Introduction

Let’s say you and your friends draw straws to see who has to do some unpleasant activity, like cleaning out the class pet’s cage. If everyone who draws before you keeps their straw, does that affect your odds of cleaning up after Fluffy the Hamster?

If you are drawing straws without replacing them, your friends’ outcomes do have an effect on yours—if there are several long straws and one unfortunate short one, your odds of drawing the short straw increase with every long straw drawn.

Introduction, *continued*

In this lesson, we will look at conditional probability—that is, the probability that an event will occur based on the condition that another event has occurred.

Key Concepts

- The **conditional probability of *B* given *A*** is the probability that event *B* occurs, given that event *A* has already occurred.
- If *A* and *B* are two events from a sample space with $P(A) \neq 0$, then the conditional probability of *B* given *A*, denoted $P(B|A)$, has two equivalent expressions:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\text{number of outcomes in } (A \text{ and } B)}{\text{number of outcomes in } A}$$

- The following slide explains these equivalent expressions.

Key Concepts, *continued*

$$P(B|A) = \frac{\text{number of outcomes in } (A \text{ and } B)}{\text{number of outcomes in } A}$$

- This uses subsets of the sample space.

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

- This uses the calculated probabilities $P(A \text{ and } B)$ and $P(A)$.

Key Concepts, *continued*

- The second formula can be rewritten as $P(A \text{ and } B) = P(A) \bullet P(B|A)$.
- $P(B|A)$ is read "the probability of *B* given *A*."
- Using set notation, conditional probability is written like this:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Key Concepts, *continued*

- The "conditional probability of *B* given *A*" only has meaning if event *A* has occurred. That is why the formula for $P(B|A)$ has the requirement that $P(A) \neq 0$.
- The conditional probability formula can be solved to obtain a formula for *P*(*A* and *B*), as shown on the next slide.

Key Concepts, *continued*

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Write the conditional probability formula.

$$P(A) \bullet P(B|A) = \frac{P(A \text{ and } B)}{P(A)} \bullet P(A)$$

Multiply both sides by *P*(*A*).

$$P(A) \bullet P(B|A) = P(A \text{ and } B)$$

Simplify.

$$P(A \text{ and } B) = P(A) \bullet P(B|A)$$

Reverse the left and right sides.

Key Concepts, *continued*

- Remember that independent events are two events such that the probability of both events occurring is equal to the product of the individual probabilities. Two events *A* and *B* are independent if and only if *P*(*A* and *B*) = *P*(*A*) • *P*(*B*). Using set notation, $P(A \cap B) = P(A) \bullet P(B)$. The occurrence or non-occurrence of one event has no effect on the probability of the other event.
- If *A* and *B* are independent, then the formula for *P*(*A* and *B*) is the equation used in the definition of independent events, as shown on the next slide.

Key Concepts, *continued*

$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$	formula for *P*(*A* and *B*)
$P(A \text{ and } B) = P(A) \cdot P(B)$	formula for *P*(*A* and *B*) if *A* and *B* are independent

Key Concepts, *continued*

- Using the conditional probability formula in different situations requires using different variables, depending on how the events are named. Here are a couple of examples.

$$P(B|A)=\frac{P(A \text{ and } B)}{P(A)}$$

Use this equation to find the probability of *B* given *A*.

$$P(A|B)=\frac{P(A \text{ and } B)}{P(B)}$$

Use this equation to find the probability of *A* given *B*.

Key Concepts, *continued*

- Another method to use when calculating conditional probabilities is dividing the number of outcomes in the intersection of *A* and *B* by the number of outcomes in a certain event:

$$P\left(A \middle| B\right) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } B}$$

Key Concepts, *continued*

- The following statements are equivalent. In other words, if any one of them is true, then the others are all true.
  - Events *A* and *B* are independent.
  - The occurrence of *A* has no effect on the probability of *B*; that is, $P(B|A) = P(B)$.
  - The occurrence of *B* has no effect on the probability of *A*; that is, $P(A|B) = P(A)$.
  - $P(A \text{ and } B) = P(A) \cdot P(B)$.

Key Concepts, *continued*

- *Note*: For real-world data, these modified tests for independence are sometimes used:
  - Events *A* and *B* are independent if the occurrence of *A* has no *significant* effect on the probability of *B*; that is, $P(B|A) \approx P(B)$.
  - Events *A* and *B* are independent if the occurrence of *B* has no *significant* effect on the probability of *A*; that is, $P(A|B) \approx P(A)$.

Key Concepts, *continued*

- When using these modified tests, good judgment must be used when deciding whether the probabilities are close enough to conclude that the events are independent.

Common Errors/Misconceptions

- thinking that $P(A|B)$ represents the probability of *A* occurring and then *B* occurring
- confusing union and intersection of sets
- incorrectly finding the probabilities of *A* or *B* or the intersection of *A* and *B*
- applying the formula for probability incorrectly

Guided Practice

Example 1

Alexis rolls a pair of number cubes. What is the probability that both numbers are odd if their sum is 6? Interpret your answer in terms of a uniform probability model.

Guided Practice: Example 1, *continued*

1. **Assign variable names to the events and state what you need to find, using conditional probability.**

   Let A be the event "Both numbers are odd."

   Let B be the event "The sum of the numbers is 6."

   You need to find the probability of A given B.

   That is, you need to find $P(A|B)$.

Guided Practice: Example 1, *continued*

2. Show the sample space.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Key: (2, 3) means 2 on the first cube and 3 on the second cube.

Guided Practice: Example 1, *continued*

3. Identify the outcomes in the events.

The outcomes for *A* are **bold and purple**.

***A*: Both numbers are odd.**

(1, 1)	(1, 2)	**(1, 3)**	(1, 4)	**(1, 5)**	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	**(3, 3)**	(3, 4)	**(3, 5)**	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	**(5, 3)**	(5, 4)	**(5, 5)**	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Guided Practice: Example 1, *continued*

The outcomes for *B* are **bold and purple**.

***B*: The sum of the numbers is 6.**

(1, 1)	(1, 2)	(1, 3)	(1, 4)	**(1, 5)**	(1, 6)
(2, 1)	(2, 2)	(2, 3)	**(2, 4)**	(2, 5)	(2, 6)
(3, 1)	(3, 2)	**(3, 3)**	(3, 4)	(3, 5)	(3, 6)
(4, 1)	**(4, 2)**	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Guided Practice: Example 1, *continued*

4. Identify the outcomes in the events $A \cap B$ and B.

Use the conditional probability formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

$A \cap B$ = the outcomes that are in *A* and also in *B*.

$$A \cap B = \{(1, 5), (3, 3), (5, 1)\}$$

B = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}

Guided Practice: Example 1, *continued*

5. Find $P(A \cap B)$ and $P(B)$.

$A \cap B$ has 3 outcomes; the sample space has 36 outcomes.

$$P(A \cap B) = \frac{\text{number of outcomes in } A \text{ and } B}{\text{number of outcomes in sample space}} = \frac{3}{36}$$

B has 5 outcomes; the sample space has 36 outcomes.

$$P(B) = \frac{\text{number of outcomes in } B}{\text{number of outcomes in sample space}} = \frac{5}{36}$$

Guided Practice: Example 1, *continued*

6. Find $P(A|B)$.

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

Write the conditional probability formula.

$$P(A|B)=\frac{\frac{3}{36}}{\frac{5}{36}}$$

Substitute the probabilities found in step 5.

$$P(A|B)=\frac{3}{36}\cdot\frac{36}{5}$$

To divide by a fraction, multiply by its reciprocal.

Guided Practice: Example 1, *continued*

$$P(A|B) = \frac{3}{\cancel{36}_1} \cdot \frac{\cancel{36}^1}{5}$$

Simplify.

$$P(A|B) = \frac{3}{5}$$

Guided Practice: Example 1, *continued*

7. Verify your answer.

Use this alternate conditional probability formula:

$$P(A|B) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } B}.$$

$A \cap B$ has 3 outcomes: {(1, 5), (3, 3), (5, 1)}.

B has 5 outcomes: {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}.

$$P(A|B) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } B}$$

$$P(A|B) = \frac{3}{5}$$

Guided Practice: Example 1, *continued*

8. **Interpret your answer in terms of a uniform probability model.**

The probabilities used in solving the problem are found by using these two ratios:

$$\frac{\text{number of outcomes in an event}}{\text{number of outcomes in the sample space}}$$

$$\frac{\text{number of outcomes in an event}}{\text{number of outcomes in a subset of the sample space}}$$

Guided Practice: Example 1, *continued*

These ratios are uniform probability models if all outcomes in the sample space are equally likely. It is reasonable to assume that Alexis rolls fair number cubes, so all outcomes in the sample space are equally likely. Therefore, the answer is valid and can serve as a reasonable predictor.

Guided Practice: Example 1, *continued*

You can predict the following: If you roll a pair of number cubes a large number of times and consider all the outcomes that have a sum of 6, then about $\frac{3}{5}$ of those outcomes will have both odd numbers.

✔

Guided Practice: Example 1, *continued*

Guided Practice

Example 3

A vacation resort offers bicycles and personal watercrafts for rent. The resort's manager made the following notes about rentals:

- 200 customers rented items in all—100 rented bicycles and 100 rented personal watercrafts.
- Of the personal watercraft customers, 75 customers were young (30 years old or younger) and 25 customers were older (31 years old or older).
- 125 of the 200 customers were age 30 or younger. 50 of these customers rented bicycles, and 75 of them rented personal watercrafts.

Guided Practice: Example 3, *continued*

Consider the following events that apply to a random customer.

Y: The customer is young (30 years old or younger).

W: The customer rents a personal watercraft.

Are *Y* and *W* independent? Compare $P(Y|W)$ and $P(W|Y)$ and interpret the results.

Guided Practice: Example 3, *continued*

1. Determine if *Y* and *W* are independent.

First, determine the probabilities of each event.

$$P(Y) = \frac{125}{200} = 0.625$$

Of 200 customers, 125 were young.

$$P(W) = \frac{100}{200} = 0.5$$

Of 200 customers, 100 rented a personal watercraft.

$$P(Y|W) = \frac{75}{100} = 0.75$$

Of 100 personal watercraft customers, 75 were young.

$$P(W|Y) = \frac{75}{125} = 0.6$$

Of 125 young customers, 75 rented a personal watercraft.

Guided Practice: Example 3, *continued*

Y and *W* are dependent because $P(Y|W) \neq P(Y)$ and $P(W|Y) \neq P(W)$.

Guided Practice: Example 3, *continued*

2. Compare $P(Y|W)$ and $P(W|Y)$.

$$P(Y|W) = 0.75$$

$$P(W|Y) = 0.6$$

0.75 > 0.6; therefore, $P(Y|W) > P(W|Y)$.

Guided Practice: Example 3, *continued*

3. Interpret the results.

$P(Y|W)$ represents the probability that a customer is young given that the customer rents a personal watercraft.

$P(W|Y)$ represents the probability that a customer rents a personal watercraft given that the customer is young.

Guided Practice: Example 3, *continued*

The dependence of the events Y and W means that a customer's age affects the probability that the customer rents a personal watercraft; in this case, being young increases that probability because $P(W|Y) > P(W)$. The dependence of the events Y and W also means that a customer renting a personal watercraft affects the probability that the customer is young; in this case, renting a personal watercraft increases that probability because $P(Y|W) > P(Y)$.

Guided Practice: Example 3, *continued*

$P(Y|W) > P(W|Y)$ means that it is more likely that a customer is young given that he or she rents a personal watercraft than it is that a customer rents a personal watercraft given that he or she is young.

✔

Guided Practice: Example 3, *continued*