Introduction

Navigators and surveyors use the properties of similar right triangles. Designers and builders use right triangles in constructing structures and objects. Cell phones and Global Positioning Systems (GPS) use the mathematical principles of algebra, geometry, and trigonometry. **Trigonometry** is the study of triangles and the relationships between their sides and the angles between these sides. In this lesson, we will learn about the ratios between angles and side lengths in right triangles. A **ratio** is the relation between two quantities; it can be expressed in words, fractions, decimals, or as a percentage.

Key Concepts

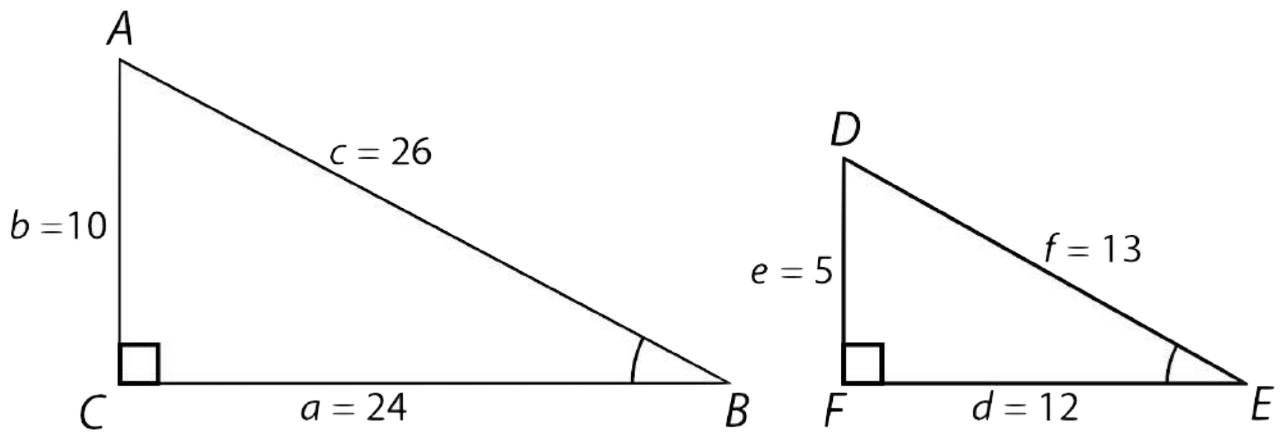
- Two triangles are similar if they have congruent angles.
- Remember that two figures are **similar** when they are the same shape but not necessarily the same size; the symbol for representing similarity is $\sim$.
- Every right triangle has one 90º angle.
- If two right triangles each have a second angle that is congruent with the other, the two triangles are similar.

Key Concepts, *continued*

- Similar triangles have proportional side lengths. The side lengths are related to each other by a scale factor.
- Examine the proportional relationships between similar triangles $\triangle ABC$ and $\triangle DEF$ in the diagram that follows. The scale factor is $k = 2$. Notice how the ratios of corresponding side lengths are the same as the scale factor.

Key Concepts, *continued*

Proportional Relationships in Similar Triangles



Corresponding sides

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

Side lengths

$$\frac{24}{12} = \frac{10}{5} = \frac{26}{13}$$

(continued)

Key Concepts, *continued*

Examine the three ratios of side lengths in $\triangle ABC$. Notice how these ratios are equal to the same ratios in $\triangle DEF$.

Corresponding sides	Side lengths
$\frac{a}{c}=\frac{d}{f}$	$\frac{24}{26}=\frac{12}{13}$
$\frac{b}{c}=\frac{e}{f}$	$\frac{10}{26}=\frac{5}{13}$
$\frac{a}{b}=\frac{d}{e}$	$\frac{24}{10}=\frac{12}{5}$

Key Concepts, *continued*

- The ratio of the lengths of two sides of a triangle is the same as the ratio of the corresponding sides of any similar triangle.
- The three main ratios in a right triangle are the sine, the cosine, and the tangent. These ratios are based on the side lengths relative to one of the acute angles.

Key Concepts, *continued*

- The **sine** of an acute angle in a right triangle is the ratio of the length of the opposite side to the length of the hypotenuse;

$$\text{the sine of } \theta = \sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}.$$

- The **cosine** of an acute angle in a right triangle is the ratio of the length of the side adjacent to the length of the hypotenuse;

$$\text{the cosine of } \theta = \cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}.$$

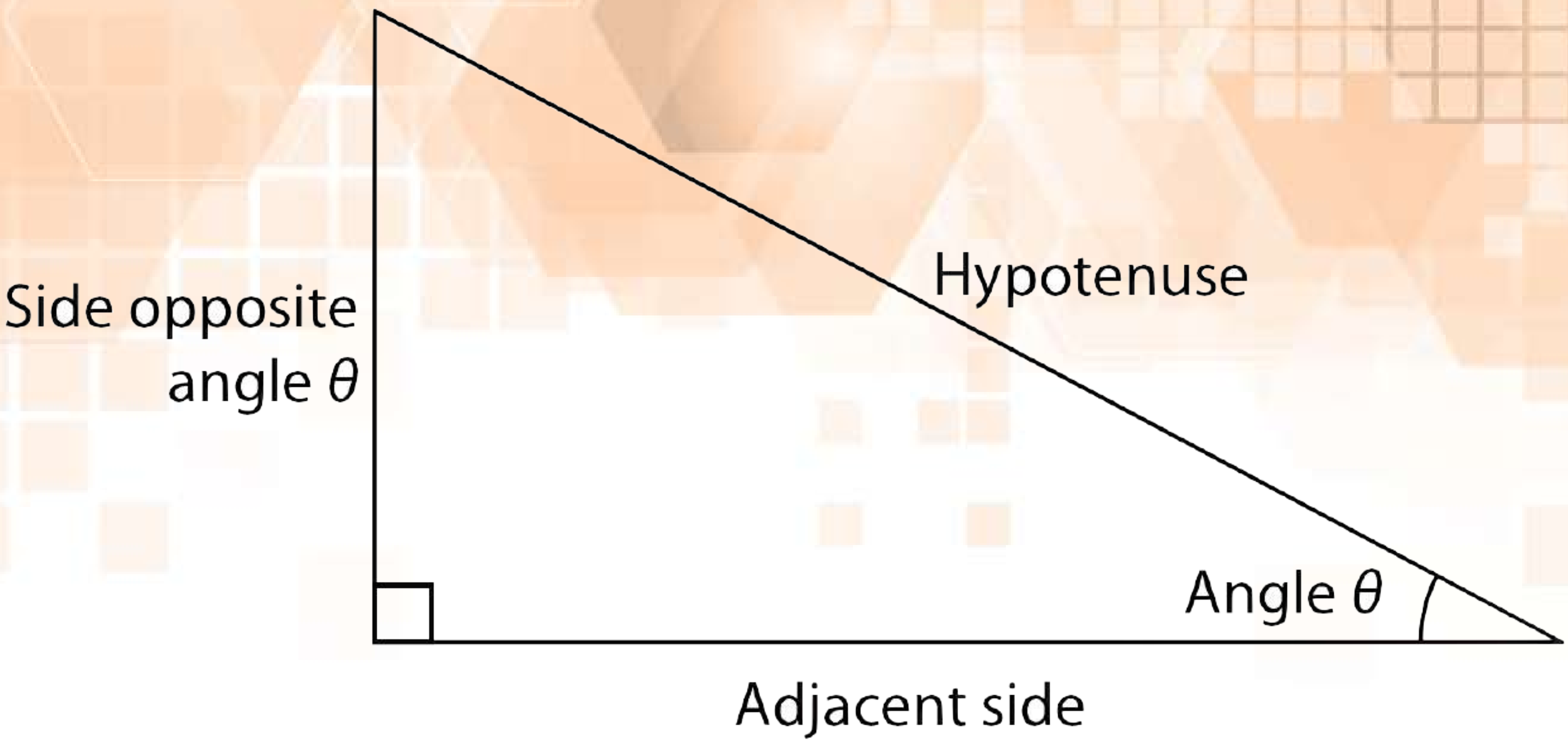
Key Concepts, *continued*

- The **tangent** of an acute angle in a right triangle is the ratio of the length of the opposite side to the length of the adjacent side;

$$\text{the tangent of } \theta = \tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}.$$

- The acute angle that is being used for the ratio is known as the **reference angle**. It is commonly marked with the symbol (*theta*).
- ***Theta*** ( ) is a Greek letter commonly used as an unknown angle measure.

Key Concepts, *continued*



Key Concepts, *continued*

- See the following examples of the ratios for sine, cosine, and tangent.

$$\text{sine of } \theta = \sin\theta = \frac{\text{length of opposite side}}{\text{length of hypotenuse}}$$

$$\text{Abbreviation: } \frac{\text{opposite}}{\text{hypotenuse}}$$

Key Concepts, *continued*

$$\text{cosine of } \theta = \cos\theta = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$$

$$\text{Abbreviation: } \frac{\text{adjacent}}{\text{hypotenuse}}$$

Key Concepts, *continued*

$$\text{tangent of } \theta = \tan\theta = \frac{\text{length of opposite side}}{\text{length of adjacent side}}$$

$$\text{Abbreviation: } \frac{\text{opposite}}{\text{adjacent}}$$

Key Concepts, *continued*

- Unknown angle measures can also be written using the Greek letter *phi* (ϕ).
- The three main ratios can also be shown as reciprocals.
- The **reciprocal** is a number that when multiplied by the original number the product is 1.

Key Concepts, *continued*

- The reciprocal of sine is **cosecant**. The reciprocal of cosine is **secant**, and the reciprocal of tangent is **cotangent**.

$$\text{cosecant of } \theta = \csc\theta = \frac{\text{length of hypotenuse}}{\text{length of opposite side}}$$

$$\text{secant of } \theta = \sec\theta = \frac{\text{length of hypotenuse}}{\text{length of adjacent side}}$$

$$\text{cotangent of } \theta = \cot\theta = \frac{\text{length of adjacent side}}{\text{length of opposite side}}$$

Key Concepts, *continued*

- Each acute angle in a right triangle has different ratios of sine, cosine, and tangent.
- The length of the hypotenuse remains the same, but the sides that are opposite or adjacent for each acute angle will be different for different reference angles.
- The two rays of each acute angle in a right triangle are made up of a leg and the hypotenuse. The leg is called the **adjacent side** to the angle. Adjacent means “next to.”

Key Concepts, *continued*

- In a right triangle, the side of the triangle opposite the reference angle is called the **opposite side**.
- Calculations in trigonometry will vary due to the variations that come from measuring angles and distances.
- A final calculation in trigonometry is frequently expressed as a decimal.
- A calculation can be made more accurate by including more decimal places.

Key Concepts, *continued*

- The context of the problem will determine the number of decimals places to which to round. Examples:
  - A surveyor usually measures tracts of land to the nearest tenth of a foot.
  - A computer manufacturer needs to measure a microchip component to a size smaller than an atom.
  - A carpenter often measures angles in whole degrees.
  - An astronomer measures angles to $\frac{1}{3600}$ of a degree or smaller.

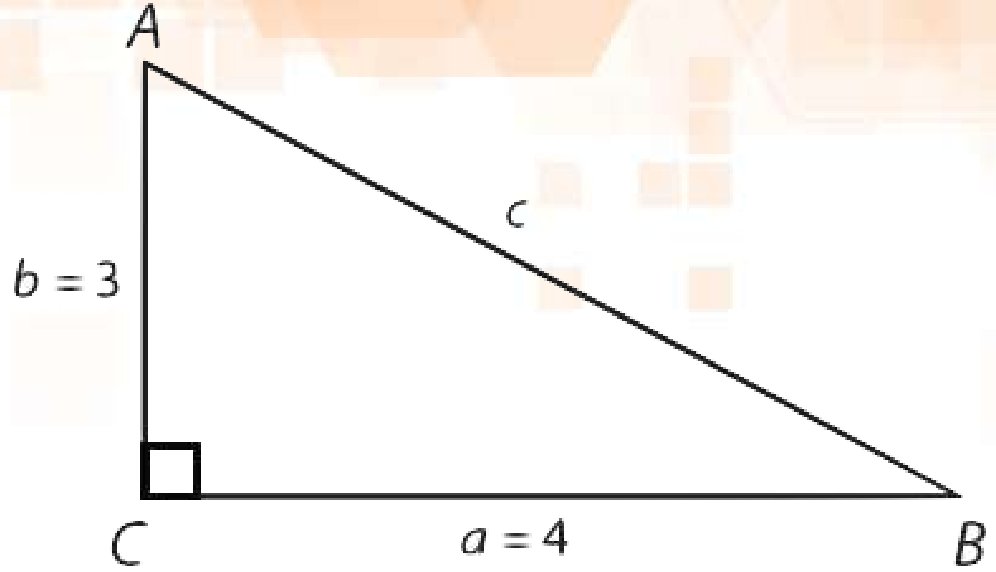
Common Errors/Misconceptions

- confusing the differences between the trigonometric ratios
- forgetting to change the adjacent and opposite sides when working with the two acute angles
- mistakenly trying to use sine, cosine, and tangent ratios for triangles that are not right triangles
- mistakenly thinking that trigonometry will find the exact length of a side or the exact measure of an angle

Guided Practice

Example 1

Find the sine, cosine, and tangent ratios for $\angle A$ and $\angle B$ in the triangle $\triangle ABC$. Convert the ratios to decimal equivalents.



Guided Practice: Example 1, *continued*

1. Find the length of the hypotenuse using the Pythagorean Theorem.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$4^2 + 3^2 = c^2$	Substitute values for *a* and *b*.
$16 + 9 = c^2$	Simplify.
$25 = c^2$	
$\pm\sqrt{25} = \sqrt{c^2}$	
$c = \pm 5$	

Since *c* is a length, use the positive value, *c* = 5.

Guided Practice: Example 1, *continued*

2. Find the sine, cosine, and tangent of $\angle A$.

Set up the ratios using the lengths of the sides and hypotenuse, then convert to decimal form.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} = 0.8$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} = 1.33\overline{3}$$

Guided Practice: Example 1, *continued*

3. Find the sine, cosine, and tangent of $\angle B$.

Set up the ratios using the lengths of the sides and hypotenuse, then convert to decimal form.

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$$

$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} = 0.8$$

$$\tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4} = 0.75$$

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Guided Practice: Example 1, *continued*

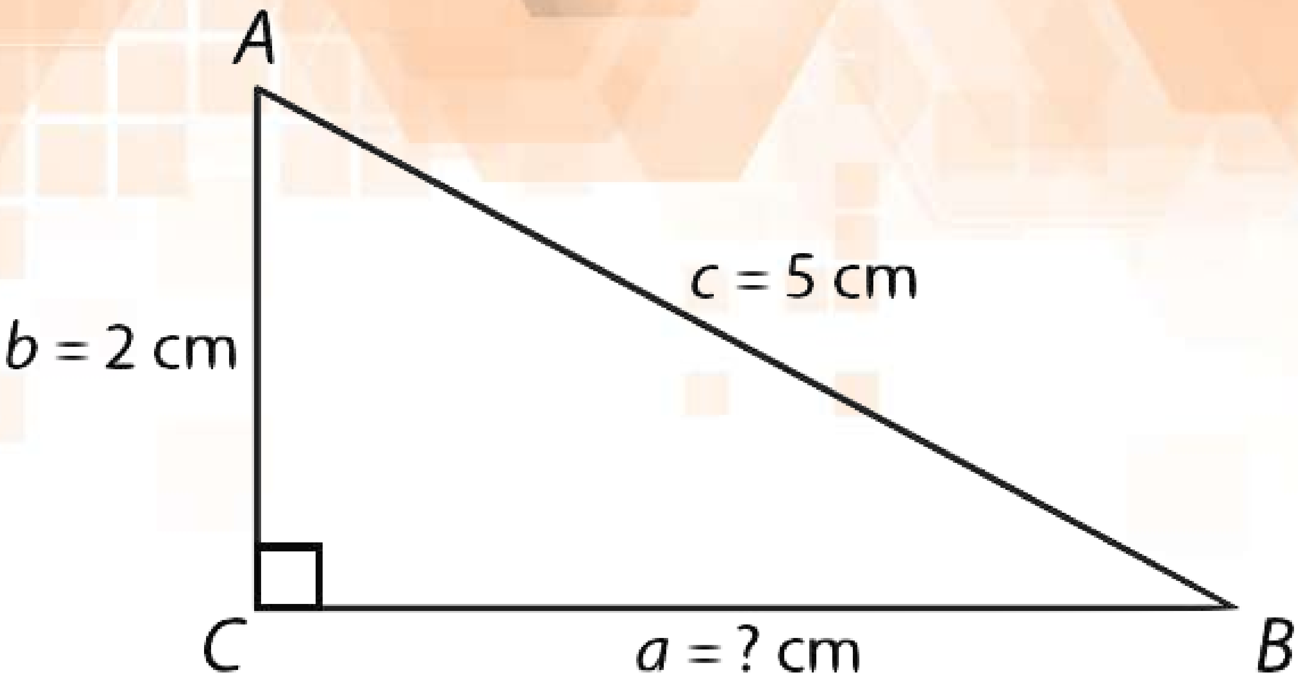
Guided Practice

Example 3

A right triangle has a hypotenuse of 5 and a side length of 2. Find the angle measurements and the unknown side length. Find the sine, cosine, and tangent for both angles. Without drawing another triangle, compare the trigonometric ratios of $\triangle ABC$ with those of a triangle that has been dilated by a factor of $k = 3$.

Guided Practice: Example 3, *continued*

1. First, draw the triangle with a ruler, and label the side lengths and angles.



Guided Practice: Example 3, *continued*

2. Find *a* by using the Pythagorean Theorem.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$a^2 + 2^2 = 5^2$	Substitute values for *b* and *c*.
$a^2 + 4 = 25$	Simplify.
$a^2 = 21$	
$a \approx 4.5826$ centimeters	

Guided Practice: Example 3, *continued*

3. **Use a protractor to measure one of the acute angles, and then use that measurement to find the other acute angle.**

$$m\angle A \approx 66.5^\circ$$

We know that $m\angle C = 90^\circ$ by the definition of right angles.

The measures of the angles of a triangle sum to 180º.

Guided Practice: Example 3, *continued*

Subtract $m\angle A$ and $m\angle C$ from 180 to find $m\angle B$.

$$m\angle B = 180 - m\angle A - m\angle C$$

$$m\angle B = 180 - (66.5) - (90)$$

$$m\angle B = 180 - 156.5$$

$$m\angle B \approx 23.5$$

Guided Practice: Example 3, *continued*

4. Find the sine, cosine, and tangent ratios for both acute angles. Express your answer in decimal form to the nearest thousandth.

Guided Practice: Example 3, *continued*

$\angle A$

$$\sin 66.5° \approx \frac{4.5826}{5} \approx 0.916$$

$$\cos 66.5° \approx \frac{2}{5} \approx 0.4$$

$$\tan 66.5° \approx \frac{4.5826}{2} \approx 2.291$$

$\angle B$

$$\sin 23.5° \approx \frac{2}{5} \approx 0.4$$

$$\cos 23.5° \approx \frac{4.5826}{5} \approx 0.916$$

$$\tan 23.5° \approx \frac{2}{4.5826} \approx 0.436$$

Guided Practice: Example 3, *continued*

5. **Without drawing a triangle, find the sine, cosine, and tangent for a triangle that has a scale factor of 3 to $\triangle ABC$. Compare the trigonometric ratios for the two triangles.**

   Multiply each side length (*a*, *b*, and *c*) by 3 to find *a*′, *b*′, and *c*′.

$$a' = 3 \bullet a = 3 \bullet (4.5826) = 13.7478$$

$$b' = 3 \bullet b = 3 \bullet (2) = 6$$

$$c' = 3 \bullet c = 3 \bullet (5) = 15$$

Guided Practice: Example 3, *continued*

Set up the ratios using the side lengths of the dilated triangle.

$$\sin 66.5^\circ \approx \frac{13.7478}{15} \approx 0.916 \qquad \sin 23.5^\circ \approx \frac{6}{15} \approx 0.4$$

$$\cos 66.5^\circ \approx \frac{6}{15} \approx 0.4 \qquad \cos 23.5^\circ \approx \frac{13.7478}{15} \approx 0.916$$

$$\tan 66.5^\circ \approx \frac{13.7478}{6} \approx 2.291 \qquad \tan 23.5^\circ \approx \frac{6}{13.7478} \approx 0.436$$

Guided Practice: Example 3, *continued*

The sine, cosine, and tangent do not change in the larger triangle. Similar triangles have identical side length ratios and, therefore, identical trigonometric ratios.

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Guided Practice: Example 3, *continued*