Accelerated Mathematics III – Unit 2: Sequences and Series

Introduction:

In earlier grades, students learned about arithmetic and geometric sequences and their relationships to linear and exponential functions, respectively. This unit builds on students' understandings of those sequences and extends students' knowledge to include arithmetic and geometric series, both finite and infinite. Summation notation and properties of sums are also introduced. Additionally, students will examine other types of sequences and, if appropriate, proof by induction. They will use their knowledge of the characteristics of the types of sequences and the corresponding functions to compare scenarios involving different sequences.

Enduring Understandings:

- All arithmetic and geometric sequences can be expressed recursively and explicitly. Some other sequences also can be expressed in both ways but others cannot.
- Arithmetic sequences are identifiable by a common difference and can be modeled by linear functions. Infinite arithmetic series always diverge.
- Geometric sequences are identifiable by a common ratio and can be modeled by exponential functions. Infinite geometric series diverge if $|r| \geq 1$ and converge is $|r| < 1$.
- The sums of finite arithmetic and geometric series can be computed with easily derivable formulas.
- Identifiable sequences and series are found in many naturally occurring objects.
- Repeating decimals can be expressed as fractions by summing appropriate infinite geometric series.
- The principle of mathematical induction is a method for proving that a statement is true for all positive integers (or all positive integers greater than a specified integer).

Unit overview:

- The launching activity begins by revisiting ideas of arithmetic sequences studied in eighth and ninth grades. Definitions, as well as the explicit and recursive forms of arithmetic sequences are reviewed. The task then introduces summations, including notation and operations with summations, and summing arithmetic series.
- The second set of tasks reviews geometric sequences and investigates sums, including infinite and finite geometric series, in the context of exploring fractals. It is assumed that students have some level of familiarity with geometric sequences and the relationship between geometric sequences and exponential functions.
- The third group addresses some common sequences and series, including the Fibonacci sequence, sequences with factorials, and repeating decimals. Additionally, mathematical induction is employed to prove that the explicit forms are valid.
- The culminating task is set in the context of applying for a job at an interior design agency. In each task, students will need to determine which type of sequence is called for, justify their choice, and occasionally prove they are correct. Students will complete the handshake problem, the salary/retirement plan problem, and some open-ended design problems that require the use of various sequences and series.

Key Standards Addressed:

MA3A9. Students will use sequences and series

a. Use and find recursive and explicit formulae for the terms of sequences.

b. Recognize and use simple arithmetic and geometric sequences.

c. Investigate limits of sequences.

d. Use mathematical induction to find and prove formulae for sums of finite series.

e. Find and apply the sums of finite and, where appropriate, infinite arithmetic and geometric series.

f. Use summation notation to explore series.

g. Determine geometric series and their limits.

Related Standards Addressed:

MA3A1. Students will explore rational function.

MA3A4. Students will investigate functions.

MA3P1. Students will solve problems (using appropriate technology).

MA3P2. Students will reason and evaluate mathematical arguments.

MA3P3. Students will communicate mathematically.

MA3P4. Students will make connections among mathematical ideas and to other disciplines.

MA3P5. Students will represent mathematics in multiple ways.

Vocabulary and formulas

Arithmetic sequence: A sequence of terms $a_1, a_2, a_3, \ldots$ with $d = a_n - a_{n-1}$. The explicit formula is given by $a_n = a_1 + (n-1)d$ and the recursive form is a_1 = value of the first term and $a_n = a_{n-1} + d$.

Arithmetic series: The sum of a set of terms in arithmetic progression $a_1 + a_2 + a_3 + \ldots$ with $d = a_n - a_{n-1}$.

Common difference: In an arithmetic sequence or series, the difference between two consecutive terms is *d*, $d = a_n - a_{n-1}$.

Common ratio: In a geometric sequence or series, the ratio between two consecutive terms is *r*, $r = \frac{a_n}{a_{n-1}}$.

Explicit formula: A formula for a sequence that gives a direct method for determining the *nth* term of the sequence. It presents the relationship between two quantities, i.e. the term number and the value of the term.

Factorial: If *n* is a positive integer, the notation *n!* (read "*n* factorial") is the product of all positive integers from *n* down through 1; that is, $n! = n(n-1)(n-2)\ldots(3)(2)(1)$. Note that 0!, by definition, is 1; i.e. $0! = 1$.

Finite series: A series consisting of a finite, or limited, number of terms.

Infinite series: A series consisting of an infinite number of terms.

Geometric sequence: A sequence of terms $a_1, a_2, a_3, \ldots$ with $r = \frac{a_n}{a_{n-1}}$. The explicit formula is given by $a_n = a_1 r^{n-1}$ and the recursive form is $a_1 = \text{value of the first term}$ and $a_n = (a_{n-1})r$.

Geometric series: The sum of a set of terms in geometric progression $a_1 + a_2 + a_3 + \ldots$ with $r = \frac{a_n}{a_{n-1}}$.

Limit of a sequence: The long-run value that the terms of a convergent sequence approach.

Partial sum: The sum of a finite number of terms of an infinite series.

Recursive formula: Formula for determining the terms of a sequence. In this type of formula, each term is dependent on the term or terms immediately before the term of interest. The recursive formula must specific at least one term preceding the general term.

Sequence: A sequence is an ordered list of numbers.

Summation or sigma notation: $\sum_{i=1}^{n} a_i$, where *i* is the index of summation, *n* is the upper limit of summation, and 1 is the lower limit of summation. This expression gives the partial sum, the sum of the first *n* terms of a sequence. More generally, we can write $\sum_{i=k}^{n} a_i$, where *k* is the starting value.

Sum of a finite arithmetic series: The sum, S_n, of the first *n* terms of an arithmetic sequence is given by $S_n = \frac{n(a_1 + a_n)}{2}$**,** where a_1 = value of the first term and a_n = value of the last term in the sequence.

Sum of a finite geometric series: The sum, S_n, of the first *n* terms of a geometric sequence is given by $S_n = \frac{a_1 - a_n r}{1-r} = \frac{a_1(1-r^n)}{1-r}$, where a_1 is the first term and *r* is the common ratio ($r \neq 1$).

Sum of an infinite geometric series: The general formula for the sum *S* of an infinite geometric series $a_1 + a_2 + a_3 + \ldots$ with common ratio *r* where $|r| < 1$ is $S = \frac{a_1}{1-r}$. If an infinite geometric series has a sum, i.e. if $|r| < 1$, then the series is called a **convergent** geometric series. All other geometric (and arithmetic) series are **divergent**.

Term of a sequence: Each number in a sequence is a term of the sequence. The first term is generally noted as a_1, the second as a_2, …, the *nth* term is noted as a_n. a_n is also referred to as the general term of a sequence.

RENAISSANCE FESTIVAL LEARNING TASK

As part of a class project on the Renaissance, your class decided to plan a renaissance festival for the community. Specifically, you are a member of different groups in charge of planning two of the contests. You must help plan the archery and rock throwing contests. The following activities will guide you through the planning process.

Group One: Archery Contest[1]

[1] Elements of these problems were adapted from *Integrated Mathematics 3* by McDougal-Littell, 2002.**)**

Before planning the archery contest, your group decided to investigate the characteristics of the target. The target being used has a center, or bull's-eye, with a radius of 4 cm, and nine rings that are each 4 cm wide.

1. The Target
   a. Sketch a picture of the center and first 3 rings of the target on the provided graph paper.

   b. Write a sequence that gives the radius of each of the concentric circles that comprise the entire target.

   c. Write a recursive formula and an explicit formula for the terms of this sequence.

   d. What would be the radius of the target if it had 25 rings? Show how you completed this problem using the explicit formula.

   e. In the past, you have studied both arithmetic and geometric sequences. What is the difference between these two types of sequences? Is the sequence in (b) arithmetic, geometric, or neither? Explain.

One version of the explicit formula uses the first term, the common difference, and the number of terms in the sequence. For example, if we have the arithmetic sequence 2, 5, 8, 11, 14, …, we see that the common difference is 3. If we want to know the value of the 20th term, or a_{20}, we could think of starting with $a_1 = 2$ and adding the difference, d = 3 a certain number of times. How many times would we need to add the common difference to get to the 20th term? _____ Because multiplication is repeated addition, instead of adding 3 that number of times, we could multiply the common difference, 3, by the number of times we would need to add it to 2.

This gives us the following explicit formula for an arithmetic sequence: $a_n = a_1 + (n-1)d$.

   f. Write this version of the explicit formula for the sequence in this problem. Show how this version is equivalent to the version above.

g. Can you come up with a reason for which you would want to add up the radii of the concentric circles that make up the target (for the purpose of the contest)? Explain.

h. Plot the sequence from this problem on a coordinate grid. What should you use for the independent variable? For the dependent variable? What type of graph is this? How does the a_n equation of the recursive formula relate to the graph? How does the parameter *d* in the explicit form relate to the graph?

i. Describe (using y-intercept and slope), but do not graph, the plots of the arithmetic sequences defined explicitly or recursively as follows:

1. $a_n = 3 + \frac{4}{3}(n-1)$

3. $a_n = 4.5 - 3.2(n-1)$

2. $\begin{cases} a_1 = -2 \\ a_n = a_{n-1} + \frac{1}{2} \end{cases}$

4. $\begin{cases} a_1 = 10 \\ a_n = a_{n-1} - \frac{2}{5} \end{cases}$

2. The Area of the Target: To decide on prizes for the archery contest, your group decided to use the areas of the center and rings. You decided that rings with smaller areas should be worth more points. But how much more? Complete the following investigation to help you decide.

a. Find the sequence of the areas of the rings, including the center. (Be careful.)

b. Write a recursive formula and an explicit formula for this sequence.

c. If the target was larger, what would be the area of the 25th ring?

d. Find the total area of the bull's eye by adding up the areas in the sequence.

e. Consider the following sum: $S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_{n-2} + a_n$. Explain why that equation is equivalent to $S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_n - 2d) + (a_n - d) + a_n$.

Rewrite this latter equation and then write it out backwards. Add the two resulting equations. Use this to finish deriving the formula for the sum of the terms in an arithmetic sequence. Try it out on a few different short sequences.

f. Use the formula for the sum of a finite arithmetic sequence in part (e) to verify the sum of the areas in the target from part (d).

g. Sometimes, we do not have all the terms of the sequence but we still want to find a specific sum. For example, we might want to find the sum of the first 15 multiples of 4. Write an explicit formula that would represent this sequence. Is this an arithmetic sequence? If so, how could we use what we know about arithmetic sequences and the sum formula in (e) to find this sum? Find the sum.

h. What happens to the sum of the arithmetic series we've been looking at as the number of terms we sum gets larger? How could you find the sum of the first 200 multiples of 4? How could you find the sum of all the multiples of 4? Explain using a graph and using mathematical reasoning.

i. Let's practice a few arithmetic sum problems.

1. Find the sum of the first 50 terms of 15, 9, 3, -3, …
2. Find the sum of the first 100 natural numbers
3. Find the sum of the first 75 positive even numbers
4. Come up with your own arithmetic sequence and challenge a classmate to find the sum.

j. Summarize what you learned / reviewed about arithmetic sequences and series during this task.

3. Point Values: Assume that each participant's arrow hits the surface of the target.

a. Determine the probability of hitting each ring and the bull's-eye.

Target Piece	**Area of Piece (in cm^2)**	**Probability of Hitting this Area**
Bull's Eye	16π	
Ring 1	48π	

Ring 2	80π	
Ring 3	112π	
Ring 4	144π	
Ring 5	176π	
Ring 6	208π	
Ring 7	240π	
Ring 8	272π	
Ring 9	304π	

b. Assign point values for hitting each part of the target, justifying the amounts based on the probabilities just determined.

c. Use your answer to (b) to determine the expected number of points one would receive after shooting a single arrow.

d. Using your answers to part (c), determine how much you should charge for participating in the contest OR for what point values participants would win a prize. Justify your decisions.