AP Calculus – Free Response Unit 6 #1
Solutions File

#1a (2005 AB3)

AP Scoring Rubric

(a) $\dfrac{T(8)-T(6)}{8-6} = \dfrac{55-62}{2} = -\dfrac{7}{2}\,{}^\circ\text{C/cm}$	1 : answer

Actual Solution Receiving Full Credit

Work for problem 3(a)

$$T'(7) = \frac{55-62}{8-6} = -3.5 \ {}^\circ\text{C/cm}$$

#1b

AP Scoring Rubric

(b) $\dfrac{1}{8}\displaystyle\int_0^8 T(x)\,dx$ Trapezoidal approximation for $\displaystyle\int_0^8 T(x)\,dx$: $A = \dfrac{100+93}{2}\cdot 1 + \dfrac{93+70}{2}\cdot 4 + \dfrac{70+62}{2}\cdot 1 + \dfrac{62+55}{2}\cdot 2$ Average temperature $\approx \dfrac{1}{8}A = 75.6875^\circ\text{C}$	$3: \begin{cases} 1: \frac{1}{8}\int_0^8 T(x)\,dx \\ 1: \text{trapezoidal sum} \\ 1: \text{answer} \end{cases}$

Actual Solution Receiving Full Credit

$$\text{Average Temp} = \frac{1}{8}\int_0^8 T(x)\,dx.$$

$$\begin{aligned}\text{Average} &= \frac{1}{8}\cdot\Big[(100+93)(1)(\tfrac{1}{2}) + (93+70)(4)(\tfrac{1}{2}) \\ &\quad + (62+70)(1)(\tfrac{1}{2}) + (55+62)(2)(\tfrac{1}{2})\Big] \\ &= 75.688\ {}^\circ\text{C}\end{aligned}$$

#1c

AP Scoring Rubric

(c) $\displaystyle\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45^\circ\text{C}$ The temperature drops 45°C from the heated end of the wire to the other end of the wire.	$2: \begin{cases} 1: \text{value} \\ 1: \text{meaning} \end{cases}$

Actual Solution Receiving Full Credit

$$\int_0^8 T'(x)\,dx = T(8) - T(0)$$
$$= 55 - 100$$
$$= -45°C$$

$\int_0^8 T'(x)\,dx$ mean the total change (drop) in temperature of the wire from 0 cm to 8 cm

#1d

AP Scoring Rubric

(d) Average rate of change of temperature on $[1, 5]$ is $\dfrac{70-93}{5-1} = -5.75$.

Average rate of change of temperature on $[5, 6]$ is $\dfrac{62-70}{6-5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some c_1 in the interval $(1, 5)$ and $T'(c_2) = -8$ for some c_2 in the interval $(5, 6)$. It follows that T' must decrease somewhere in the interval (c_1, c_2). Therefore T'' is not positive for every x in $[0, 8]$.

$2 : \begin{cases} 1 : \text{two slopes of secant lines} \\ 1 : \text{answer with explanation} \end{cases}$

Units of $°C/cm$ in (a), and $°C$ in (b) and (c)

1 : units in (a), (b), and (c)

Actual Solution Receiving Full Credit

$T''(x) > 0 \Rightarrow T'(x)$ is increasing over the period

from $x = 0$ to 1
slope $\Rightarrow -7$

$x = 1$ to 5
slope $\Rightarrow \dfrac{70-93}{5-1} = -5.75$

$x = 5$ to 6
slope $\Rightarrow \dfrac{62-70}{6-5} = -8.$

$x = 6$ to 8
slope $\Rightarrow \dfrac{55-62}{8-6} = -3.5$

By MVT

$\therefore$ between 5 to 6 there is a point with slope -8 which means a decrease of $T'(x)$

$\Downarrow$

$T'(x)$ is not always increasing

$\therefore T''(x) > 0$ is not consistent in the table data.

END OF PART A OF SECTION II

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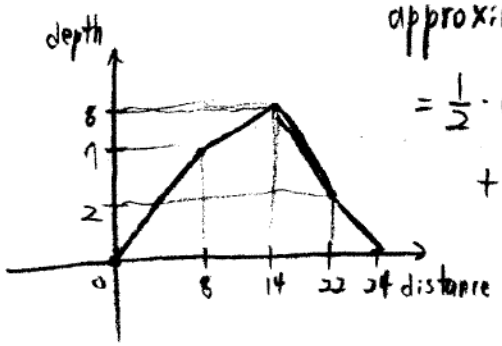
#2a (2008 ABB3)

AP Scoring Rubric

(a) $\dfrac{(0+7)}{2}\cdot 8 + \dfrac{(7+8)}{2}\cdot 6 + \dfrac{(8+2)}{2}\cdot 8 + \dfrac{(2+0)}{2}\cdot 2$ $= 115 \text{ ft}^2$	1 : trapezoidal approximation

Actual Solution Receiving Full Credit

Work for problem 3(a)



approximation of area using trapezoidal sum

$$= \frac{1}{2}\cdot(8)\cdot 7 + \frac{1}{2}\underbrace{(7+8)}_{=15}\cdot\underbrace{(14-8)}_{=6}$$

$$+ \frac{1}{2}\cdot\underbrace{(8+2)}_{=10}\cdot\underbrace{(22-14)}_{=8} + \frac{1}{2}\cdot 2\cdot\underbrace{(24-22)}_{=2}$$

$$= 4\cdot 7 + 3\cdot 15 + 5\cdot 8 + 2$$

$$= 28 + 45 + 40 + 2$$

$$= 73 + 42 = 115$$

$\underline{115\ (ft)^2}$

#2b

AP Scoring Rubric

(b) $\dfrac{1}{120}\displaystyle\int_0^{120} 115v(t)\,dt$ $= 1807.169$ or $1807.170 \text{ ft}^3/\text{min}$	3 : { 1 : limits and average value constant 1 : integrand 1 : answer

Actual Solution Receiving Full Credit

Work for problem 3(b) Average value of volumetric flow at Picnic Point

$$= \frac{1}{120-0}\left(\int_0^{120} v(t)\,dt\right)\cdot 115 = \frac{115}{120}\int_0^{120}\left(16 + 2\sin\left(\sqrt{t+10}\right)\right)dt$$

$$= \underline{1807.16491\ (ft)^3/min}$$

nd this border.

#2c

AP Scoring Rubric

(c) $\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right) dx = 122.230$ or 122.231 ft^2

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Actual Solution Receiving Full Credit

Work for problem 3(c)

$$\text{Area} = \int_0^{24} 8\sin\left(\frac{\pi x}{24}\right) dx$$

$\cos\pi = -1$

$\cos(0) = 1$

$$= \left[-\frac{24}{\pi}\cdot 8\cos\left(\frac{\pi x}{24}\right)\right]_0^{24} = -\frac{24\cdot 8}{\pi}\cos(\pi) + \frac{24\cdot 8}{\pi}\cos(0)$$

$$= -\frac{24\cdot 8}{\pi}\cdot(-1) + \frac{24\cdot 8}{\pi}\cdot(1)$$

$$= 2\cdot\frac{24\cdot 8}{\pi} = 122.23049\,(ft)^2$$

#2d

AP Scoring Rubric

(d) Let C be the cross-sectional area approximation from part (c). The average volumetric flow is

$\frac{1}{20}\int_{40}^{60} C\cdot v(t)\,dt = 2181.912$ or 2181.913 ft^3/min.

Yes, water must be diverted since the average volumetric flow for this 20-minute period exceeds 2100 ft^3/min.

$3 : \begin{cases} 1 : \text{volumetric flow integral} \\ 1 : \text{average volumetric flow} \\ 1 : \text{answer with reason} \end{cases}$

Actual Solution Receiving Full Credit

Average value of volumetric flow during $40 \leq t \leqq 60$

$$= \frac{1}{60-40}\left(\int_{40}^{60}\left(16+2\sin\left(\sqrt{t+10}\right)\right)dt\right)\cdot(122.23099)$$

$$= \frac{122.23099}{20}\cdot\int_{40}^{60}\left(16+2\sin\left(\sqrt{t+10}\right)\right)dt$$

$$= \underline{2181.91253\ (ft)^3/min} > 2100\ ft)^3/min$$

Thus, water must be diverted. //

#3a (2010 AB3)

AP Scoring Rubric

(a) $\int_0^3 r(t)\,dt = 2\cdot\frac{1000+1200}{2} + \frac{1200+800}{2} = 3200$ people	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

Actual Solution Receiving Full Credit

Work for problem 3(a)

initially 700 ppl present

$\int_0^3 r(t)\,dt$ ← amount arrived from t=0 to t=3

$= \frac{(1000+1200)\times 2}{2} + \frac{(800+1200)\times 1}{2}$

$= 3200$ ppl

#3b

AP Scoring Rubric

(b) The number of people waiting in line is increasing because people move onto the ride at a rate of 800 people per hour and for $2 < t < 3$, $r(t) > 800$.	1 : answer with reason

Actual Solution Receiving Full Credit

Between t=2 and t=3

The rate of ppl arriving is greater than the rate in which ppl move onto the ride

Therefore the number of ppl waiting in line is increasing between t=2 and t=3.

#3c

AP Scoring Rubric

(c) $r(t) = 800$ only at $t = 3$

For $0 \le t < 3$, $r(t) > 800$. For $3 < t \le 8$, $r(t) < 800$.

Therefore, the line is longest at time $t = 3$.

There are $700 + 3200 - 800 \cdot 3 = 1500$ people waiting in line at time $t = 3$.

$3 : \begin{cases} 1 : \text{identifies } t = 3 \\ 1 : \text{number of people in line} \\ 1 : \text{justification} \end{cases}$

Actual Solution Receiving Full Credit

Work for problem 3(c)

The line is the longest at $t=3$. since from $t=0$ to $t=3$ $r(t) >$ the rate ppl move onto the ride. From $t=3$ to $t=8$, $r(t) <$ the rate ppl move onto the ride so the lineup will be shorter.

Amount of ppl in line

$700 + \int_0^3 r(t)\,dt - (800 \times 3)$

$= 3900 - 2400$

$= 1500$ ppl in line at $t=3$

#3d

AP Scoring Rubric

(d) $0 = 700 + \int_0^t r(s)\,ds - 800t$

$3 : \begin{cases} 1 : 800t \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Actual Solution Receiving Full Credit

$700 + \int_0^t r(x)\,dx - \int_0^t 800\,dx = 0$

or

$700 + \int_0^t (r(x) - 800)\,dx = 0$

#4a . (2006 ABB4)

AP Scoring Rubric

(a) $f'(22) = \dfrac{15-3}{20-24} = -3$ calories/min/min

1 : $f'(22)$ and units

Actual Solution Receiving Full Credit

$f'(22) =$ Gradient of straight line from $t = 20$ to $t = 24$

$= \dfrac{3-15}{24-20}$

$= \dfrac{-12}{4}$

$= -3$ calories/minute^2

#4b

AP Scoring Rubric

(b) f is increasing on $[0, 4]$ and on $[12, 16]$.

On $(12, 16)$, $f'(t) = \dfrac{15-9}{16-12} = \dfrac{3}{2}$ since f has constant slope on this interval.

On $(0, 4)$, $f'(t) = -\dfrac{3}{4}t^2 + 3t$ and

$f''(t) = -\dfrac{3}{2}t + 3 = 0$ when $t = 2$. This is where f' has a maximum on $[0, 4]$ since $f'' > 0$ on $(0, 2)$ and $f'' < 0$ on $(2, 4)$.

On $[0, 24]$, f is increasing at its greatest rate when $t = 2$ because $f'(2) = 3 > \dfrac{3}{2}$.

$$4 : \begin{cases} 1 : f' \text{ on } (0, 4) \\ 1 : \text{shows } f' \text{ has a max at } t = 2 \text{ on } (0, 4) \\ 1 : \text{shows for } 12 < t < 16,\ f'(t) < f'(2) \\ 1 : \text{answer} \end{cases}$$

Actual Solution Receiving Full Credit

From graph, we see that f is only increasing in the intervals $0 \leq t \leq 4$ and $12 \leq t \leq 16$

Rate of increment for $12 \leq t \leq 16$

$$= \frac{15-9}{16-12}$$

$$= \frac{6}{4} = 1.5$$

For $0 \leq t \leq 4$,

$f'(t) = -\frac{3}{4}t^2 + 3t$

$f''(t) = -\frac{3}{2}t + 3$

At greatest rate of increment, $f''(t) = 0 \Rightarrow t = 2$

$f'''(t) = -\frac{3}{2} < 0 \Rightarrow$ At $t = 2$, rate of increment is greatest and not smallest

$f'(2) = -\frac{3}{4}(2)^2 + 3(2) = 3 > 1.5$

$\therefore f$ is increasing at its greatest rate ~~than~~ at $t = 2$

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#4c

AP Scoring Rubric

(c) $\int_6^{18} f(t)\,dt = 6(9) + \frac{1}{2}(4)(9+15) + 2(15)$

$= 132$ calories

$2 : \begin{cases} 1 : \text{method} \\ 1 : \text{answer} \end{cases}$

Actual Solution Receiving Full Credit

~~From 6 to 12 min~~

~~For $6 \le t \le 12$~~,

Total number of calories burned $= \int_6^{18} f(t)\,dt$

$\cancel{= \int_6^{12} 9\,dt + \int_{12}^{16}}$

$= 9(12-6) + \frac{1}{2}(9+15)(16-12) + 15(18-16)$

$= 54 + 48 + 30 = 132$ calories

#4d

AP Scoring Rubric

(d) We want $\frac{1}{12}\int_6^{18} (f(t) + c)\,dt = 15.$

This means $132 + 12c = 15(12)$. So, $c = 4$.

OR

Currently, the average is $\frac{132}{12} = 11$ calories/min.

Adding c to $f(t)$ will shift the average by c.

So $c = 4$ to get an average of 15 calories/min.

$2 : \begin{cases} 1 : \text{setup} \\ 1 : \text{value of } c \end{cases}$

Actual Solution Receiving Full Credit

Before setting is changed, average calories in $6 \le t \le 18$

$= \frac{1}{18-6}\int_6^{18} f(t)\,dt = \frac{132}{12} = 11$ calories

Now, $\frac{1}{18-6}\int_6^{18} [f(t) + c]\,dt = 15$

$\Rightarrow \frac{1}{12}\int_6^{18} f(t)\,dt + \frac{1}{12}\int_6^{18} c\,dt = 15$

$\Rightarrow \frac{1}{12}[cx]_6^{18} = 15 - 11 = 4$

$\frac{1}{12}(18-6)c = 4$

$\Rightarrow c = 4$