Introduction

The graph of an equation in x and y is the set of all points (x, y) in a coordinate plane that satisfy the equation. Some equations have graphs with precise geometric descriptions. For example, the graph of the equation $y = 2x + 3$ is the line with a slope of 2, passing through the point (0, 3). This geometric description uses the familiar concepts of line, slope, and point. The equation $y = 2x + 3$ is an algebraic description of the line.

Introduction, *continued*

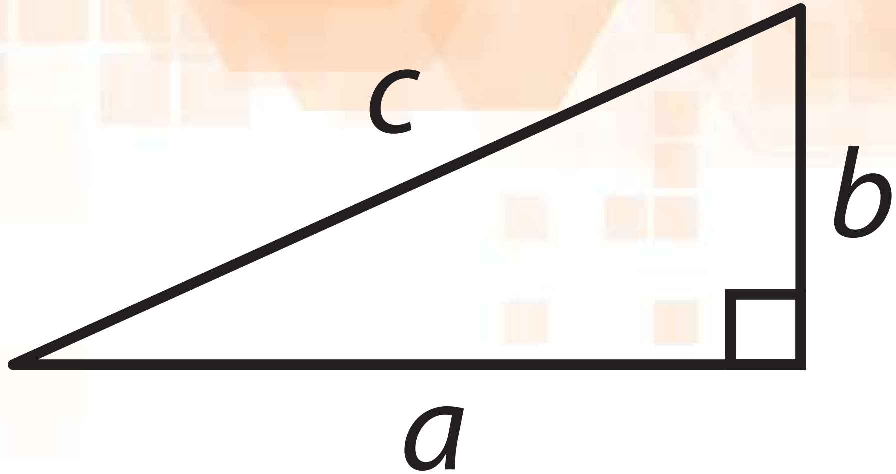
In this lesson, we will investigate how to translate between geometric descriptions and algebraic descriptions of circles. We have already learned how to use the Pythagorean Theorem to find missing dimensions of right triangles. Now, we will see how the Pythagorean Theorem leads us to the distance formula, which leads us to the standard form of the equation of a circle.

Key Concepts

- The standard form of the equation of a circle is based on the distance formula.
- The distance formula, in turn, is based on the Pythagorean Theorem.

Key Concepts, *continued*

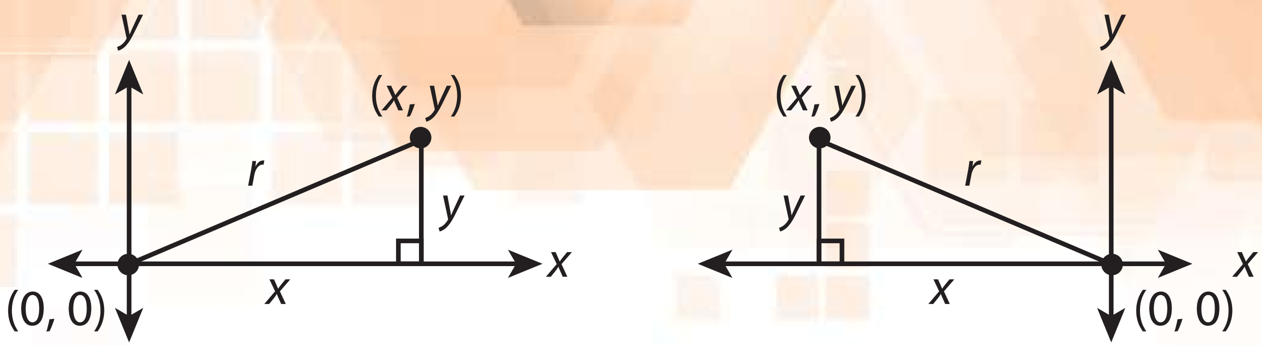
- The **Pythagorean Theorem** states that in any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.



$$a^2 + b^2 = c^2$$

Key Concepts, *continued*

- If r represents the distance between the origin and any point (x, y), then $x^2 + y^2 = r^2$.



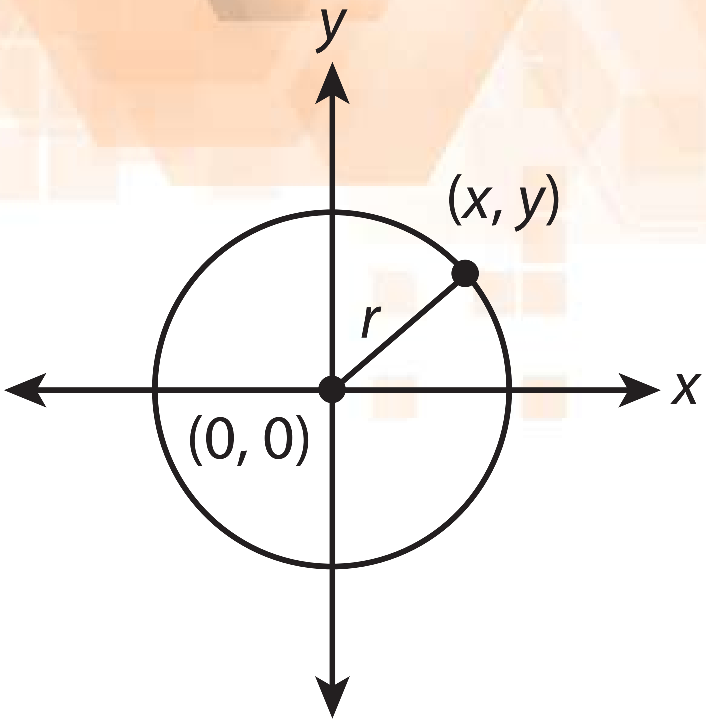
  - x can be positive, negative, or zero because it is a coordinate.
  - y can be positive, negative, or zero because it is a coordinate.
  - r cannot be negative because it is a distance.

Key Concepts, *continued*

- A **circle** is the set of all points in a plane that are equidistant from a reference point in that plane, called the center. The set of points forms a 2-dimensional curve that measures 360°.
- The **center of a circle** is the point in the plane of the circle from which all points on the circle are equidistant. The center is in the interior of the circle.
- The **radius** of a circle is the distance from the center to a point on the circle.

Key Concepts, *continued*

- For a circle with center (0, 0) and radius r, any point (x, y) is on that circle if and only if $x^2 + y^2 = r^2$.

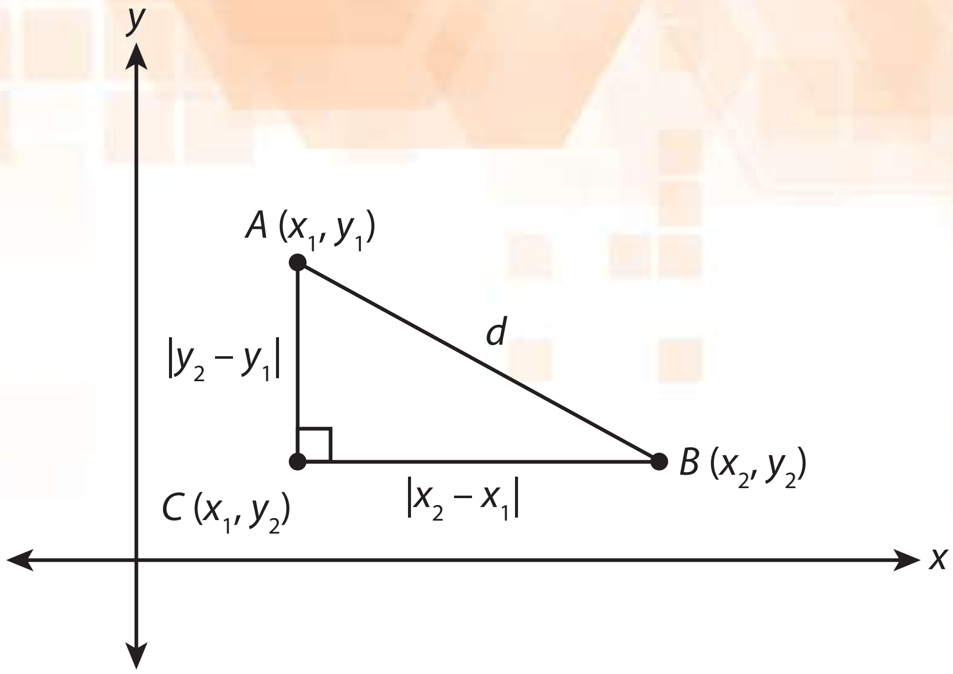


Key Concepts, *continued*

- The distance formula is used to find the distance between any two points on a coordinate plane.
- The **distance formula** states that the distance d between A (x_1, y_1) and B (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The distance formula is based on the Pythagorean Theorem.

Key Concepts, *continued*

- Look at this diagram of a right triangle with points *A*, *B,* and *C*. The distance *d* between points *A* and *B* is unknown.



Key Concepts, *continued*

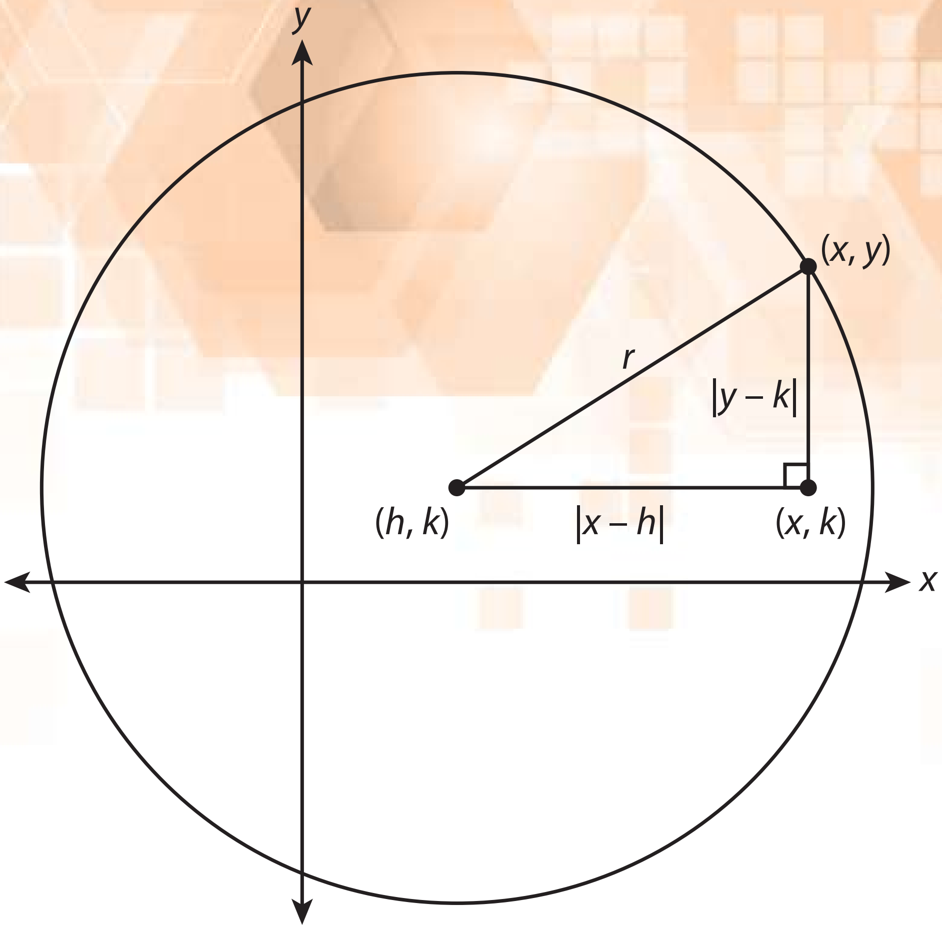
- The worked example below shows how the distance formula is derived from the Pythagorean Theorem, using the points from the diagram to find *d*:

$AB^2 = BC^2 + AC^2$	Pythagorean Theorem
$d^2 = \lvert x_2 - x_1 \rvert^2 + \lvert y_2 - y_1 \rvert^2$	Substitute values for sides *AB*, *BC*, and *AC* of the triangle.
$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$	Simplify. All squares are nonnegative.
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Take the square of each side of the equation to arrive at the distance formula.

Key Concepts, *continued*

- For a circle with center (h, k) and radius r, any point (x, y) is on that circle if and only if $\sqrt{(x-h)^2+(y-k)^2}=r$. Squaring both sides of this equation yields the **standard form of an equation of a circle** with center (h, k) and radius r: $(x - h)^2 + (y - k)^2 = r^2$.

Key Concepts, *continued*



Key Concepts, *continued*

- If a circle has center (0, 0), then its equation is $(x - 0)^2 + (y - 0)^2 = r^2$, or $x^2 + y^2 = r^2$.
- If the center and radius of a circle are known, then either of the following two methods can be used to write an equation for the circle:
  - Apply the Pythagorean Theorem to derive the equation.
  - Or, substitute the center coordinates and radius directly into the standard form.

Key Concepts, *continued*

- The **general form of an equation of a circle** is $Ax^2 + By^2 + Cx + Dy + E = 0$, where $A = B$, $A \neq 0$, and $B \neq 0$.
- If any one of the following three sets of facts about a circle is known, then the other two can be determined:
  - center (h, k) and radius r
  - standard equation: $(x - h)^2 + (y - k)^2 = r^2$
  - general equation: $Ax^2 + By^2 + Cx + Dy + E = 0$
- The general form of the equation of a circle comes from expanding the standard form of the equation of the circle.

Key Concepts, *continued*

- The standard form of the equation of a circle comes from completing the square from the general form of the equation of a circle.
- Every **perfect square trinomial** has the form

$x^2 + bx + \left(\frac{b}{2}\right)^2$ because it is the square of a binomial:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

Key Concepts, *continued*

- **Completing the square** is the process of determining the value of $\left(\frac{b}{2}\right)^2$ and adding it to $x^2 + bx$ to form the perfect square trinomial, $x^2 + bx + \left(\frac{b}{2}\right)^2$.

Common Errors/Misconceptions

- confusing the radius with the square of the radius
- forgetting to square half the coefficient of *x* when completing the square
- neglecting to square the denominator when squaring a fraction

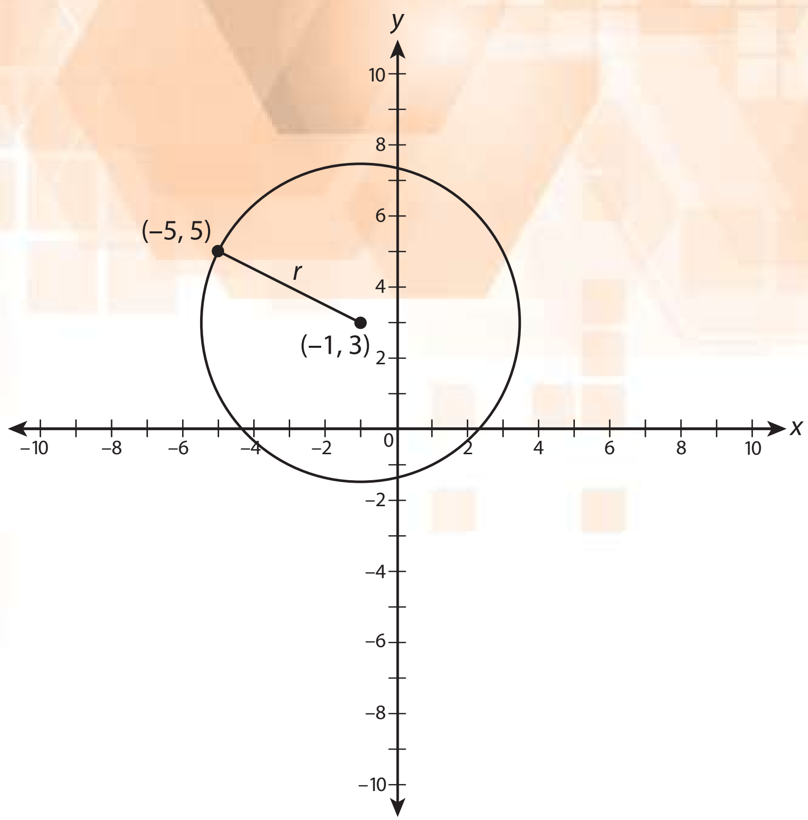
Guided Practice

Example 3

Write the standard equation and the general equation of the circle that has center (–1, 3) and passes through (–5, 5).

Guided Practice: Example 3, *continued*

1. Sketch the circle.



Guided Practice: Example 3, *continued*

2. **Use the distance formula to find the radius, *r*.**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ Distance formula

$$r = \sqrt{[(-5)-(-1)]^2 + [5-3]^2}$$ Substitute (–1, 3) and (–5, 5) for (x_1, y_1) and (x_2, y_2).

$$r = \sqrt{(-4)^2 + (2)^2}$$ Simplify.

$$r = \sqrt{16+4}$$

$$r = \sqrt{20}$$

Guided Practice: Example 3, *continued*

$r = \sqrt{4 \bullet 5}$	Write 20 as a product with a perfect square factor.
$r = \sqrt{4} \bullet \sqrt{5}$	Apply the property $\sqrt{ab} = \sqrt{a}\sqrt{b}$.
$r = 2\sqrt{5}$	Simplify.

Guided Practice: Example 3, *continued*

3. Substitute the center and radius directly into the standard form.

$(x-h)^2+(y-k)^2=r^2$	Standard form
$[x-(-1)]^2+(y-3)^2=(2\sqrt{5})^2$	Substitute values into the equation, using the center (–1, 3), and the radius $2\sqrt{5}$.
$(x+1)^2+(y-3)^2=20$	Simplify to obtain the standard equation.

The standard equation is $(x+1)^2+(y-3)^2=20$.

Guided Practice: Example 3, *continued*

4. Square the binomials and rearrange terms to obtain the general form.

$(x+1)^2+(y-3)^2=20$	Standard equation
$(x+1)(x+1)+(y-3)(y-3)=20$	Expand the factors.
$x^2+2x+1+y^2-6y+9=20$	Square the binomials to obtain trinomials.
$x^2+2x+y^2-6y+10=20$	Combine the constant terms on the left side of the equation.

Guided Practice: Example 3, *continued*

$x^2 + 2x + y^2 - 6y - 10 = 0$	Subtract 20 from both sides to get 0 on the right side.
$x^2 + y^2 + 2x - 6y - 10 = 0$	Rearrange terms in descending order to obtain the general equation.

The general equation is

$x^2 + y^2 + 2x - 6y - 10 = 0.$

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Guided Practice: Example 3, *continued*

Guided Practice

Example 5

Find the center and radius of the circle described by the equation $4x^2 + 4y^2 + 20x - 40y + 116 = 0$.

Guided Practice: Example 5, *continued*

1. Rewrite the equation in standard form.

$4x^2 + 4y^2 + 20x - 40y + 116 = 0$	General form of the equation
$x^2 + y^2 + 5x - 10y + 29 = 0$	Divide each term on both sides by 4 to make the leading coefficient 1.
$x^2 + y^2 + 5x - 10y = -29$	Subtract 29 from both sides to get the constant term on one side.
$x^2 + 5x + y^2 - 10y = -29$	Combine like terms.

Guided Practice: Example 5, *continued*

Next, complete the square for both variables. Add the same values to both sides of the equation, as shown below:

$$x^2+5x+\left(\frac{5}{2}\right)^2+y^2-10y+\left(\frac{-10}{2}\right)^2=-29+\left(\frac{5}{2}\right)^2+\left(\frac{-10}{2}\right)^2$$

Simplify the equation.

$$x^2+5x+\frac{25}{4}+y^2-10y+25=-29+\frac{25}{4}+25$$

Guided Practice: Example 5, *continued*

$$\left(x+\frac{5}{2}\right)^2+(y-5)^2=\frac{9}{4}$$

Write the perfect square trinomials as squares of binomials.

The standard equation is $\left(x+\frac{5}{2}\right)^2+(y-5)^2=\frac{9}{4}$.

Guided Practice: Example 5, *continued*

2. Determine the center and radius.

$$\left(x+\frac{5}{2}\right)^2+(y-5)^2=\frac{9}{4}$$

Write the standard equation from step 1.

$$\left[x-\left(-\frac{5}{2}\right)\right]^2+(y-5)^2=\left(\frac{3}{2}\right)^2$$

Rewrite to match the form $(x-h)^2+(y-k)^2=r^2$.

Guided Practice: Example 5, *continued*

For the equation $(x - h)^2 + (y - k)^2 = r^2$, the center is (h, k) and the radius is r, so for the equation $\left[x-\left(-\frac{5}{2}\right)\right]^2+(y-5)^2=\left(\frac{3}{2}\right)^2$, the center is $\left(-\frac{5}{2}, 5\right)$ and the radius is $\frac{3}{2}$.

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Guided Practice: Example 5, *continued*