this model, we find the product equals $900 + 210 + 30 + 7 = 1147$. Verify this solution using a calculator. Are we correct?

Rewrite these similarly and use area models to calculate each of the following products:

b. $(52)(57)$

c. $(16)(13)$

d. $(48)(42)$

e. $(72)(75)$

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2. All of the previous products involved addition, how do you think it would be different if they also included subtraction? What if the products involved both addition and subtraction?

a. $(27)(37)$ can be thought of as $(30 - 3)(30 + 7)$.

	30	-3
30	900	-90
7	210	-21

Using this model, we find the product equals $900 + 210 + -90 + -21 = 999$. Verify this solution using a calculator. Are we correct?

Use both addition and subtraction to rewrite these similarly and use the area models to calculate each of the following products:

b. $(46)(57)$

c. $(16)(25)$

d. $(38)(42)$

e. $(62)(75)$

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3. Look at problem 2d above; is there anything special about the binomials that you wrote and the answer that you got?

a. With a partner compose three other multiplication questions that use the same idea. Explain your thinking. What must always be true for this special situation to work?

Now calculate each of the following using what you have learned about these special binomials.

b. $(101)(99)$

c. $(22)(18)$

d. $(45)(35)$

e. $(2.2)(1.8)$

4. In Question 3, you computed several products of the form $(x + y)(x - y)$ verifying that the product is always of the form $x^2 - y^2$.

a. If we choose values for x and y so that $x = y$ what will the product be?

b. Is there any other way to choose numbers to substitute for x and y so that the product $(x + y)(x - y)$ will equal 0?

c. In general, if the product of two numbers is zero, what must be true about one of them?

d. These products are called conjugates. Give two examples of other conjugates.

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e. $(x + y)(x - y) = \underline{\hspace{2cm}}$ is called a polynomial **identity** because this statement of equality is true for all values of the variables.

f. Polynomials in the form of $a^2 - b^2$ are called the **difference of two squares**. Factor the following using the identity you wrote in problem 4e:

$$x^2 - 25$$

$$x^2 - 121$$

$$x^4 - 49$$

$$4x^4 - 81$$

5.. Previously, you've probably been told you couldn't factor the **sum of two squares**. These are polynomials that come in the form $a^2 + b^2$. Well you can factor these; just not with real numbers.

a. Recall $\sqrt{-1} = i$. What happens when you square both sides?

b. Now multiply $(x + 5i)(x - 5i) = \underline{\hspace{2cm}}$. Describe what you see.

c. I claim that you can factor the sum of two squares just like the difference of two squares, just with i 's after the constant terms. Do you agree? Why or why not?

d. This leads us to another polynomial identity for the sum of two squares.

$$a^2 + b^2 = \underline{\hspace{2cm}}$$

e. Factor the following using the identity you wrote in problem 5d:

$$x^2 + 25$$

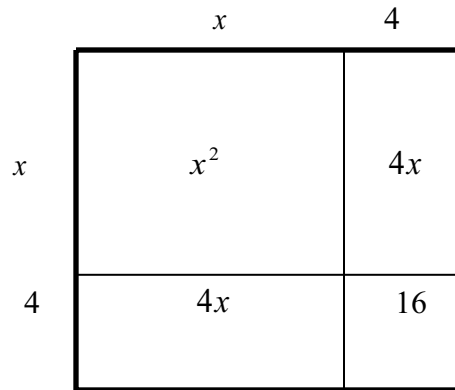
$$x^2 + 121$$

$$x^2 + 49$$

$$4x^2 + 81$$

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6. Now, let's consider another special case to see what happens when the numbers are the same. Start by considering the square below created by adding 4 to the length of each side of a square with side length x .



- a. What is the area of the square with side length $= x$?
 - b. What is the area of the rectangle with length $= x$ and width $= 4$?
 - c. What is the area of the rectangle with length $= 4$ and width $= x$?
 - d. What is the area of the square with side length $= 4$?
 - e. What is the total area of the square in the model above?
- f. Draw a figure to illustrate the area of a square with side length $(x + y)$ assuming that x and y are positive numbers. Use your figure to explain the identity for a **perfect square trinomial**: $(x + y)^2 = x^2 + 2xy + y^2$

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7. This identity gives a rule for squaring a sum. For example, 103^2 can be written as $(100 + 3)(100 + 3)$. Use this method to calculate each of the following by making convenient choices for x and y .

a. 302^2

b. 54^2

c. 65^2

d. 2.1^2

8. Determine the following identity: $(x - y)^2 = \underline{\hspace{4cm}}$. Explain or show how you came up with your answer.

9. We will now extend the idea of identities to cubes.

a. What is the volume of a cube with side length 4?

b. What is the volume of a cube with side length x ?

c. Now we'll determine the volume of a cube with side length $x + 4$.

First, use the rule for squaring a sum to find the area of the base of the cube:

Now use the distributive property to multiply the area of the base by the height, $x + 4$, and simplify your answer:

d. Repeat part 8c for a cube with side length $x + y$. Write your result as a rule for the cube of a sum.

First, use the rule for squaring a sum to find the area of the base of the cube:

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Now use the distributive property to multiply the area of the base by the height, $x + y$, and simplify your answer:

e. So the identity for a **binomial cubed** is $(x + y)^3 =$ _____

f. Determine the following identity: $(x - y)^3 =$ _____ .
Explain or show how you came up with your answer.

10. Determine whether the cube of a binomial is equivalent to the sum of two cubes by exploring the following expressions:

a. Simplify $(x + 2)^3$

b. Simplify $x^3 + 2^3$

c. Is your answer to 10a equivalent to your answer in 10b?

d. Simplify $(x + 2)(x^2 - 2x + 4)$

e. Is your answer to part b equivalent to your answer in part d?

f. Your answers to parts b and d should be equivalent. They illustrate two more commonly used polynomial identities:

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The Sum of Two Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

The Difference of Two Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

g. Simplify the following and describe your results in words:

$$(x - 3)(x^2 + 3x + 9)$$

$$(2x + 5)(4x^2 - 10x + 25)$$

11. Complete the table of polynomial identities to summarize your findings:

Description	Identity
Difference of Two Squares	$(a + b)(a - b) =$
Sum of Two Squares	$(a + bi)(a - bi) =$
Perfect Square Trinomial	$(a + b)^2 =$
Perfect Square Trinomial	$(a - b)^2 =$
Binomial Cubed	$(a + b)^3 =$
Binomial Cubed	$(a - b)^3 =$
Sum of Two Cubes	$a^3 + b^3 =$
Difference of Two Cubes	$a^3 - b^3 =$

12. Finally, let's look further into how we could raise a binomial to any power of interest. One way would be to use the binomial as a factor and multiply it by itself n times. However, this process could take a long time to complete. Fortunately, there is a quicker way. We will now explore and apply the binomial theorem, using the numbers in Pascal's triangle, to expand a binomial in $(a + b)^n$ form to the n^{th} power.

Binomial Expansion	Pascal's Triangle	n^{th} row
$(a + b)^0$	1	$n = 0$
$(a + b)^1$	1 1	$n = 1$
$(a + b)^2$	1 2 1	$n = 2$
$(a + b)^3$	1 3 3 1	$n = 3$
$(a + b)^4$	1 4 6 4 1	$n = 4$

a. Use the fourth row of Pascal's triangle to find the numbers in the fifth row:

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Use the fifth row of Pascal's triangle to find the numbers in the sixth row:

Use the sixth row of Pascal's triangle to find the numbers in the seventh row:

b. The binomial coefficients from the third row of Pascal's Triangle are 1, 3, 3, 1, so the expansion of $(x + 2)^3 = (1)(x^3)(2^0) + (3)(x^2)(2^1) + (3)(x^1)(2^2) + (1)(x^0)(2^3)$. Describe the pattern you see, and then simplify the result:

c. Use Pascal's triangle in order to expand the following:

$$(x + 5)^3 =$$

$$(x + 1)^4 =$$

$$(x + 3)^5 =$$

d. To expand binomials representing differences, rather than sums, the binomial coefficients will remain the same but the signs will alternate beginning with positive, then negative, then positive, and so on. Simplify the following and compare the result part b.

$$(x - 2)^3 = (1)(x^3)(2^0) - (3)(x^2)(2^1) + (3)(x^1)(2^2) - (1)(x^0)(2^3)$$

e. Use Pascal's triangle in order to expand the following:

$$(x - 5)^3 =$$

$$(x - 2)^4 =$$

$$(x - 10)^5 =$$

Rewriting a Rational Expression

Math Goals

- Rewrite simple rational expressions using long division
- Rewrite simple rational expression using synthetic division

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MGSE9-12.A.APR.6 Rewrite simple rational expressions in different forms using inspection, long division, or a computer algebra system; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$.

Standards for Mathematical Practice

1. Attend to precision
2. Look for and make use of structure
3. Look for and express regularity in repeated reasoning

Introduction

This task teaches students how to complete polynomial long division as well as synthetic division. Students will use long division to help them find slant asymptotes in a later task. A key point to synthetic division is that it can only be completed when the divisor is linear. This is a good task to relate back to the previous unit and The Remainder Theorem.

Rewriting a Rational Expression

A Rational Function is defined as the quotient of two polynomials. It follows that rational functions can be rewritten in various forms after division is performed. Let's use the quotient $\frac{x^3 + 2x^2 - 5x - 6}{x - 2}$ to illustrate this idea.

First, let's think about something we learned in elementary school, long division.

Think about the problem $46 \overline{)3768}$. What did you think about to start the division problem? Try to complete the entire long division problem below. Talk to your partner about the steps and what operations you use to complete the problem.

Answers will vary. Students need to think about the entire long division process including the repetition of "multiply, subtract, and bring down".

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Now, we are going to use the same idea to divide polynomials. Specifically,

$$x - 2 \overline{) x^3 + 2x^2 - 5x - 6}$$

$$\begin{array}{r} x^2 + 4x + 3 \\ x-2 \overline{) x^3 + 2x^2 - 5x - 6} \\ \underline{-(x^3 - 2x^2)} \\ 4x^2 - 5x \\ \underline{-(4x^2 - 8x)} \\ 3x - 6 \\ \underline{-(3x - 6)} \\ 0 \end{array}$$

How did you know when you were finished with the problem? Was there a remainder? If so, how do you write the final solution to show that there was a remainder?

The problem was finished when there was nothing left to bring down. There is no remainder in this problem.

Now try this one: $x + 3 \overline{) x^3 - 6x^2 + x + 10}$

$$\begin{array}{r} x^2 - 9x + 28 + \frac{-74}{x+3} \\ x+3 \overline{) x^3 - 6x^2 + x + 10} \\ \underline{-(x^3 + 3x^2)} \\ -9x^2 + x \\ \underline{-(-9x^2 - 27x)} \\ 28x + 10 \\ \underline{-(28x + 84)} \\ -74 \end{array}$$

How did you know when you were finished with the problem? Was there a remainder? If so, how do you write the final solution to show that there was a remainder?

The problem was finished when there was nothing left to bring down. This problem did have a remainder. In order to incorporate the remainder into the final solution, you must write it as the numerator of a fraction with the divisor as the denominator. You will then add this fraction to the end of the polynomial that was obtained as the quotient, $x^2 - 9x + 28 + \frac{-74}{x+3}$.

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As you can see, long division can be quite tedious. However, when the divisor is linear, there is a short cut. Let us consider another way to show this division called synthetic division. The next part of this task will explore how it works and why it only works when there is a linear divisor.

The following excerpt is adapted from:

J.M. Kingston, *Mathematics for Teachers of the Middle Grades*, John Wiley & Sons, Inc. NY, 1966, p. 203-205.

The labor of dividing a polynomial by $x - t$ can be reduced considerably by eliminating the symbols that occur repetitiously in the procedure. Let us consider the following division:

$$\begin{array}{r}
 4x^3 + 5x^2 + 3x + 2 \\
 x - 2 \overline{) 4x^4 - 3x^3 - 7x^2 - 4x - 9} \\
 \underline{4x^4 - 8x^3} \\
 5x^3 - 7x^2 \\
 \underline{5x^3 - 10x^2} \\
 3x^2 - 4x \\
 \underline{3x^2 - 6x} \\
 2x - 9 \\
 \underline{2x - 4} \\
 -5
 \end{array}$$

Notice that the math is being performed with the coefficients of each power of x . The powers of x themselves are redundant because the position of the term signifies its power. If you remove the powers of x , the problem is greatly simplified:

$$\begin{array}{r}
 4 \quad 5 \quad 3 \quad 2 \\
 -2 \overline{) 4 \quad -3 \quad -7 \quad -4 \quad -9} \\
 \underline{-8} \\
 5 \\
 \underline{-10} \\
 3 \\
 \underline{-6} \\
 2 \\
 \underline{-4} \\
 -5
 \end{array}$$

You can “collapse” the problem and write all the arithmetic in one row like this:

$$\begin{array}{r} -2 \overline{) 4 - 3 - 7 - 4 - 9} \\ \underline{-8 - 10 - 6 - 4} \\ 4 \quad 5 \quad 3 \quad 2 - 5 \end{array}$$

Note that

$$\begin{aligned} -8 &= 4(-2) \\ -10 &= 5(-2) \\ -6 &= 3(-2) \\ -4 &= 2(-2) \end{aligned}$$

When completing a long division problem you “subtract and bring down” over and over again. In this simplified version it will be easier to think of subtraction as addition so we must change our divisor to 2 to make this accommodation.

$$\begin{array}{r} 2 \overline{) 4 - 3 - 7 - 4 - 9} \\ \underline{8 \quad 10 \quad 6 \quad 4} \\ 4 \quad 5 \quad 3 \quad 2 - 5 \end{array}$$

We now have a final streamlined process called Synthetic Division where you bring down the first coefficient and then “multiply by t and add”.

There are a few things to consider. The number in the upper left-hand corner is t , if we are dividing by $x - t$. What would be the divisor if we are dividing by $x + t$?

The divisor would be $-t$. It is important for students to see that we are actually using the opposite of t in the upper left-hand corner.

The top row consists of the coefficients of the terms of the dividend polynomial in descending order. Since the order of the coefficients denotes its corresponding power of x what do you think happens if the dividend is missing a term in the sequence? For instance, how would we represent $5x^4 + 2x - 3$ as a dividend in synthetic division?

If a power of x is missing, we must leave a space for it. You must insert 0 to act as placeholders for missing powers. $5x^4 + 2x - 3$ would be represented by 5 0 0 2 -3 in synthetic division.

Take a moment to go back to the original long division problems in this task. Complete both of them using Synthetic Division. What do you notice about the right-hand number in the final row of the problem?

$$\begin{array}{r|rrrr} 2 & 1 & 4 & 3 & \\ & \downarrow & 2 & 8 & 6 \\ \hline & 1 & 2 & -5 & -6 \\ & & & & \boxed{0} \end{array}$$

$$x^2 + 4x + 3$$

$$\begin{array}{r|rrrr} -3 & 1 & -6 & 1 & 10 \\ & \downarrow & -3 & 27 & -84 \\ \hline & 1 & -9 & 28 & \boxed{-74} \end{array}$$

$$x^2 - 9x + 28 + \frac{-74}{x+3}$$

Students should notice that the final number in the final row is the remainder. You can relate this back to the previous unit and the Remainder Theorem.

To finally rewrite our original rational function in a new way we must reunite our coefficients and their corresponding powers of x while also making sure to show a remainder if necessary. Take this completed synthetic division problem and write the original rational function in the form $\frac{a(x)}{b(x)}$ as well as the “new” form $q(x) + \frac{r(x)}{b(x)}$:

$$\begin{array}{r|rrrr} 2 & 3 & 0 & -6 & 14 \\ & & 6 & 12 & 12 \\ \hline & 3 & 6 & 6 & 2 \end{array}$$

Students will first need to figure out what the original division problem is by looking at the worked out solution, $\frac{3x^3 - 6x^2 + 6x + 14}{x - 2}$. Then they will need to write the quotient, $3x^2 + 6x + 6 + \frac{14}{x - 2}$.

Now that we’ve learned the process let’s practice some more synthetic division problems. Rewrite the following rational expressions using synthetic division.

a. $\frac{10x^3 - 17x^2 - 7x + 2}{x - 2}$

$$\begin{array}{r|rrrr} 2 & 10 & -17 & -7 & 2 \\ & \downarrow & 20 & 6 & -2 \\ \hline & 10 & 3 & -1 & \boxed{0} \end{array}$$

$$10x^2 + 3x - 1$$

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b. $\frac{x^3 + 3x^2 - 10x - 24}{x + 4}$

Handwritten long division showing the division of $x^3 + 3x^2 - 10x - 24$ by $x + 4$. The quotient is $x^2 - x - 6$ and the remainder is 0.

$$\begin{array}{r} -4 \overline{) 1 \ 3 \ -10 \ -24} \\ \underline{\downarrow -4 \ 4 \ 24} \\ 1 \ -1 \ -6 \ \underline{0} \\ x^2 - x - 6 \end{array}$$

c. $\frac{x^3 - 7x - 6}{x + 1}$

Handwritten long division showing the division of $x^3 - 7x - 6$ by $x + 1$. The quotient is $x^2 - x - 6$ and the remainder is 0.

$$\begin{array}{r} -1 \overline{) 1 \ 0 \ -7 \ -6} \\ \underline{\downarrow -1 \ 1 \ 6} \\ 1 \ -1 \ -6 \ \underline{0} \\ x^2 - x - 6 \end{array}$$

Rewriting a Rational Expression:

A Rational Function is defined as the quotient of two polynomials. It follows that rational functions can be rewritten in various forms after division is performed. Let's use the quotient $\frac{x^3 + 2x^2 - 5x - 6}{x - 2}$ to illustrate this idea.

First, let's think about something we learned in elementary school, long division. Think about the problem $46 \overline{)3768}$. What did you think about to start the division problem? Try to complete the entire long division problem below. Talk to your partner about the steps and what operations you use to complete the problem.

Now, we are going to use the same idea to divide polynomials. Specifically,

$$x - 2 \overline{)x^3 + 2x^2 - 5x - 6}$$

How did you know when you were finished with the problem? Was there a remainder? If so, how do you write the final solution to show that there was a remainder?

Now try this one: $x + 3 \overline{)x^3 - 6x^2 + x + 10}$

How did you know when you were finished with the problem? Was there a remainder? If so, how do you write the final solution to show that there was a remainder?

As you can see, long division can be quite tedious. However, when the divisor is linear, there is a short cut. Let us consider another way to show this division called synthetic division. The next part of this task will explore how it works and why it only works when there is a linear divisor.

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The labor of dividing a polynomial by $x - t$ can be reduced considerably by eliminating the symbols that occur repetitiously in the procedure. Let us consider the following division:

$$\begin{array}{r}
 4x^3 + 5x^2 + 3x + 2 \\
 \hline
 x - 2 \overline{) 4x^4 - 3x^3 - 7x^2 - 4x - 9} \\
 \underline{4x^4 - 8x^3} \\
 5x^3 - 7x^2 \\
 \underline{5x^3 - 10x^2} \\
 3x^2 - 4x \\
 \underline{3x^2 - 6x} \\
 2x - 9 \\
 \underline{2x - 4} \\
 -5
 \end{array}$$

Notice that the math is being performed with the coefficients of each power of x . The powers of x themselves are redundant because the position of the term signifies its power. If you remove the powers of x , the problem is greatly simplified:

$$\begin{array}{r}
 4 \ 5 \ 3 \ 2 \\
 \hline
 -2 \overline{) 4 \ -3 \ -7 \ -4 \ -9} \\
 \underline{-8} \\
 5 \\
 \underline{-10} \\
 3 \\
 \underline{-6} \\
 2 \\
 \underline{-4} \\
 -5
 \end{array}$$

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You can “collapse” the problem and write all the arithmetic in one row like this:

$\underline{-2} \mid$	4	-3	-7	-4	-9	Note that	-8 = 4(-2)
		-8	-10	-6	-4		-10 = 5(-2)
		4	5	3	2	-5	-6 = 3(-2)
							-4 = 2(-2)

When completing a long division problem you “subtract and bring down” over and over again. In this simplified version it will be easier to think of subtraction as addition so we must change our divisor to 2 to make this accommodation.

$$\begin{array}{r} \underline{2} \mid 4 \quad -3 \quad -7 \quad -4 \quad -9 \\ 8 \quad 10 \quad 6 \quad 4 \\ \hline 4 \quad 5 \quad 3 \quad 2 \quad -5 \end{array}$$

We now have a final streamlined process called Synthetic Division where you bring down the first coefficient and then “multiply by t and add”.

There are a few things to consider. The number in the upper left-hand corner is t , if we are dividing by $x - t$. What would be the divisor if we are dividing by $x + t$?

The top row consists of the coefficients of the terms of the dividend polynomial in descending order. Since the order of the coefficients denotes its corresponding power of x what do you think happens if the dividend is missing a term in the sequence? For instance, how would we represent $5x^4 + 2x - 3$ as a dividend in synthetic division?

Take a moment to go back to the original long division problems in this task. Complete both of them using Synthetic Division. What do you notice about the right-hand number in the final row of the problem?

To finally rewrite our original rational function in a new way we must reunite our coefficients and their corresponding powers of x while also making sure to show a remainder if necessary. Take this completed synthetic division problem and write the original rational function in the form $\frac{a(x)}{b(x)}$ as well as the “new” form $q(x) + \frac{r(x)}{b(x)}$:

$$\begin{array}{r} \underline{2} \mid \quad 3 \quad 0 \quad -6 \quad 2 \\ \quad \quad 6 \quad 12 \quad 12 \\ \hline 3 \quad 6 \quad 6 \quad 14 \end{array}$$

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Now that we've learned the process let's practice some more synthetic division problems. Rewrite the following rational expressions using synthetic division.

a.
$$\frac{10x^3 - 17x^2 - 7x + 2}{x - 2}$$

b.
$$\frac{x^3 + 3x^2 - 10x - 24}{x + 4}$$

c.
$$\frac{x^3 - 7x - 6}{x + 1}$$

Finding Inverses Task (Learning Task)

Formally :Leading to Logarithms (Learning Task)

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Build new functions from existing functions

MGSE9-12.F.BF.4 Find inverse functions.

MGSE9-12.F.BF.4a Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2(x^3)$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.

Math Goals

In order to understand that a logarithm is an inverse of an exponential, a student should be able to calculate the inverses of simple functions. This task leads a student through the procedure of finding inverses of one-to-one functions. Logarithms will be discussed in unit 5.

Suppose we have a function f that takes x to y , so that

$$f(x) = y.$$

An inverse function, which we call f^{-1} , is another function that takes y back to x . So

$$f^{-1}(y) = x.$$

For f^{-1} to be an inverse of f , this needs to work for every x that f acts upon.

Key Point

The inverse of the function f is the function that sends each $f(x)$ back to x . We denote the inverse of f by f^{-1} .

Working out f^{-1} by reversing the operations of f

One way to work out an inverse function is to reverse the operations that f carries out on a Number.

Example: We shall set $f(x) = 4x$, so that f takes a number x and multiplies it by 4:

$$f(x) = 4x \text{ (multiply by 4).}$$

We want to define a function that will take 4 times x , and send it back to x . This is the same as saying that $f^{-1}(x)$ divides x by 4. So

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There is an important point about notation here. You should notice that $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$. For this example, $\frac{1}{f(x)}$ would be $\frac{1}{4x}$ with the x in the denominator, and that is not the same.

Here is a slightly more complicated **example**. Suppose we have $f(x) = 3x + 2$.

We can break up this function into a series of operations. First the function multiplies by 3, and then it adds on 2.

- x
- Times 3
- Then, add 2

To get back to x from $f(x)$, we would need to reverse these operations. So we would need to take away 2, and then divide by 3. When we undo the operations, we have to reverse the order as well.

- $x - 2$
- Then, divide by 3

$$f^{-1}(x) = \frac{x-2}{3}$$

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Here is one more **example** of how we can reverse the operations of a function to find its inverse.

Suppose we have

$$f(x) = 7 - x^3.$$

It is easier to see the sequence of operations to be carried out on x if we rewrite the function as

$$f(x) = -x^3 + 7.$$

So the first operation performed by f takes x and cubes it; then the result is multiplied by -1 ; and finally 7 is added on.

- x^3
- Times -1
- Plus, 7

So to get from $f(x)$ to x , we need to

- Subtract 7
- Then, divide by -1
- And, take the cube root

$$\text{So, } f^{-1}(x) = \sqrt[3]{-x + 7}$$

Key Point

We can work out f^{-1} by reversing the operations of f . If there is more than one operation, then we must reverse the order as well as reversing the individual operations.

Exercises

Work out the inverses of the following functions:

(a) $f(x) = 6x$

$$f^{-1}(x) = \frac{x}{6}$$

(b) $f(x) = 3 + 4x^3$

$$f^{-1}(x) = \sqrt[3]{\frac{x-3}{4}}$$

(c) $f(x) = 1 - 3x$

$$f^{-1}(x) = \frac{-x+1}{3}$$

Using algebraic manipulation to work out inverse functions

Another way to work out inverse functions is by using algebraic manipulation. We can demonstrate this using our second example, $f(x) = 3x + 2$.

Now the inverse function takes us from $f(x)$ back to x . If we set $y = f(x) = 3x + 2$, then f^{-1} is the function that takes y to x . So to work out f^{-1} we need to know how to get to x from y . If we rearrange the expression to solve for x

$$y = 3x + 2,$$

$$y - 2 = 3x$$

so that

$$x = \frac{y - 2}{3}$$

So, reversing x and y yields $f^{-1}(x) = \frac{x - 2}{3}$

We can use the method of algebraic manipulation to work out inverses when we have slightly trickier functions than the ones we have seen so far. Let us take

$$f(x) = \frac{3x}{2x - 1}, x \neq \frac{1}{2}.$$

We have made the restriction $x \neq \frac{1}{2}$ because at $x = \frac{1}{2}$ the function does not have a value. This is because the denominator is zero when $x = \frac{1}{2}$.

Now we set $y = \frac{3x}{2x - 1}$. Multiplying both sides by $2x - 1$ we get

$$y(2x - 1) = 3x,$$

and then multiplying out the bracket gives

$$2yx - y = 3x.$$

We want to rearrange this equation so that we can express x as a function of y , and to do this we take the terms involving x to the left-hand side, giving

$$2yx - 3x = y.$$

Now we can then take out x as a factor on the left-hand side to get

$$x(2y - 3) = y,$$

and dividing throughout by $2y - 3$ we finally obtain

$$x = \frac{y}{2y - 3}$$

So the by reversing the x and y the inverse function is $f^{-1}(x) = \frac{x}{2x - 3}$

In the last example, it would not have been possible to work out the inverse function by trying to reverse the operations of f . This example shows how useful it is to have algebraic manipulation to work out inverses.

Key Point

Algebraic manipulation is another method that can be used to work out inverse functions. The key points are to solve the function for x and then reverse the x and the y .

This last form is the way to find the inverse of an exponential function.

Exercises

Find the inverse of the following using this algebraic manipulation method.

(a) $f(x) = -5x - 1$

$$f^{-1}(x) = \frac{-x-1}{5}$$

(b) $f(x) = \frac{3x+7}{2}$

$$f^{-1}(x) = \frac{2x-7}{3}$$

(c) $f(x) = \frac{2x}{5x-1}$

$$f^{-1}(x) = \frac{x}{5x-2}$$

(d) $f(x) = \frac{x-1}{2x+3}$

$$f^{-1}(x) = \frac{-3x-1}{2x-1}$$

More in depth practice with inverses can be found at

http://www.khanacademy.org/math/trigonometry/functions_and_graphs/function_inverses/v/introduction-to-function-inverses

Finding Inverses Task (Learning Task)

Formally: Leading to Logarithms (Learning Task)

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Build new functions from existing functions

MGSE9-12.F.BF.4 Find inverse functions.

MGSE9-12.F.BF.4a Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2(x^3)$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.

Math Goals

In order to understand that a logarithm is an inverse of an exponential, a student should be able to calculate the inverses of simple functions. This task leads a student through the procedure of finding inverses of one-to-one functions. Logarithms will be discussed in unit 5.

Suppose we have a function f that takes x to y , so that

$$f(x) = y.$$

An inverse function, which we call f^{-1} , is another function that takes y back to x . So

$$f^{-1}(y) = x.$$

For f^{-1} to be an inverse of f , this needs to work for every x that f acts upon.

Key Point

The inverse of the function f is the function that sends each $f(x)$ back to x . We denote the inverse of f by f^{-1} .

Working out f^{-1} by reversing the operations of f

One way to work out an inverse function is to reverse the operations that f carries out on a Number.

Example: We shall set $f(x) = 4x$, so that f takes a number x and multiplies it by 4:

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